Systems & Control: Foundations & Applications

Peter I. Kogut
Günter R. Leugering

Optimal Control Problems for Partial Differential Equations on Reticulated Domains

Approximation and Asymptotic Analysis

Birkhäuser

Systems & Control: Foundations & Applications

Series Editor
Tamer Başar, University of Illinois at Urbana-Champaign

Editorial Board
Karl Johan Åström, Lund University of Technology, Lund, Sweden
Han-Fu Chen, Academia Sinica, Beijing, China
Bill Helton, University of California, San Diego, CA, USA
Alberto Isidori, University of Rome, Rome, Italy;
Washington University, St. Louis, MO, USA
Miroslav Krstic, University of California, San Diego, CA, USA
Alexander Kurzhanski, University of California, Berkeley, CA, USA;
Russian Academy of Sciences, Moscow, Russia
H. Vincent Poor, Princeton University, Princeton, NJ, USA
Mete Soner, Koç University, Istanbul, Turkey

For further volumes:
www.springer.com/series/4895

Peter I. Kogut • Günter R. Leugering

Optimal Control Problems for Partial Differential Equations on Reticulated Domains

Approximation and Asymptotic Analysis

Birkhäuser

Peter I. Kogut
Oles Honchar Dnipropetrovsk
National University
Faculty of Mathematics and Mechanics
Department of Differential Equations
Gagarin av., 72
49010 Dnipropetrovsk
Ukraine
p.kogut@i.ua

Günter R. Leugering
Friedrich-Alexander Universität
Erlangen-Nürnberg
Department of Mathematics
Chair of Applied Mathematics II
Martensstr. 3
91058 Erlangen
Germany
leugering@am.uni-erlangen.de

ISBN 978-0-8176-8148-7 e-ISBN 978-0-8176-8149-4
DOI 10.1007/978-0-8176-8149-4
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011912819

Mathematics Subject Classification (2010): 35B27, 35J25, 49J20, 93C20

© Springer Science+Business Media, LLC 2011
All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.
The use in this publication of trade names, trademarks, service marks, and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed on acid-free paper

www.birkhauser-science.com

To our wives, Ljuba and Barbara,
with love and thanks

Preface

Optimal control of partial differential equations (PDEs) is by now, after more than 50 years of ever-increasing scientific interest, a well-established discipline in mathematics with many interfaces to science and engineering. During its development, the complexity of the systems to be controlled has also increased significantly, so that today, for example, fluid-structure interactions, magneto-hydromechanical, and electromagnetical as well as chemical and civil engineering problems can be dealt with. However, the numerical realization of optimal controls based on, say, optimality conditions together with the simulation of states has also become an issue in scientific computing, as the number of variables involved may easily exceed a couple of millions and, hence, structure-exploiting discretization and corresponding adaptive algorithms have to be developed.

There are several trends to be observed in this discipline. One is to increase the complexity of the system description in terms of genuinely nonlinear partial differential equations and hybrid couplings to ordinary or even event-driven dynamics together with constraints not only on the controls but also on the states. This kind of investigation typically focuses on a more accurate modeling with respect to the physical description and typically subsumes simple domains for the components. On the other hand, we observe a development, in particular in multiscale modeling, where even classical equations are considered on very complicated domains such that the focus is on the increasing complexity of the geometry of the underlying domain. It obviously is desirable, but also mostly prohibitive, to ask for everything: a very detailed modeling of the process, the handling of very complex geometries, and a timely or even real-time capable numerical realization. However, in the context of modern industrial or science applications, this often turns out to be impossible if one insists on very high accuracy.

Model reduction or effective modeling for optimal control problems involving systems of PDEs on complicated domains, therefore, has been the focus of many research initiatives in the last decade. The dynamical system describing the behavior of states may be replaced by a low-dimensional one using, for in-

stance, a technique that has come to be known as the reduced basis approach or, possibly greedy, proper orthogonal decomposition. Here, the geometry is built into the simulation tools that provide the empirical bases using "snapshots" which, in turn, makes this approach difficult for hierarchically ordered domains. On the other hand, one may use "surrogate models" on simple domains, models that are simplifying approximations of both the state equation and the domain.

A third avenue, the one along which we will proceed in this monograph, is based on asymptotic analysis. The method we describe and further develop aims at combining techniques of homogenization and approximation in order to cover optimal control problems defined on reticulated domains, such as lattice structures, honeycomb structures, hierarchical structures, or networked domains in general. Here, error estimates are introduced in order to control the quality of the controls obtained on the approximation level.

Our interest is mainly in problems where the control is exerted at, for example, highly oscillating boundaries or interfaces associated with such structures, but we also ask for "controls in the coefficients", that is, for controlling material parameters on the microscopic ε-level. Our research is strongly motivated by recent developments in multiscale modeling and simulation in a variety of applications. However, from a mathematical point of view, only the aspect ratio – that is, the relation of, say, thickness versus length – is relevant (as long as one does not enter molecular dynamics). In that respect, we can also relate our research to networked structure mechanics in civil engineering, such as flexible structures, masts of all kinds, and gas, water, and traffic networks.

As for material sciences, metallic, ceramic, or polymeric foams are particularly interesting because of their mechanical properties, such as being extremely lightweight and at the same time adequately stiff. Similarly, complex conductors on the micro-level exhibit graphlike structures, and carbon nanotubes are used in many applications. They themselves serve as gridlike domains supporting processes like electromagnetic wave propagation. However, properly assembled in thousands or millions on a waferlike substrate, they can be used as reactors for catalytic processes.

Mathematically speaking, elliptic, parabolic, and hyperbolic systems on reticulated domains are used to model these applications which, in turn, exhibit a genuine multiscale character. Engineers define cost or merit functions that are to be optimized with respect to various control actions along the boundary of the structure and, even more challenging, with respect to material properties. Certainly, both states and controls have to be taken into account as being constrained.

In particular, state constraints genuinely give rise to Lagrange-multipliers that are measures. As a result, in a general setting, such PDEs on reticulated domains should be dealt with in the context of descriptions allowing for measures on the "right-hand side" as well as in the system equations and the geometries. This means that providing an abstract approach to optimal

control problems for PDEs in such domains is not the result of an intrinsically motivated mathematical desire to achieve the highest degree of generality—it follows the needs dictated by the applications!

Moreover, the type of reticulated structures to be considered varies drastically with the application context or with the degree of approximation needed in a given context. Namely, in a gas network or a sewer system, civil engineers trust in one-dimensional models for the gas-flow, and consequently in network-flow problems on graphs. In a river system a fully three-dimensional treatment may be necessary, but, still, the problem would be one on a networked domain—a "fat graph". The same rationale applies to elastic multistructures, foams, and all the way down the scales to nanotubes. Percolation networks carrying microflows can be modeled as perforated domains, but, in a further approximation, also as fat graphs.

Obviously, the aspect ratio and the cell size for the periodic structure serve as scaling parameters. An optimal control problem defined on such structures, therefore, genuinely inherits a number of scales that may be separated or even not separable at all. The fundamental problem now is: How do such optimal control problems "behave" upon changing these parameters? More precisely, what happens if we go up the scales to a "continuum" description? However, why should we do that to begin with? This question brings us back to the question of model reduction and its interplay with approximation properties. For a small scale, for example, a fine gridstructure, the numerical effort related to a discretization in order to reveal a resolution according to the scale is typically prohibitive. On the other hand, a coarse graining may miss the effects looked for. Asymptotic analysis resolves this antagonism elegantly, in that the limiting problem and approximations thereof can be taken as a surrogate optimal control problem on a simple domain such that error estimates show how far the true fine-scale solution is away from the "homogenized" solution which, in turn, can be termed suboptimal.

In order for all of this to become a mathematically sound theory, it is not sufficient to apply asymptotic expansions to all components of such an optimal control problem—namely the cost function, the state equation, the domains, and the control and observation instruments individually—and replace it by its low-order parts. An asymptotic analysis of optimal control problems as such is in order. Meanwhile, an ill-posed PDE problem—in the sense of Hadamard—due to the freedom of choosing proper control inputs may turn out to be well-posed as an optimization problem, while a well-posed PDE problem easily can exhibit ill-posedness, once integrated into an optimal control problem. Hence, well-posedness of optimal control problems is a new and interesting issue.

The book focuses on all of these aspects from two perspectives. First, a rigorous and mostly self-contained mathematical theory of PDEs on reticulated domains together with well-posedness for the governing optimal control problems is described. The concept of optimal control problems for PDEs in varying such domains, and hence in varying Banach spaces, is developed, fol-

lowed by appropriate concepts for convergence of optimal control problems in variable spaces. This comprises Part I of the book. Even though there are by now a number of textbooks for PDE-constrained optimal control available, this monograph contains a unique collection of results that are necessary to treat the optimal control problem on varying structures which are not available in a textbook otherwise. In Part II, particular examples and applications are investigated with the tools established in Part I of the book. These examples are strongly motivated by applications in mechanics and material sciences as explained above, but they can be understood without any knowledge from those fields. In order to accomplish this, the models are somewhat simplified such that, for example, only quasi-static flow in cylindrically perforated domains is considered instead of the fully time-dependent Navier–Stokes flow. Additionally, the elliptic second order problems on thin or fat graphs are scalar instead of being vectorial, which would be necessary in order to be directly applicable to problems of elasticity.

Overall, the book's first part can be seen as an advanced textbook for abstract optimal control problems, in particular on reticulated domains, which can be used in graduate courses, while its second part serves as a research monograph, using somewhat stratified applications in an exemplary manner. Part II can be of use also in seminars, building on the knowledge from a graduate course taught from Part I. For the reader's convenience, in Part II, we sometimes reintroduce the basic concepts that are dealt with in Part I on an abstract level; however, they are explicitly geared towards the particular application. Admitting some potential redundancy, we thereby keep the chapters in the second part of the book self-contained for researchers in the field.

Dnipropetrowsk and Erlangen, *Peter I. Kogut*
May 2011 *Günter R. Leugering*

Contents

1 Introduction 1

Part I Asymptotic Analysis of Optimal Control Problems for Partial Differential Equations: Basic Tools

2 Background Material on Asymptotic Analysis of Extremal Problems 15
2.1 Measure theory and basic notation 15
2.1.1 Hausdorff measures 17
2.2 Sobolev spaces and boundary value problems 19
2.2.1 Weak derivatives 19
2.2.2 Sobolev spaces 20
2.2.3 Vector-valued spaces of the type $\mathbf{L^p(a,b;X)}$ 25
2.2.4 Lax–Milgram's lemma 27
2.2.5 General setting of the variational formulation of boundary value problems 28
2.3 Spaces of periodic functions 33
2.4 Weak and weak-$*$ convergence in Banach spaces 34
2.4.1 Weak convergence of measures 39
2.4.2 Weak convergence in $L^1(\Omega)$ 43
2.5 Elements of capacity theory 44
2.6 On the space $W_0^{1,p}(\Omega) \cap L^p(\Omega, \mathrm{d}\mu)$ and its properties 47
2.7 Sobolev spaces with respect to a measure 49
2.8 Boundary value problems in Sobolev spaces with measures . . . 53
2.9 On weak compactness of a class of bounded sets in Banach spaces 58

3 Variational Methods of Optimal Control Theory 63
3.1 The general setting 64
3.2 Abstract extremal problems 72

3.3 Extremal problems for steady-state processes 78
3.3.1 Dirichlet and Neumann boundary control problems . . . 78
3.3.2 Ill-posed control objects 83
3.3.3 Optimal control of the Cauchy problem for an elliptic equation 84
3.3.4 Controls with hard constraints 86
3.4 Optimal control problems for parabolic equations 88
3.4.1 Distributed control 88
3.4.2 Control in the initial conditions 91
3.4.3 Neumann boundary control 94
3.5 Optimal control problems for hyperbolic equations 98
3.6 Optimality system to optimal control problems 100
3.6.1 The general setting of the Lagrange multiplier principle 100
3.6.2 Necessary optimality conditions in the form of variational inequalities 106
3.7 Optimal control of distributed singular systems 109

4 Suboptimal and Approximate Solutions to Extremal Problems 113
4.1 The notion of suboptimal and approximate solutions 114
4.2 Regularization of optimal control problems 117
4.3 ε-Suboptimal solutions to optimal control problems 123
4.4 Approximate solutions to distributed singular systems 129

5 Introduction to the Asymptotic Analysis of Optimal Control Problems: A Parade of Examples 133
5.1 Component-by-component limit analysis 134
5.2 Limit analysis of optimality conditions 143
5.3 Limit analysis of optimal control problems by Γ-convergence 150
5.4 Direct variational convergence of optimal control problems 156

6 Convergence Concepts in Variable Banach Spaces 161
6.1 General setting 161
6.2 Weak convergence in variable L^p-spaces 167
6.3 Two-scale convergence in variable L^p-spaces 171
6.4 Variable Sobolev spaces and two-scale convergence 181
6.4.1 p-Connected measures and their properties 181
6.4.2 Degenerate measures 184
6.4.3 Two-scale convergence in variable Sobolev spaces 185
6.5 Approximation of singular measures by smoothing and its application 189

6.6 Two-scale convergence with respect to a variable measure 194
6.7 Some properties of the strong convergence in spaces $L^2(\square, \mathrm{d}\mu^h)$ associated with thin periodic structures 198
6.8 On approximative properties of Hilbert spaces with respect to a periodic Borel measure μ^h 203
6.9 The homothetic mean value property on periodically perforated domains 207
6.9.1 A measure approach to the description of the sets Ω_ε 209
6.9.2 The homothetic mean value property 212

7 Convergence of Sets in Variable Spaces 217
7.1 Set convergence in $\mathbb{R}^n$ via the limit properties of characteristic functions 217
7.2 On some limit properties of characteristic functions for nonperiodically perforated domains 224
7.3 Convergence of sets associated with thin periodic structures 230
7.4 Set convergence in the sense of Kuratowski and in the Hausdorff metric 234
7.5 Parametrical convergence of open sets and of associated mappings 241
7.6 Kuratowski set convergence in variable spaces 247
7.7 γ_p–Convergence of open sets and Mosco convergence of the associated Sobolev spaces 257

8 Passing to the Limit in Constrained Minimization Problems 263
8.1 A short survey of Γ-convergence theory in metric spaces 264
8.2 Γ-Convergence of functionals defined on a Banach space 270
8.3 Variational convergence of constrained minimization problems in Banach spaces 275
8.4 Variational convergence of minimization problems in variable spaces 281
8.5 Asymptotic analysis of a Dirichlet optimal control problem . . 286
8.5.1 The statement of the optimal control problem and preliminary results 286
8.5.2 On modifications of the optimal control problem with controllability constraints 288
8.5.3 Passing to the limit in the modified optimal control problem 292
8.6 On homogenization of Dirichlet optimal control problems in perforated domains 299

Part II Optimal Control Problems on Periodic Reticulated Domains: Asymptotic Analysis and Approximate Solutions

9 Suboptimal Control of Linear Steady-State Processes on Thin Periodic Structures with Mixed Boundary Controls . . . 311
9.1 A measure-theoretic approach to the description of the network Ω_ε . . . 315
9.2 Statement of the optimal control problem . . . 320
9.3 Convergence in the variable space $\mathbb{Z}_\varepsilon$ . . . 325
9.4 Definition of the limit problem and its properties . . . 340
9.5 Main convergence theorem . . . 343
9.6 Identification of the limit optimal control problem . . . 347
9.7 On suboptimal controls for $\mathbb{P}_\varepsilon$-problems . . . 350

10 Approximate Solutions of Optimal Control Problems for Ill-Posed Parabolic Problems on Thin Periodic Structures . . . 357
10.1 Statement of the problem . . . 360
10.2 On the solvability of $\mathbb{P}_\varepsilon$ and its C-extension . . . 362
10.3 On the description of $\mathbb{CP}_\varepsilon$ in terms of singular measures . . . 369
10.4 Convergence in the variable space $\mathbb{Z}_\varepsilon$ . . . 374
10.4.1 Convergence formalism for Dirichlet boundary controls . . . 378
10.4.2 w-Convergence of admissible solutions of the C-extended problems . . . 380
10.5 The limiting optimal control problem and its properties . . . 381
10.6 Main convergence theorem . . . 383
10.7 The limit analysis of the **C**-extended optimal control problem . . . 388
10.8 Recovery of the limiting singular optimal control problem $\mathbb{P}^{hom}$ . . . 394
10.9 On suboptimal controls for $\mathbb{P}_\varepsilon$-problems . . . 398
10.10 Optimal control problem for systems on thin lattice structures with blowup . . . 402

11 Asymptotic Analysis of Optimal Control Problems on Periodic Singular Graphs . . . 409
11.1 ε-Periodic graphlike structures in $\mathbb{R}^2$ and their description . . . 410
11.2 Statement of an optimal control problem on ε-periodic graphs . . . 412
11.3 Convergence formalism in the variable spaces associated with ε-periodic graphs . . . 416

11.4 Variational convergence of constrained minimization problems on varying graphs . . . 421
11.5 Asymptotic analysis of optimal control problems on ε-periodic graphs . . . 423
11.6 Modeling of suboptimal controls . . . 432
11.7 An example of an optimal control problem on ε-periodic square grid . . . 434

12 Suboptimal Boundary Control of Elliptic Equations in Domains with Small Holes . . . 441
12.1 Statement of the problem . . . 441
12.2 Reformulation of the original problem in terms of singular measures . . . 445
12.3 Convergence in the variable space $\mathbb{X}_\varepsilon$ . . . 449
12.4 Definition of a limit problem and its property . . . 456
12.5 Convergence theorem and correctors . . . 456
12.6 Identification of the limiting optimal control problem . . . 467
12.7 Optimality conditions for the limit problem and suboptimal controls for $\mathbb{P}_\varepsilon$-problem . . . 472

13 Asymptotic Analysis of Elliptic Optimal Control Problems in Thick Multistructures with Dirichlet and Neumann Boundary Controls . . . 477
13.1 Statement of the problem and basic notation . . . 478
13.2 Description of the optimal control problem in terms of singular measures . . . 481
13.3 The choice of topology . . . 484
13.4 Definition of the limit problem and its properties . . . 491
13.5 Analytical representation of the limit set of admissible solutions . . . 492
13.6 Identification of the limiting cost functional . . . 502
13.7 Modeling of suboptimal controls . . . 506

14 Gap Phenomenon in Modeling of Suboptimal Controls to Parabolic Optimal Control Problems in Thick Multistructures . . . 515
14.1 On solvability of the original optimal control problems . . . 518
14.2 Formalism of convergence in variable Banach spaces . . . 522
14.2.1 The convergence concept for the $\mathbb{P}^a_\varepsilon$-problems . . . 523
14.2.2 The convergence concept for the $\mathbb{P}^b_\varepsilon$-problems . . . 527
14.3 Definition of the limit problems and their properties . . . 528
14.4 Analytical representation of the limit sets of admissible solutions . . . 532
14.4.1 Recovery of the set Ξ_a . . . 532
14.4.2 Recovery of the set Ξ_b . . . 538
14.5 Identification of the cost functionals I_a and I_b . . . 540

15 Boundary Velocity Suboptimal Control of Incompressible Flow in Cylindrically Perforated Domains 547
15.1 Preliminaries and notation 551
15.2 Admissible controls and regularity of solutions to the boundary value problem for Navier–Stokes equations . . 552
15.3 On solvability of the optimal boundary control problem 554
15.4 Reformulation of the problem $(\mathbb{P}_\varepsilon)$ 555
15.5 Convergence in the variable space $\mathbb{X}_\varepsilon$ 559
15.6 Definition of suboptimal controls 563
15.7 Convergence theorem 565
15.8 Identification of the limit optimal control problem 576
15.9 Suboptimal controls and their approximation properties 582

16 Optimal Control Problems in Coefficients: Sensitivity Analysis and Approximation 585
16.1 Notation and preliminaries 586
16.2 H-Convergence and a counterexample of Murat 590
16.3 Setting of the optimal control problem and existence theorem 593
16.4 H^c- and t-admissible domain perturbations for optimal control problems in coefficients 601
16.5 p-Perturbation of elliptic optimal control problems in coefficients 609
16.6 Mosco-stability of optimal control problems 612

References 621

Index 635

1

Introduction

The aim of this work is to study the asymptotic behavior of various classes of optimal control problems governed by partial differential equations (PDEs) on domains with a lattice-type structure. The investigation of optimal control problems for such structures is important to researchers working with cellular materials (lightweight materials) such as honeycomb structures, ceramic, metal, or polymeric foams which are used in automotive industry, aircraft design, robotics, and micro-mechanics. Lattice-type materials are also important in life sciences. In particular, bones and, correspondingly, bio-mimetic materials are subject of major research initiatives. Other modern engineering applications are flexible multistructures and civil engineering technologies for transportation. Because of the complicated geometry of lattice-type structures, a direct numerical simulations of solutions, let alone of optimal controls, in such a domain is extremely demanding due to the excessive number of variables. Consequently, model reduction is in order. One way of thinking about model reduction is to consider geometrically motivated or material-related scale parameters that tend to zero in order to achieve a new so-called effective model. In the asymptotic limit, the problem may be lower dimensional, as for the 2D-plate models generated via asymptotic analysis out of 3D elasticity, or they can be higher dimensional, as in the case of thin graphlike structures when the spacing scale tends to zero. Regardless of the dimension of the limiting system versus the dimension of the original model, the limiting problem should be easier to access numerically. In this monograph, asymptotic analysis is the main approach to study optimal control of boundary value problems in such complex domains because it gives not only the possibility to replace the original problems by the corresponding limit problems defined in "simpler" domains but also to use an optimal solution to the limit problem as the basis for the construction of suboptimal controls for the original control problem.

As we will see later, asymptotic analysis of optimal control problems is typically not a trivial task. One of main reasons for this is the fact that there is a principle difference between the theory of optimal control problems for systems with distributed parameters and the theory of partial differential

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_1,
© Springer Science+Business Media, LLC 2011

equations. Indeed, let us consider the following example. Let Ω be a bounded open subset of $\mathbb{R}^n$, and let $\partial\Omega$ be its boundary. The optimal control problem we consider can be described as follows: minimize the cost functional

$$I = \int_\Omega |y(x)|^p \, \mathrm{d}x + \|u\|_{L^\infty(\Omega)} \tag{1.1}$$

subject to the constraints

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(u(x)\frac{\partial y}{\partial x_i}\right) = f(x), \quad x \in \Omega, \tag{1.2}$$

$$y|_{\partial\Omega} = 0, \tag{1.3}$$

$$u \in U = \{\, v \in L^\infty(\Omega) \mid \ 0 < \alpha \le v(x) \le \beta \text{ a.e. on } \ \Omega\}\,. \tag{1.4}$$

Here,

$$p \ge 2, \quad f \in L^q(\Omega), \quad \text{and} \ \ p^{-1} + q^{-1} = 1.$$

It is well known that for every fixed control function $u \in U$, the boundary value problem (1.2)–(1.3) admits a unique weak solution $y(u) \in W_0^{1,p}(\Omega)$. Hence, the problem (1.2)–(1.3) is well-posed in the Hadamard sense from the theory of boundary value problems point of view. However, the corresponding optimal control problem (1.1)–(1.4) has no solution, in general (see Lions [172] and Murat [193]). In [172], this fact was formulated as an example of an ill-posed optimization problem. Thus, a well-posed problem in the sense of PDEs turns out to be ill-posed in the sense of optimization theory.

On the other hand, let us consider in the same domain Ω the following optimal control problem: minimize the cost functional

$$I = \int_\Omega |y(x)|^p \, \mathrm{d}x + \delta \int_{\partial\Omega} |u|^q \, \mathrm{d}\mathcal{H}^{n-1} \quad (\delta > 0) \tag{1.5}$$

subject to the constraints

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial y}{\partial x_i}\right) = f(x), \quad x \in \Omega, \tag{1.6}$$

$$\left.\frac{\partial y}{\partial \nu}\right|_{\partial\Omega} = u(x), \quad x \in \partial\Omega, \tag{1.7}$$

$$u \in U = L^q(\partial\Omega), \tag{1.8}$$

where

$$\frac{\partial y}{\partial \nu} = \sum_{i=1}^n \frac{\partial y}{\partial x_i}\,\nu_i(x), \quad p \ge 2, \quad f \in L^q(\Omega), \ \text{ and } \ p^{-1} + q^{-1} = 1.$$

We recall that a pair $(u, y) \in U \times W^{1,p}(\Omega)$ satisfying the conditions (1.6)–(1.7) is called admissible, and the problem (1.5)–(1.8) is regular if its set of admissible pairs is nonempty.

It is easy to see that the boundary value problem (1.6)–(1.7) is noncoercive unless certain compatibility conditions hold for the data. Hence, we cannot assert that this problem has a solution $y(u)$ for every $u \in U$. So, we deal with a potentially ill-posed problem from the the point of view of PDEs. Nevertheless, the corresponding optimization problem (1.5)–(1.8) is regular, coercive, and, hence, solvable. Indeed, to verify the regularity property, it is enough to consider the problem (1.6) with the Dirichlet boundary condition (1.3). In this case, the existence of a unique solution $y^* \in W_0^{1,p}(\Omega)$ is well known. Further, acting on y^* by the operator $\dfrac{\partial}{\partial \nu_A}$ and, taking the corresponding trace on the boundary $\partial\Omega$, we obtain

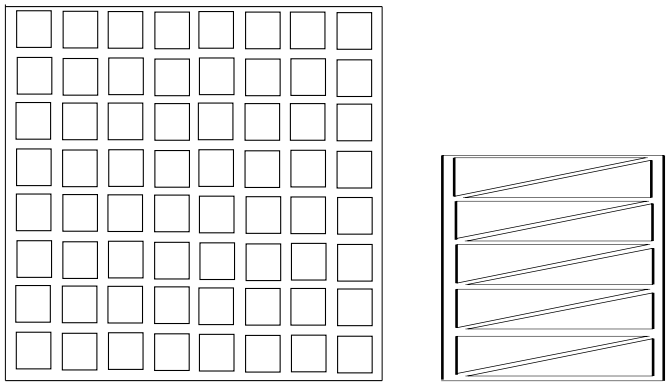
$$u^* = \left.\frac{\partial y^*}{\partial \nu_A}\right|_{\partial\Omega} \in W^{-1,q}(\partial\Omega) \Rightarrow u^* \in L^q(\partial\Omega),$$

and, hence, the pair (u^*, y^*) is admissible for the problem (1.5)–(1.8).

Thus, the examples given above show that well-posedness for optimal control problems governed by PDEs and well-posedness for PDEs themselves are different notions, and consequently, the corresponding theories are not mutually included. It is expected that this difference also surfaces when it comes to asymptotic analysis of optimal control problems governed by PDEs versus asymptotic analysis of PDEs without controls. Asymptotic analysis of optimal control problems for PDEs is a comparatively modern theory that, as a mathematical discipline, took shape only in the last three decades, whereas the physically motivated ideas of homogenization for PDEs date back at least to [69, 183, 192, 215, 218]. A very good historical record of works related to homogenization until 1975 can be found in [13] and the references therein. Recently, there is renewed interest in such asymptotic results due to the growing importance of multiscale models.

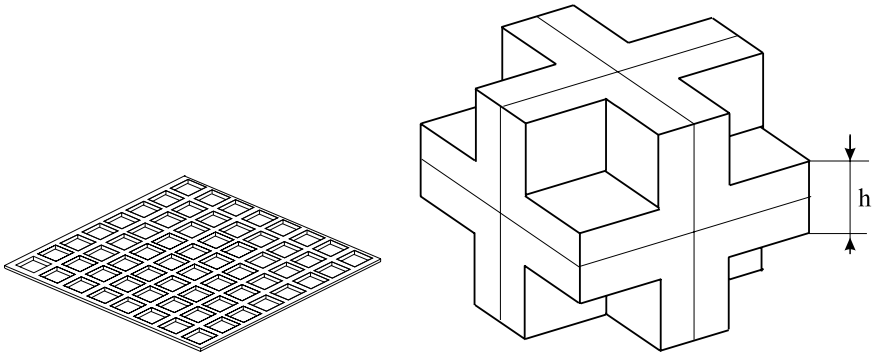
Let us now focus on the geometry of the lattice-type structures. Typically, these depend on two parameters mutually related to each other; namely, ε defines the periodicity cell and εh is the thickness of constituting elements of the structure. We have several types of such structures that consist of identical cells, periodically distributed in all directions, *reticulated structures* (see Fig. 1.1(a)); or in two dimensions, *gridworks* (see Fig. 1.2(a)); or in only one direction, *tall structures* (see Fig. 1.1(b)). If the material is concentrated along layers, then we speak about *honeycomb structures* (Fig. 1.2(b)), and if it is along bars, then we speak of *reinforced structures* (see Fig. 1.3). Concerning methods for handling such structures in a classical engineering sense, we refer to Gibson and Ashby [114].

The above mentioned 3D lattice-type structures can be described as follows: Let $\mathcal{F}$ be a periodic box or rod graph structure in $\mathbb{R}^3$ with zero thickness ($h = 0$) for its components. We assume that $\mathcal{F}$ consists of the coordinate planes or axes and their shifts by integer-valued vectors. We call $\mathcal{F}$ a singular structure. Let the cube $\square = [0, 1)^3$ be the cell of periodicity for $\mathcal{F}$.



(a) A lattice plate (b) A tall structure

Fig. 1.1. Examples of 2D and 1D lattice-type structures



(a) A gridwork (b) A honeycomb structure

Fig. 1.2. Examples of 3D lattice-type structures

Let $\mathcal{F}^h$ be the $\square$-periodic structure composed of infinite plates (or rods) whose thickness is $h > 0$, having the corresponding plane (or straight line) from the structure $\mathcal{F}$ as the middle plane or median. Fragments of the structures $\mathcal{F}$ and $\mathcal{F}^h$ within the limits of the cell of periodicity $\square$ are depicted in Fig. 1.2(b) (see also Fig. 1.3).

The structure $\mathcal{F}^h$ is said to be thin. Let ε be a small parameter. We will always suppose that ε varies in a strictly decreasing sequence of positive numbers which converges to 0. Then the lattice-type structure $\mathcal{F}^h_\varepsilon$ in $\mathbb{R}^3$ can be represented as the homothetic contraction of the structure $\mathcal{F}^h$ with a factor of ε^{-1} (i.e., $\mathcal{F}^h_\varepsilon = \varepsilon\mathcal{F}^h$). In the 2D case, we call $\mathcal{F}^h_\varepsilon$ the periodic networks (see Fig. 1.1(a)).

In this work, we suppose that the thickness εh $(0 < \varepsilon h < \varepsilon/2)$ of constituting elements of $\mathcal{F}^h_\varepsilon$ is always supposed to be dependent on ε. As a result,

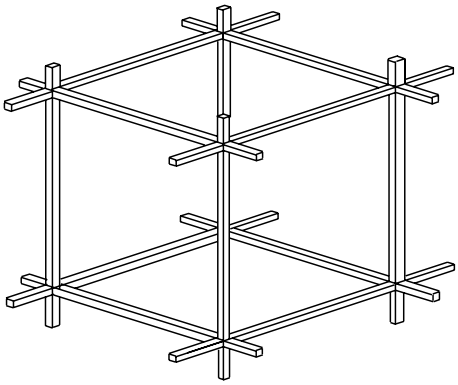


Fig. 1.3. A reinforced structure

we come to the following definition: We say that a domain Ω_ε has a periodic lattice-type structure if

$$\Omega_\varepsilon^{h(\varepsilon)} = \Omega \cap \mathcal{F}^h(\varepsilon), \quad 0 < h(\varepsilon) < \frac{1}{2}.$$

Let us point out that for fixed ε and h, the lattice structure Ω_ε can be viewed as a perforated domain with rapidly oscillating boundary. However, there is a principal difference between perforated domains and thin structures. For the perforated domains, the parameter h is typically either independent of ε or such that

$$\liminf_{\varepsilon \to 0} h(\varepsilon) = h^* > 0.$$

At the same time, for the thin domains the parameters ε and $h = h(\varepsilon)$ are related by the supposition $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. As we will see later, the asymptotic behavior of the optimal control problems essentially depends on how $h(\varepsilon)$ tends to h^* as $\varepsilon \to 0$. In view of this, we introduce the classifications of lattice-type structures as they were proposed by Zhikov [257]: If $\lim_{\varepsilon \to 0} h(\varepsilon) = 0$, then $\Omega_\varepsilon^{h(\varepsilon)}$ is said to be a domain with thin periodic structure; otherwise, $\Omega_\varepsilon^{h(\varepsilon)}$ is a periodically perforated domain. A periodic structure is characterized as follows:

(i) Sufficiently thick, when

$$\lim_{\varepsilon \to 0} h(\varepsilon)\varepsilon^{-1} = \infty.$$

(ii) Sufficiently thin, when

$$\lim_{\varepsilon \to 0} h(\varepsilon)\varepsilon^{-1} = 0.$$

(iii) A structure of critical thickness, when

$$\lim_{\varepsilon \to 0} h(\varepsilon)\varepsilon^{-1} = \theta > 0.$$

Given this characterization, the main problem is to relate the geometry of lattice-type structures to the properties of the solutions to different classes of optimal control problems that can be considered in these domains.

Note that, as the small parameter ε varies, it is often possible to foresee some "limit" behavior and "guess" that we may substitute the complex, degenerate problems we started with by a new one, simpler and with a more comprehensible behavior, possibly of a completely different type. Since the numerical computation of the optimal states and controls on each ε-level is very costly due to the complexity of the geometry of lattice-type structures, the main question is to study the asymptotic behavior of optimal solutions as the small parameter ε tends to 0 and to view the approximation as a *suboptimal control* for the original problem. Hence, the crucial point of this book is the development of methods for the regularization and approximation of optimal solutions in lattice-type structures using the properties of the limit optimal control problems.

We decompose the content of this monograph in two parts. The first part gives an up-to-date and in-depth, mathematically self-contained account on asymptotic analysis of optimal control problems on a general level, whereas the second part concentrates on more specific problems on periodic lattice-type structures. In the first part, we dispense with providing proofs for most of the results that are considered well known, providing detailed proofs only for those results that are typically not contained in a textbook, whereas in the second part, we do provide proofs for most of the results, some of which have not been published elsewhere.

We now describe the content of this book in more detail.

Part I. Basic Tools of Asymptotic Analysis of Optimal Control Problems. In Part I, we follow, as a guideline, the variational treatment of the main common boundary value problems and related optimal control problems. We introduce basic elements of variational analysis which allow one to study these problems for thin and reticulated structures and the closely related questions of their asymptotic analysis.

Chapter 2 contains an in-depth exposition of the main notions, concepts, and results which play a fundamental role in the variational approach to the asymptotic analysis of optimal control problems. To this end, we recall some main concepts and basic results from functional analysis, measure theory, Sobolev spaces, boundary value problems, and capacity theory, which are important constituents of modern mathematical PDE control theory.

Chapter 3 provides an exposition of the mathematical theory of parameterized optimal control problems. We discuss the general setting of optimal control theory for PDEs, or distributed parameter systems, in variable spaces.

We show the importance of the *Direct Method of Calculus of Variation* and emphasize its basic topological ingredients: lower semicontinuity, coercivity, and inf-compactness. We discuss the topological properties of solutions, optimality conditions for a wide class of extremal problems in the form of variational inequalities, and also questions related to the construction of approximative solutions to the different classes of optimal control problems.

Our main interest in Chap. 4 is the regularization and approximation of solutions to general optimal control problems. We provide precise definitions of ε-approximate solutions, weak ε-approximate solutions, weakened ε-approximate solutions, ε-suboptimal solutions, and approximate solutions of such problems. We also provide illustrative examples.

Chapter 5 deals with the main object of this book, namely parameterized optimal control problems depending on some small parameter ε. We discuss different approaches to the study of the asymptotic behavior of a wide class of optimal control problems when the parameter ε tends to 0.

In Chap. 6, we describe the main convergence concepts in variable Banach spaces. The aim of this chapter is to give a systematic exposition of the main properties of weak and strong convergence in variable L^p- and $W^{1,p}$-spaces for $p > 1$ with respect to Radon measures which can be associated with different classes of thin and reticulated structures.

The objective of Chap. 7 is to study problems where the knowledge of "the limit set" is important. We discuss different analytical frameworks and concepts that form the basis of set convergence in variable spaces. In this context, we introduce different notions of limits for sequences of nonempty sets and give some applications to the theory of thin periodic and reticulated structures.

Chapter 8 provides an exposition of variational methods for constrained minimization problems in varying spaces. We show that in many practical situations it is more appropriate to study the asymptotic behavior of the family of minimum problems by defining an appropriate limit minimum problem, which also serves as a "good approximation" to the original problem, rather than via the study of the "limit" properties of their solutions. As an example, we study the asymptotic behavior of an optimal control problem for linear elliptic equation with partial exact controllability constraints. We show that this problem can be viewed as a particular example covered by the approach described in this chapter.

Part II. Approximate Solutions of Optimal Control Problems on Periodic Lattice Structures. The second part of this monograph includes Chaps. 9 through 16 and deals with the second objective which is to present new results in optimal control theory for PDEs on reticulated structures and networks.

In Chap. 9, we study an optimal control problem for a linear steady-state process on a thin periodic structure with mixed boundary controls. We focus our attention on optimal control problems with different types of control and state constraints. Having provided the asymptotic analysis of such problems

as the parameter of periodicity ε tends to 0, we construct so-called asymptotically suboptimal controls with respect to the original problems. For the sake of simplicity, we restrict our analysis to the case of a 2D lattice structure. It is well known that homogenization of most of the boundary value problems on thin structures with Dirichlet and Neumann boundary conditions reveals a classical form: If strong convergence of the corresponding solutions holds, the limit function does not contain an oscillating variable and the limit problem does not depend on the type of the thin structure. In contrast to this, we show that for the considered optimal control problems, the structure of the corresponding limit problem depends essentially on how the thickness parameter $h(\varepsilon)$ tends to 0 as $\varepsilon \to 0$; we have a so-called "scaling effect."

The next observation deals with the "structural stability" of the limiting optimal control problem. We show that the passage to the limit in the original problem, with boundary controls of two different types, leads to an optimal control problem with only Dirichlet controls. The Neumann boundary control can only appear in the limit problem if the thin domain has a critical thickness $\xi^* = \theta > 0$.

In Chap. 10, we focus on control objects which are described by singular parabolic equations with Robin boundary conditions at the boundary of holes, and with two different types of boundary controls – Dirichlet and Neumann controls – on the external boundary of the thin periodic structure Ω_ε. Having admitted that a blowing-up phenomenon can appear in the original problem, we provide its asymptotic analysis, as the parameter ε tends to 0. It is shown that the structure of the limiting problem depends essentially on how the thickness $h(\varepsilon)$ tends to zero as $\varepsilon \to 0$ ("scaling effect"). The characteristic feature of this control problem is the fact that the original initial-boundary value problem has no global solution for some admissible controls. This is in contrast to the assumption which usually plays a fundamental role in classical approaches of the homogenization theory for PDEs.

In this chapter, we propose an "indirect approach" to the asymptotic analysis of boundary optimal control problems for singular systems. The main idea is to introduce a couple of new optimal control problems for parabolic equations on the reticulated structure Ω_ε. The first of them is a virtual extension of the original problem. Having introduced a fictitious "distributed" control into the original problem and having modified its cost functional, we prove that the new problem always has a nonempty set of admissible solutions (in contrast to the original one), and moreover that for every $\varepsilon > 0$, there is a one-to-one correspondence between the sets of optimal solutions for the original and virtual problems. The second problem is the so-called C-extension of the virtual problem. The consideration of this problem is motivated by the fact that the state equation of the C-extended problem has a linear structure. It is clear that in the linear case, the C-extended problem always coincides with the original one. As a result, we propose a new scheme for the asymptotic analysis of optimal control problems for parabolic equations with potential blowup.

By analogy with the previous chapter, we show that for optimal control problems with Robin conditions at the boundary of holes, the structure of the corresponding limit problem depends essentially on how the parameter $h(\varepsilon)$ tends to 0 as $\varepsilon \to 0$; in particular for critical thickness, some additional terms appear in the limiting state equation. At the same time, we do not have any such artifacts if the thin structure Ω_ε is sufficiently thick. However, as for sufficiently thin domains Ω_ε, a limiting optimal control problem does not exist, in general. However, in this case, the asymptotic analysis is possible only if homogeneous Neumann conditions are given instead of the Robin conditions along the boundary S^ε_{int}.

We also derive conditions under which in the limit we do not obtain an optimal control problem, but rather some initial-boundary value problem with or without controls. In conclusion, we construct asymptotically suboptimal controls for the original problem and show an approximation property of such controls for small enough ε.

Another important aspect of asymptotic methods concerns the study of optimal control problems on singular structures, like periodic graphs. In Chap. 11, we examine the statements and properties of different classes of optimization problems for 1D PDEs on periodic graphs. We concentrate on the asymptotic analysis of optimal control problems for 1D elliptic equations on periodic graphs with "distributed" and "boundary" controls, as the period ε of the graph tends to 0. Note that if the small parameter ε is changed, then all components of the original control problem, including the ε-periodic graph Ω_ε, the control constraint sets, the cost functional, and the set, where we seek its infimum, are changed as well. This means that the original problem (on the graph) for different values of the parameter ε "lives" in different function spaces. Moreover, the Lebesgue measure of the "material" included in the periodic graph Ω_ε is equal to 0 for every $\varepsilon > 0$, whereas there exists a set Ω which is filled up by this planar graph in the limit, as $\varepsilon \to 0$.

Using approaches of variational convergence of minimization problems in variable spaces and relying on the description of optimal control problems for planar networks in terms of singular measures, we show that the problem on the periodic graph tends to some standard optimal control problem for a 2D system, and its solution can be used as suboptimal control to the original one.

Our prime interest in Chap. 12 deals with the construction of suboptimal solutions to a class of boundary optimal control problems in ε-periodically perforated domains with small holes. We suppose that the support of the controls is contained in the boundaries of the holes. This set is divided into two parts: On one part the controls are of Dirichlet type, whereas on the other part the controls are of Neumann type. Using the ideas of the theory of Γ-convergence and the concept of variational convergence of constrained minimization problems, we show that the limit problem can be recovered in an explicit analytical form. However, in contrast to the case of thin periodic structures (see Chap. 9), the control through the boundary of small holes leads us in the limit to an optimal control problem with drastically different properties.

In particular, we show that in this case, a "strange term" appears both in the limit equation *and* in the cost functional, and these terms depend on the geometry of the holes. Moreover, the characteristic feature of the obtained limiting control problem is also the fact that it contains two independent distributed control functions which can be used as suboptimal controls for the original one.

In Chap. 13, we study the asymptotic behavior of an optimal control problem for an elliptic equation in a thick multistructure with different types and classes of admissible boundary controls. This thick multistructure consists of a domain (the domain of a basis structure) and a large number of ε-periodically situated thin cylinders. We mentioned in the Preface that such structures are very important in certain applications where processes are studied (e.g., in nanoreactors). We consider two types of boundary controls – namely Dirichlet $H^{1/2}$-controls on the bases Γ_ε of thin cylinders, and Neumann L^2-controls on their vertical sides. We present some ideas and results concerning the asymptotic analysis for such problems as $\varepsilon \to 0$ and derive conditions under which the homogenized problem can be recovered in an explicit form. We show that the mathematical description of the limit optimal boundary control problem is different from the original one. These differences appear not only in the control constraints, limit cost functional, state equations, and boundary conditions, but also in the type of admissible controls for the limit problem.

In Chap. 14, we study the asymptotic behavior of a parabolic optimal control problem in a domain $\Omega_\varepsilon \subset \mathbb{R}^n$ whose boundary $\partial\Omega_\varepsilon$ contains a highly oscillating part. We consider this problem with two different classes of Dirichlet boundary controls and provide its asymptotic analysis with respect to different topologies of variational convergence. It is shown that the mathematical descriptions of the limiting optimal control problems have different forms and these differences appear not only in the state equation and boundary conditions, but also in the control constrains and the limit cost functional.

In Chap. 15, we focus on suboptimal controls for an optimal boundary control problem for the 3D steady-state Navier–Stokes equation in a cylindrically perforated domain Ω_ε. The control is the boundary velocity field supported on the "vertical" sides of the thin cylinders. We minimize the vorticity of a viscous flow through the perforated domain. We show that an optimal solution to the limit problem in a nonperforated domain can be used as a basis for the construction of suboptimal controls for the original control problem. It is worth noticing that the limit problem may take the form of either a variational calculation problem or an optimal control problem for Brinkman's law with another cost functional, depending on the cross-section of the thin cylinders.

The last topic, considered in Chap. 16, is a classical Dirichlet optimal control problem for an elliptic equation with controls in the coefficients taken in $L^\infty(\Omega)$. Since problems of this type have no solutions in general, we make a special assumption on the coefficients of the state equation and introduce the class of the so-called solenoidal controls. Using the direct method of cal-

culus of variations, we prove an existence result for this problem. We also provide the sensitivity analysis of this problem with respect to domain perturbations. With this aim, we introduce the concept of the Mosco-stability for such problems and study the variational properties of Mosco-stable problems with respect to different types of domain perturbations. We show that this is one of the possible ways for the approximation of optimal solutions in domains with a complicated geometry. It is clear that such problems are of great importance in material optimization, which is an emerging field for PDE-constrained optimization.

Part I

Asymptotic Analysis of Optimal Control Problems for Partial Differential Equations: Basic Tools

2

Background Material on Asymptotic Analysis of Extremal Problems

This chapter is intended to provide various facts, notions, and concepts which play a fundamental role in modern asymptotic analysis of optimization problems. We recall some main concepts and basic results of measure theory, Sobolev spaces, and boundary value problems which are used later. We include proofs only if the line of arguments is of importance for the understanding of subsequent remarks. For a deeper insight in the subject, we refer to the books of Adams [2], Bucur and Buttazzo [38], Evans and Gariepy [106], Kantorovich and Akilov [128], Lions and Magenes [173], Maz'ya [185], Yosida [251], Ziemer [267], and so on.

2.1 Measure theory and basic notation

Let $\Omega \subset \mathbb{R}^n$ be a nonempty set. We say that a collection $\mathcal{E}$ of subsets of Ω is a σ-algebra on Ω if

$$\emptyset \in \mathcal{E}, \quad \Omega \setminus A \in \mathcal{E} \quad \text{whenever} \quad A \in \mathcal{E},$$
$$\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{E} \quad \text{whenever} \quad A_k \in \mathcal{E} \quad \text{for every} \quad k \in \mathbb{N}.$$

Given a σ-algebra on Ω, we say that the pair $(\Omega, \mathcal{E})$ is a measure space.

We denote by $\mathcal{B}(\Omega)$ the intersection of all σ-algebras on Ω containing the open subsets of Ω. It turns out that $\mathcal{B}(\Omega)$ is actually the smallest σ-algebra on Ω containing the open subsets of Ω, and it is called the σ-algebra of Borel subsets of Ω and its elements are called Borel sets.

Let $(\Omega, \mathcal{B}(\Omega))$ be a Borel measure space. We define measures as set functions.

Definition 2.1. *A function $\mu : \mathcal{B}(\Omega) \to \mathbb{R}$ is a Borel measure on Ω (or simply a measure) if $\mu(\emptyset) = 0$ and μ is countably additive in the sense that*

$$A = \bigcup_{k\in\mathbb{N}} A_k, \ A_k \cap A_j = \emptyset \ \text{ if } \ k \neq j \quad \Rightarrow \quad \mu(A) = \sum_{k\in\mathbb{N}} \mu(A_k). \tag{2.1}$$

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_2,
© Springer Science+Business Media, LLC 2011

The set of such measures will be denoted by $\mathcal{M}(\Omega)$. *We also say that a Borel measure is positive if it takes its values in* $[0,\infty)$. *The set of positive Borel measures will be denoted by* $\mathcal{M}_+(\Omega)$.

We observe that, in the case of measures, the series in (2.1) must necessary converge absolutely, since the union on the left-hand side of (2.1) does not depend on the order in which the sets $A_1, A_2, \ldots$ are listed.

Let $\mu : 2^\Omega \to \mathbb{R}$ be a set function. We define the restriction $\mu \llcorner A$ of μ to $A \subset \Omega$ by $\mu \llcorner A(B) = \mu(A \cap B)$ for all $B \in 2^\Omega$.

Definition 2.2. *A positive Borel measure on* Ω *that is finite on each compact subset of* Ω *is said to be a Radon measure on* Ω.

The restriction of the Lebesgue measure to $\mathcal{B}(\mathbb{R}^n)$ is a classical example of a Radon measure on $\mathbb{R}^n$. Note also that the Lebesgue measure on $\mathbb{R}^n$ can be defined as the unique positive Radon measure $\mathcal{L}_n$ on $\mathbb{R}^n$ satisfying

$$\mathcal{L}_n([0,1]^n) = 1 \quad \text{and} \quad \mathcal{L}_n(a + tA) = t^n \mathcal{L}_n(A) \quad \forall\, a \in \mathbb{R}^n,\ A \in \mathcal{B}(\mathbb{R}^n), t > 0. \tag{2.2}$$

We denote $|A| = \mathcal{L}_n(A)$.

Definition 2.3. *For* $\mu \in \mathcal{M}(\Omega)$ *and* $A \in \mathcal{B}(\Omega)$, *we define the total variation of* μ *on* A *by*

$$|\mu|(A) = \sup\left\{ \sum_{k\in\mathbb{N}} |\mu(A_k)| \ :\ A = \bigcup_{k\in\mathbb{N}} A_k,\ A_k \cap A_j = \emptyset \ \text{ if } \ k \neq j \right\}.$$

It is well known that the total variation of a measure is a positive measure taking only finite values $|\mu(A)| \le |\mu|(A)$ for all $A \in \mathcal{B}(\Omega)$, and the total variation can be viewed as a norm on the set of measures on Ω.

We will indicate by $\mathcal{M}_b(\Omega)$ the space of Radon measures on Ω with finite total variation. Note that $\mathcal{M}_b(\Omega)$ is a Banach space (i.e., a complete linear normed space) when equipped with the norm $\|\mu\|_{\mathcal{M}_b(\Omega)} := |\mu|(\Omega)$.

Definition 2.4. *The support of* $\mu \in \mathcal{M}(\Omega)$ *is defined as*

$$\operatorname{spt}\mu = \{x \in \Omega \ :\ |\mu|(B_r(x)) > 0 \text{ for all open balls } \ B_r(x) \subset \Omega\}.$$

Theorem 2.5. *Every measure* $\mu \in \mathcal{M}_+(\Omega)$ *is regular in the following sense:*

$$\mu(A) = \inf\{\mu(B) \ :\ A \subset B, B \ \text{ open}\}, \tag{2.3}$$

$$\mu(A) = \sup\{\mu(C) \ :\ C \subset A, C \ \text{ closed}\} \tag{2.4}$$

for all $A \in \mathcal{B}(\Omega)$.

Note that approximating closed sets with compact sets, we also have

$$\mu(A) = \sup\{\mu(C) \ :\ K \subset A, K \ \text{ compact}\}.$$

Definition 2.6. *Let $\mu \in \mathcal{M}_+(\Omega)$ and $\lambda \in \mathcal{M}(\Omega)$. We say that λ is absolutely continuous with respect to μ (and we write $\lambda \ll \mu$) if $\lambda(A) = 0$ for every $A \in \mathcal{B}(\Omega)$ with $\mu(A) = 0$.*

We say that λ is singular with respect to μ, if there exists a set $E \in \mathcal{B}(\Omega)$ such that $\mu(E) = 0$ and $\lambda(A) = 0$ for all $A \in \mathcal{B}(\Omega)$ with $A \cap E = \emptyset$ (in this case, we say that λ is concentrated on E).

For $\mu \in \mathcal{M}(\Omega)$ we adopt the usual notation $L^p(\Omega, \mathrm{d}\mu)$ to indicate the space of p-summable functions with respect to μ on Ω. We omit μ if it is the Lebesgue measure. Let us observe that if $f \in L^1(\Omega, \mathrm{d}\mu)$ and $\mu \in \mathcal{M}(\Omega)$ then we can define the measure $f\mu \in \mathcal{M}(\Omega)$ by

$$f\mu(A) = \int_A f \, \mathrm{d}\mu.$$

Hence, $f\mu \ll |\mu|$ and $|f\mu| = |f||\mu|$. In particular, if $\lambda \ll \mu$, then $\lambda = f\mu$ for some $f \in L^1(\Omega, \mathrm{d}\mu)$.

Theorem 2.7. *(Radon–Nikodym) For $\lambda \in \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}_+(\Omega)$ there exists a function $f \in L^1(\Omega, \mathrm{d}\mu)$ and a measure λ^s, singular with respect to λ, such that*

$$\lambda = f\mu + \lambda^s.$$

This relation is called the Radon–Nikodym decomposition of λ with respect to μ.

Definition 2.8. *Let $\mu = \mathcal{L}_n$ and let $f \in L^1(\Omega)$. Then each point $x \in \Omega$ with*

$$\lim_{r \to 0+} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \, \mathrm{d}\mu(y) = 0$$

is called a Lebesgue point for f. Note that the set of Lebesgue points of f depends on the particular choice of the representative in the equivalence class of $L^1(\Omega)$. Hence, we will always take a particular choice of the representative of f whenever we consider Lebesgue points.

The expression "μ a.e." means "almost everywhere with respect to the measure μ" – that is, except possibly on a set A with $\mu(A) = 0$.

2.1.1 Hausdorff measures

We next introduce certain "lower-dimensional" measures on $\mathbb{R}^n$, which allow one to measure certain "very small" subsets of $\mathbb{R}^n$. The idea is that A is an "s-dimensional subset" of $\mathbb{R}^n$ if there is a so-called Hausdorff measure $\mathcal{H}^s$ such that $0 < \mathcal{H}^s(A) < +\infty$ even if A is very complicated geometrically. To do so, we will construct a positive measure $\mathcal{H}^s$ on $\mathbb{R}^n$ from set functions, following a procedure due to Carathéodory, which we briefly recall.

Definition 2.9. *A set function $\lambda : 2^{\Omega} \to [0, +\infty]$ is an outer measure if $\lambda(\emptyset) = 0$ and λ is countably subadditive, that is,*

$$\lambda(B) = \sum_{k\in\mathbb{N}} \lambda(B_k) \quad \text{for} \quad B \subseteq \bigcup_{k\in\mathbb{N}} B_k.$$

We say that a set M is measurable for λ if

$$\lambda(B) = \lambda(B \cap M) + \lambda(B \setminus M) \quad \forall\, B \subseteq \Omega.$$

Let $\mathcal{M}_\lambda$ be a family of all measurable sets for λ. If λ is an outer measure, then $\mathcal{M}_\lambda$ is a σ-algebra and $\lambda|_{\mathcal{M}_\lambda}$ is countably additive (see Evans [106]). So, if $\mathcal{B}(\Omega) \subseteq \mathcal{M}_\lambda$ and $\lambda(\Omega) < +\infty$, then $\mu = \lambda|_{\mathcal{M}_\lambda} \in \mathcal{M}_+(\Omega)$. In order to see that an outer measure generates a measure by the above construction, we have to prove that Borel sets are measurable. The following proposition provides a simple criterion for measurability.

Proposition 2.10. *Let λ be an outer measure. Then Borel sets are measurable for λ if and only if*

$$\operatorname{dist}(A, B) > 0 \quad \Rightarrow \quad \lambda(A) + \lambda(B) = \lambda(A \cup B) \tag{2.5}$$

for all $A, B \subseteq \Omega$, where $\operatorname{dist}(A, B) = \inf\{|x - y| \ : \ x \in A, y \in B\}$.

We now apply Carathéodory's construction to define the measure which will be of use in the sequel.

Definition 2.11. *Let $\alpha \geq 0$ and $\delta > 0$. For all $E \subset \mathbb{R}^n$ we define the pre-Hausdorff measure $\mathcal{H}^\alpha_\delta$ of E as*

$$\mathcal{H}^\alpha_\delta(E) = \frac{\omega_\alpha}{2^\alpha} \inf\left\{ \sum_{k\in\mathbb{N}} (\operatorname{diam} E_k)^\alpha \ : \ \operatorname{diam} E_k < \delta, \ E \subseteq \bigcup_{k\in\mathbb{N}} E_k \right\},$$

where $\omega_\alpha = \pi^{\alpha/2}/\Gamma(\alpha/2 + 1)$ and $\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1} \exp(-s)\, \mathrm{d}s$ is the Euler function, which coincides with the Lebesgue measure of the unit ball in $\mathbb{R}^n$ if α is integer.

Note that $\mathcal{H}^\alpha_\delta(E)$ is decreasing in δ. Consequently, the limit

$$\mathcal{H}^\alpha(E) = \lim_{\delta\to 0+} \mathcal{H}^\alpha_\delta(E) = \sup_{\delta>0} \mathcal{H}^\alpha_\delta(E)$$

exists and is in $[0, +\infty]$. The value $\mathcal{H}^\alpha(E)$ is called the α-dimensional Hausdorff measure of E.

Remark 2.12. Let us note the following:

(a) $\mathcal{H}^\alpha$ is a regular Borel measure ($0 \leq \alpha < +\infty$).
(b) $\mathcal{H}^\alpha$ is the null measure if $\alpha > n$.

(c) $\mathcal{H}^\alpha(\lambda A) = \lambda^\alpha \mathcal{H}^\alpha(A)$ for all $\lambda > 0$, $A \subset \mathbb{R}^n$.
(d) If $\mathcal{H}^\alpha(A) < \infty$, then $\mathcal{H}^\beta(A) = 0$ for all $0 \le \alpha < \beta < \infty$.
(e) If $\mathcal{H}^\beta(A) > 0$, then $\mathcal{H}^\alpha(A) = +\infty$ for all $0 \le \alpha < \beta < \infty$.

Thus, if $\alpha < n$, then $\mathcal{H}^\alpha(A) = +\infty$ for all nonempty open sets $A \subset \mathbb{R}^n$, and $\mathcal{H}^\alpha(A)$ agrees with the ordinary "k-dimensional surface area" on nice sets $A \in \mathcal{B}(\mathbb{R}^n)$. Thus, $\mathcal{H}^\alpha$ is not a positive Radon measure on $\mathbb{R}^n$ if $\alpha < n$, since $\mathbb{R}^n$ is not σ-finite with respect to $\mathcal{H}^\alpha$. However, if $\mathcal{H}^\alpha(B) < +\infty$ for some $B \in \mathcal{B}(\Omega)$, then $\mathcal{H}^\alpha \llcorner B \in \mathcal{M}_+(\Omega)$. Moreover, it can be proved that $\mathcal{H}^n = \mathcal{L}_n$ in $\mathbb{R}^n$.

2.2 Sobolev spaces and boundary value problems

Here, we briefly outline some basic facts from the theory of Sobolev spaces and the theory of boundary value problems, which are widely used throughout the book. Most of these result are well known; therefore, we just formulate them without proof. For details and complete proofs, the reader may turn to the numerous textbooks on the subject – e.g., Adams [2], Kantorovich and Akilov [128], Lions and Magenes [173], Maz'ya [185], Sobolev [232], Smirnov [231], and so forth.

2.2.1 Weak derivatives

Let Ω be a bounded open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, let $C^k(\Omega)$ be the space of k-times continuously differentiable functions $u : \Omega \to \mathbb{R}$, and let $C_0^k(\Omega)$ be the functions in $C^k(\Omega)$ with compact support in Ω. Hence, $C_0^\infty(\Omega)$ is the set of all real-valued infinitely differentiable functions with compact support in Ω. In this case, for any $x_0 \in \Omega$ and $\varepsilon > 0$ small enough, the function

$$\varrho_\varepsilon(x) = \varrho\left(\frac{x - x_0}{\varepsilon}\right), \text{ where } \varrho(x) = \begin{cases} \exp\left\{-\dfrac{1}{1 - \|x\|^2_{\mathbb{R}^n}}\right\} & \text{if } \|x\|_{\mathbb{R}^n} < 1, \\ 0, & \text{otherwise,} \end{cases}$$

is in $C_0^\infty(\Omega)$. The support of ϱ_ε is the ball with center x_0 and radius ε.

We define the set $C_0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the uniform topology. It is a separable Banach space if equipped with the $\|\cdot\|_\infty$-norm.

Denote by $L^p(\Omega)$, $1 \le p < +\infty$, the space of functions defined on Ω and pth-power summable in the sense of Lebesgue. An element of $L^p(\Omega)$ is an equivalence class of functions different only on a set of zero $\mathcal{L}_n$-measure. The space $L^p(\Omega)$ equipped with norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, \mathrm{d}x\right)^{1/p}$$

is a Banach space.

Introduce the space $L^\infty(\Omega)$ of essentially bounded (i.e., bounded by a constant almost everywhere) real-valued functions in Ω that are Lebesgue measurable. The space $L^\infty(\Omega)$ is equipped with the norm

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{esssup}_{x\in\Omega} |u(x)| = \inf_{A\subset\Omega,\ |A|=0} \sup_{x\in\Omega\setminus A} |u(x)|;$$

that is, "ess sup" means the supremum up to a set of zero measure. In the case of a bounded domain Ω, we have

$$\|u\|_{L^\infty(\Omega)} = \lim_{p\to\infty} \|u\|_{L^p(\Omega)},$$

which justifies the notation $L^\infty(\Omega)$.

Suppose $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of order $|\alpha| = \alpha_1 + \dots + \alpha_n = k$. We call a function φ belonging to $C_0^\infty(\Omega)$ a test function. We say that $v \in L^1(\Omega)$ is the αth-weak partial derivative of $u \in L^1(\Omega)$, denoted $D^\alpha u = v$, provided

$$\int_\Omega u D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\varphi \, \mathrm{d}x \tag{2.6}$$

for all test functions $\varphi \in C_0^\infty(\Omega)$. If a vector $\mathbf{v} = \{v_1, \dots, v_n\}$, $v_i \in L^1(\Omega)$, is the gradient of a function $u \in L^1(\Omega)$ in the weak sense, then we denote it either by ∇u or by $\partial u/\partial x$.

2.2.2 Sobolev spaces

Having fixed $1 \le p < +\infty$ and k, we denote by $W^{k,p}(\Omega)$ the Sobolev space formed by all functions $u \in L^p(\Omega)$ such that for each multi-index α with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha|\le k} |D^\alpha u(x)|^p \, \mathrm{d}x \right)^{1/p} \tag{2.7}$$

is finite. If $p = 2$, we usually write $H^k(\Omega) = W^{k,2}(\Omega)$ $(k = 0, 1, \dots)$. By convention we set $H^0(\Omega) = L^2(\Omega)$ and $D^0 v = v$. The letter H is used, since $H^k(\Omega)$ is a Hilbert space equipped with the scalar product

$$(u_1, u_2)_{H^k(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha u_1(x) D^\alpha u_2(x) \, \mathrm{d}x.$$

The space $W_0^{k,p}(\Omega)$ is the closure of the set $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. The space $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ is naturally associated with the Dirichlet problem, since the inclusion $u \in H_0^1(\Omega)$ represents an equivalent formulation of the boundary condition $u|_{\partial\Omega} = 0$. To clarify the presentation further, we will consider only the Sobolev space $W^{1,p}(\Omega)$, which is a separable Banach space for the norm (2.7) and reflexive if $1 < p < +\infty$.

We recall that a Banach space X is said to be *compactly embedded* in a Banach space Y provided (i) $\|x\|_Y \le C\|x\|_X$ $(x \in X)$ for some constant C and (ii) each bounded sequence in X is precompact in Y. The following definition expresses in precise terms the intuitive notion of a regular boundary (C^∞, C^k, or Lipschitz).

Definition 2.13. *(i) Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say that Ω is a bounded open set with C^k, $k \ge 1$, boundary, if for every $x \in \partial\Omega$, there exist a neighborhood $U \subset \mathbb{R}^n$ of x and a one-to-one and onto map $T : Q \mapsto U$, where*

$$Q = \{x \in \mathbb{R}^n \ : \ |x_j| < 1, \ j = 1, 2, \dots, n\},$$
$$T \in C^k(\overline{Q}), T^{-1} \in C^k(\overline{U}), T(Q_+) = U \cap \Omega, T(Q_0) = U \cap \partial\Omega$$

with $Q_+ = \{x \in Q \ : \ x_n > 0\}$ and $Q_0 = \{x \in Q \ : \ x_n = 0\}$.

(ii) If T is in $C^{k,\alpha}$, $0 < \alpha \le 1$, we will say that Ω is a bounded open set with $C^{k,\alpha}$ boundary.

(iii) If T is in $C^{0,1}$, we will say that Ω is a bounded open set with Lipschitz boundary.

Thus, an open set $\Omega \subset \mathbb{R}^n$ has Lipschitz boundary if for every $x \in \partial\Omega$, there exists a neighborhood $U \subset \mathbb{R}^n$ of x such that $U \cap \partial\Omega$ is a graph, in a suitable coordinate system, of a Lipschitz continuous function whose epigraph contains $U \cap \Omega$; see Fig. 2.1. Note that every polyhedron has a Lipschitz boundary, whereas the unit ball in $\mathbb{R}^n$ has a C^∞ boundary. If Ω is an open convex set, then Ω has a Lipschitz boundary as well. If Ω has a Lipschitz boundary, then for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$, there exists the outward unit vector normal to $\partial\Omega$, which we denote by $\mathbf{n}_\Omega$ (see Nečas [205]).

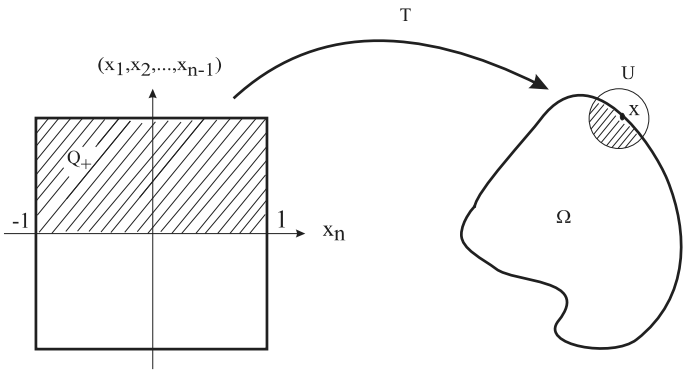


Fig. 2.1. A regular boundary

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary. Then, by the Sobolev embedding theorem and the Rellich–Kondrachov theorem (see Adams [2], Maz'ya [185]), we have the following:

- If $1 \le p < n$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ with
  - compact embedding for $q \in [1, p^*)$, where $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$, and
  - continuous embedding for $q = p^*$;
- If $p = n$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ with compact embedding for $q \in [1, \infty)$;
- If $p > n$, $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ with compact embedding.

Hereafter we call the constant $p^* = np/(n-p)$ the Sobolev conjugate of p.

Note that in all cases (i.e., $1 \le p \le \infty$), the embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is compact. These results are still valid for $W_0^{1,p}(\Omega)$ without any regularity assumption on $\partial\Omega$. However, examples of domains Ω between two spirals can be constructed where the embedding $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is not compact [91]. If Ω is unbounded, then the compactness of that embeddings is lost. Indeed, as in P. L. Lions [174, 175], we concentrate on the case when $\Omega = \mathbb{R}^n$.

Theorem 2.14. *Assume that for $n \ge 3$,*

$$f_k \to f \quad \textit{strongly in} \quad L^2_{loc}(\mathbb{R}^n), \quad Df_k \rightharpoonup Df \quad \textit{in} \quad \left[L^2(\mathbb{R}^n)\right]^n.$$

Suppose further that

$$|Df_k|^2 \rightharpoonup \mu \quad \textit{in} \quad \mathcal{M}(\mathbb{R}^n), \quad |f_k|^{2^*} \rightharpoonup \nu \quad \textit{in} \quad \mathcal{M}(\mathbb{R}^n).$$

Then there exist an at most countable index set J, distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^n$, and non-negative weights $\{\mu_j, \nu_j\}$ such that

$$\nu = |f|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \mu \ge |Df|^2 + \sum_{j\in J} \mu_j \delta_{x_j},$$

where $\delta_{x_j} \in \mathcal{M}_b(\mathbb{R}^n)$ denotes the Dirac measure located at the point x_j.

Note that a typical function $y \in W^{1,p}(\Omega)$ is not, in general, continuous and is only defined almost everywhere on Ω. Since $\partial\Omega$ has n-dimensional Lebesgue measure 0, there is no direct meaning we can give to the expression "u restricted to $\partial\Omega$." The notion of a *trace operator* resolves this problem – namely if $\partial\Omega$ is Lipschitz continuous, then there exists a unique linear continuous map

$$\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that for any $y \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, one has $\gamma(y) = y|_{\partial\Omega}$. The function $\gamma(y)$ is called the trace of y on $\partial\Omega$.

Remark 2.15. It is worth noting that when Ω is an open set with cusps on the boundary, then the existence of the trace operator with the above properties may fail. Indeed, consider, for instance, in $\mathbb{R}^2$ the domain

$$\Omega = \left\{(x_1, x_2) \ : \ 0 < x_1 < 1, 0 < x_2 < x_1^5\right\}.$$

Then for the function $u(x_1, x_2) = 1/x_1$, we have

$$\frac{\partial u}{\partial x_1} = -\frac{1}{x_1^2}, \quad \frac{\partial u}{\partial x_2} = 0 \ \text{ in } \ \Omega$$

in the distributional sense. So

$$\begin{aligned}\int_\Omega \left\{u^2 + |\nabla u|^2\right\} \mathrm{d}x &= \int_\Omega \left\{\frac{1}{x_2^2} + \frac{1}{x_1^4}\right\} \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \int_1^1 \mathrm{d}x_1 \int_0^{x_1^5} \left\{\frac{1}{x_2^2} + \frac{1}{x_1^4}\right\} \mathrm{d}x_2 \\ &= \int_0^1 \left\{x_1^3 + x_1\right\} \mathrm{d}x_1 = \frac{3}{4}.\end{aligned}$$

Thus, $u \in H^1(\Omega)$, but the trace of u on $\partial\Omega$ does not belong to $L^2(\partial\Omega)$ since $\frac{1}{x_1} \notin L^2(0,1)$.

It is clear now that the space $H_0^1(\Omega)$ can be defined as

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega), \ \gamma(u) = 0\right\}.$$

However, we note that γ is not onto $L^p(\partial\Omega)$ (i.e., there exist functions in $L^p(\partial\Omega)$ which are not traces of any element of $W^{1,p}(\Omega)$). In particular, if $p = 2$, then this leads us to the following set $H^{1/2}(\partial\Omega) := \gamma(H^1(\Omega))$, where $H^{1/2}(\partial\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{H^{1/2}(\partial\Omega)} = \left(\|u\|^2_{L^2(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+1}} \,\mathrm{d}x \,\mathrm{d}y\right)^{1/2}$$

with compact embedding $H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$.

As mentioned earlier, if $\partial\Omega$ is Lipschitz continuous, then the unit outward normal vector $\mathbf{n} = (n_1, \ldots, n_n)$ to Ω is well defined almost everywhere. As a result, the well-known *Green formula* for smooth functions can be extended to Sobolev spaces. Indeed, in this case we have

$$\int_\Omega u \frac{\partial v}{\partial x_i} \,\mathrm{d}x = -\int_\Omega v \frac{\partial u}{\partial x_i} \,\mathrm{d}x + \int_{\partial\Omega} \gamma(u)\gamma(v)\, n_i \,\mathrm{d}s, \quad i = 1, \ldots, n,$$

for any $u, v \in H^1(\Omega)$.

For any bounded domain Ω, the Friedrichs inequality

$$\int_\Omega y^2 \,\mathrm{d}x \le C \int_\Omega |\nabla y|^2 \,\mathrm{d}x, \quad \forall\, y \in H_0^1(\Omega), \tag{2.8}$$

holds with a constant C independent of y. Inequality (2.8) implies that the functional $\|y\|_1 = \left(\int_\Omega |\nabla y|^2 \,\mathrm{d}x\right)^{1/2}$ can be taken as an equivalent norm in $H_0^1(\Omega)$, and, indeed, we will always consider $\|y\|_1$ as a norm in this space.

Remark 2.16. It should be stressed that we need to impose a condition of the type $y = 0$ on $\partial\Omega$ (which comes from the hypothesis $y \in H_0^1(\Omega)$) to avoid constant functions y (which imply $\nabla y = 0$), otherwise inequality (2.8) would be trivially false.

If $u \in H^1(\Omega)$ and Ω is a bounded connected open domain with Lipschitz boundary, then the Poincaré inequality

$$\int_\Omega y^2 \,\mathrm{d}x \le C \left\{ \left(\int_\Omega y \,\mathrm{d}x \right)^2 + \int_\Omega |\nabla y|^2 \,\mathrm{d}x \right\}, \quad \forall\, y \in H^1(\Omega), \tag{2.9}$$

is valid, and, by the Rellich–Kondrachov compactness theorem, the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Sometimes the Poincaré inequality appears in the following form. If $1 \le p \le \infty$ and if we set

$$\mathcal{M}_\Omega(y) = \frac{1}{|\Omega|} \int_\Omega y(x) \,\mathrm{d}x,$$

then there exists a constant $C(\Omega, p) > 0$ so that

$$\|y - \mathcal{M}_\Omega(y)\|_{L^p(\Omega)} \le C(\Omega, p) \|\nabla y\|_{L^p(\Omega)}, \quad \forall\, y \in W_{1,p}(\Omega).$$

The dual space of $H_0^1(\Omega)$ (i.e., the set of all continuous linear functionals on $H_0^1(\Omega)$) is denoted by $H^{-1}(\Omega)$. If f is an element of $H^{-1}(\Omega)$, then $\langle f, y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$ stands for the value of the functional f applied to the element $y \in H_0^1(\Omega)$. Note that for any element $F \in H^{-1}(\Omega)$, there exist $n+1$ functions $f_0, f_1, \ldots, f_n$ such that $F = f_0 + \sum_{k=1}^n \partial f_k / \partial x_k$ in the sense of distributions, that is,

$$\langle F, y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega f_0 y \,\mathrm{d}x - \sum_{k=1}^n \int_\Omega f_k \frac{\partial y}{\partial x_k} \,\mathrm{d}x.$$

Moreover,

$$\|F\|^2_{H^{-1}(\Omega)} = \inf \sum_{k=0}^n \|f_k\|^2_{L^2(\Omega)},$$

where the infimum is taken over all vectors

$$(f_0, f_1, \ldots, f_n) \in \left[L^2(\Omega) \right]^{n+1}$$

such that the representation for F given above holds true.

One can give an example of a linear functional on $H_0^1(\Omega)$, setting

$$\langle f_0, y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega f_0 \varphi \,\mathrm{d}x, \quad f_0 \in L^2(\Omega).$$

Similarly, if $\varphi \in H^{1/2}(\partial\Omega)$ and $f \in L^2(\partial\Omega)$, one also has

$$\langle f, y\rangle_{(H^{1/2}(\partial\Omega))^*, H^{1/2}(\partial\Omega)} = \int_{\partial\Omega} f\varphi \,\mathrm{d}s.$$

Suppose that $\partial\Omega$ is Lipschitz continuous. Then one has $L^2(\Omega) \subset H^{-1}(\Omega)$ with compact injection. Hence, the following embeddings are compact (so-called Gelfand–Lions triplet of Sobolev spaces):

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega).$$

Remark 2.17. It is clear that the restriction of any element of $(H^1(\Omega))^*$ to $H_0^1(\Omega)$ is in $H^{-1}(\Omega)$. However, the dual space $(H^1(\Omega))^*$ is not contained in $H^{-1}(\Omega)$ since $(H^1(\Omega))^*$ can be identified with the direct sum $H^{-1}(\Omega) \oplus H^{-1/2}(\partial\Omega)$, where $H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^*$. Moreover, if $\partial\Omega$ is Lipschitz continuous, $\mathbf{y} \in H(\Omega, \mathrm{div}) = \left\{\mathbf{y} : \mathbf{y} \in \left[L^2(\Omega)\right]^n, \mathrm{div}\,\mathbf{y} \in L^2(\Omega)\right\}$, and $w \in H^1(\Omega)$, then $\mathbf{y}\cdot\mathbf{n} \in H^{-1/2}(\partial\Omega)$, the map

$$H(\Omega, \mathrm{div}) \ni \mathbf{y} \mapsto \mathbf{y}\cdot\mathbf{n} \in H^{-1/2}(\partial\Omega)$$

is linear and continuous, and

$$-\int_\Omega (\mathrm{div}\mathbf{y}) w \,\mathrm{d}x = \int_\Omega \mathbf{y}\cdot\nabla w \,\mathrm{d}x + \langle \mathbf{y}\cdot\mathbf{n}, w\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}. \tag{2.10}$$

For any vector $\mathbf{p} \in \mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^n$, the *divergence* is an element of the space $H^{-1}(\Omega)$ defined by

$$\langle \mathrm{div}\,\mathbf{p}, \varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = -\int_\Omega \mathbf{p}\cdot\nabla\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in H_0^1(\Omega), \tag{2.11}$$

where "$\cdot$" denotes the scalar product of two vectors. The following estimate is evident:

$$\|\mathrm{div}\,\mathbf{p}\|_{H^{-1}(\Omega)} = \sup_{\|\varphi\|_1 = 1} \int_\Omega \mathbf{p}\cdot\nabla\varphi \,\mathrm{d}x \le \|\mathbf{p}\|_{\mathbf{L}^2(\Omega)}. \tag{2.12}$$

A vector field $\mathbf{p}$ is said to be *solenoidal* if $\mathrm{div}\,\mathbf{p} = 0$. We say that a vector field $\mathbf{v} \in \mathbf{L}^2(\Omega)$ is *potential* if $\mathbf{v}$ can be represented in the form $\mathbf{v} = \nabla u$, where $u \in H_0^1(\Omega)$.

2.2.3 Vector-valued spaces of the type $\mathbf{L^p(a, b; X)}$

Let $\mathbf{X}$ be a Banach space, $\Omega \subset \mathbb{R}^n$, and p such that $1 \le p \le +\infty$. We denote by $L^p(\Omega; \mathbf{X})$ the set of measurable functions $y : \Omega \to \mathbf{X}$ such that $\|u(\cdot)\|_{\mathbf{X}} \in L^p(\Omega)$. Similarly, one can also define the set of distributions $\mathcal{D}'(\Omega; \mathbf{X})$ on Ω with values in X. $L^p(\Omega; \mathbf{X})$ is a Banach space with respect to the norm

$$\|y\|_{L^p(\Omega;\mathbf{X})} = \left(\int_\Omega \|u(x)\|_{\mathbf{X}}^p \,\mathrm{d}x\right)^{1/p}.$$

If $\mathbf{X}$ is reflexive and $1 < p < \infty$, the space $L^p(\Omega; \mathbf{X})$ is reflexive, too. Moreover, if $\mathbf{X}$ is separable and $1 \leq p < \infty$, then $L^p(\Omega; \mathbf{X})$ is separable.

This type of spaces is well adapted to the study of problems where one of the variables plays a special role. For instance, this occurs for the variable "time" in time-dependent problems. For various results on vector-valued functions, we refer to Schwartz [224] and Lions and Magenes [173].

Let B_0 and B be two Banach spaces such that the embedding $B_0 \hookrightarrow B$ is compact. A natural question is whether the embedding $L^p(a, b; B_0) \hookrightarrow L^p(a, b; B)$ is also compact. Actually, one can prove that this is not true, in general. However, if we have three Banach spaces $B_0 \hookrightarrow B \hookrightarrow B_1$ such that B_0 and B_1 are reflexive and the embedding $B_0 \hookrightarrow B$ is compact, then the embedding $W \hookrightarrow L^{p_0}(a, b; B)$ is compact too, where

$$W = \left\{ y \ : \ y \in L^{p_0}(a, b; B_0), \frac{\partial y}{\partial t} \in L^{p_1}(a, b; B_1) \right\}, \quad 1 < p_0, p_1 < +\infty,$$

is a Banach space with respect to the graph norm

$$\|y\|_W = \|y\|_{L^{p_0}(a,b;B_0)} + \left\| \frac{\partial y}{\partial t} \right\|_{L^{p_1}(a,b;B_1)}.$$

Here, the derivative $\partial y/\partial t$ is the distribution in $\mathcal{D}'(a, b; B_1)$ defined by

$$\frac{\partial y}{\partial t}(\varphi) = -\int_a^b y \frac{\partial \varphi}{\partial t} \, \mathrm{d}t, \quad \forall \, \varphi \in \mathcal{D}(a, b).$$

The following theorem plays an important role in the study of PDEs.

Theorem 2.18. *Let us define the Banach spaces*

$$\mathcal{W} = \left\{ y \ : \ y \in L^2(a, b; H_0^1(\Omega)), \frac{\partial y}{\partial t} \in L^2(a, b; H^{-1}(\Omega)) \right\},$$
$$\mathcal{W}_1 = \left\{ y \ : \ y \in L^2(a, b; L^2(\Omega)), \frac{\partial y}{\partial t} \in L^2(a, b; H^{-1}(\Omega)) \right\},$$

equipped with the norm of the graph. Then the following properties holds true:

(a) The embeddings $\mathcal{W} \hookrightarrow L^2(a, b; L^2(\Omega))$, $\mathcal{W}_1 \hookrightarrow L^2(a, b; H^{-1}(\Omega))$ are compact.

(b) One has the embedding

$$\mathcal{W} \hookrightarrow C([a, b]; L^2(\Omega)), \quad \mathcal{W}_1 \hookrightarrow C([a, b]; H^{-1}(\Omega)),$$

where, for $\mathbf{X} = L^2(\Omega)$ or $\mathbf{X} = H^{-1}(\Omega)$, one denotes by $C([a, b]; \mathbf{X})$ the space of measurable functions on $[a, b] \times \Omega$ such that $y(t, \cdot) \in \mathbf{X}$ for any $t \in [a, b]$ and such that the map $t \in [a, b] \mapsto y(t, \cdot) \in \mathbf{X}$ is continuous.

(c) For any $u, v \in \mathcal{W}$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} u(t,x)v(t,x)\,\mathrm{d}x = \langle u'(t,\cdot), v(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle v'(t,\cdot), u(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,.$$

Let $y \in L^2\left(a,b;H_0^1(\Omega)\right) \cap C\left([a,b];L^2(\Omega)\right)$. Then the following density result holds: For any $\delta > 0$, there exists $\Phi \in C^\infty([a,b];\mathcal{D}(\Omega))$ such that

$$\|y - \Phi\|_{C([a,b];L^2(\Omega))} \le \delta, \quad \|\nabla y - \nabla\Phi\|_{L^2((a,b)\times\Omega)} \le \delta.$$

To end this subsection, we recall one property, useful in the sequel, concerning the space $L^2(\Omega; C_{\mathrm{per}}(Y))$, where $C_{\mathrm{per}}(Y)$ denotes the subset of $C(\overline{Y})$ of Y-periodic functions – namely the space $L^2(\Omega; C_{\mathrm{per}}(Y))$ is separable and dense in $L^2(\Omega; L^2(Y)) = L^2(\Omega \times Y)$.

2.2.4 Lax–Milgram's lemma

Consider a real Hilbert space V. Let $a(u,v)$ be a bilinear form on V (i.e., $a : V \times V \to \mathbb{R}$ is linear with respect to each argument). Suppose that the form $a(\cdot,\cdot)$ is continuous and coercive – that is, the following inequalities are satisfied:

$$\begin{aligned} &a(u,v) \le \nu_1\|u\|_V\|v\|_V, \quad \forall\, u,v \in V,\ \nu_1 > 0,\\ &a(u,u) \ge \nu_2\|u\|_V^2, \quad \forall\, u \in V,\ \nu_2 > 0. \end{aligned} \tag{2.13}$$

Let V^* be the dual space of V. For any fixed $u \in V$, the linear form $a(u,v)$ is continuous with respect to $v \in V$ and represents an element of V^* denoted by $\mathcal{A}u$. Thus, we obtain a linear operator defined by the formula

$$\langle \mathcal{A}u, v\rangle_{V^*,V} = a(u,v), \quad \mathcal{A} : V \to V^*.$$

Inequalities (2.13) show that the operator $\mathcal{A}$ is bounded and coercive: $\|\mathcal{A}\| \le \nu_2$, $\langle \mathcal{A}u, u\rangle_{V',V} \ge \nu_1\|u\|_V^2$. For any given $f \in V^*$, consider the following problem: Find an element $u \in V$ such that

$$a(u,v) = \langle f, v\rangle_{V^*,V}\,, \quad \forall\, v \in V. \tag{2.14}$$

In fact, this problem is equivalent to establishing solvability of the equation $\mathcal{A}u = f$. The following assertion generalizes the Riesz representation theorem.

Lemma 2.19. *(Lax–Milgram) The problem (2.14) has a solution u which is unique and satisfies the estimate $\|u\|_V \le \nu_1^{-1}\|f\|_{V^*}$. In other words, the bounded coercive operator $\mathcal{A}$ is an isomorphism between the spaces V and V^* and the norm of the inverse operator is bounded by ν_1^{-1}.*

If the bilinear form $a(u,v)$ is symmetric, then $\mathcal{A}u = f$ is the Euler equation associated with the following variational problem:

$$E = \inf_{v\in V} F(v), \quad F(v) = \frac{1}{2}a(v,v) - \langle f, v\rangle_{V^*,V}\,.$$

Indeed, let $u = \mathcal{A}^{-1}f$. Then

$$\begin{aligned} 2(F(v)-F(u)) &= \langle \mathcal{A}v, v\rangle_{V^*,V} - \langle \mathcal{A}u, u\rangle_{V^*,V} - 2\langle f, v\rangle_{V^*,V} \\ +2\langle f, u\rangle_{V^*,V} &= \langle \mathcal{A}v, v\rangle_{V^*,V} - \langle \mathcal{A}u, u\rangle_{V^*,V} + 2\langle f, u\rangle_{V^*,V} \\ -2\langle \mathcal{A}u, v\rangle_{V^*,V} &= \langle \mathcal{A}v, v\rangle_{V^*,V} - 2\langle \mathcal{A}u, v\rangle_{V^*,V} + \langle \mathcal{A}u, u\rangle_{V^*,V} \\ &= \langle \mathcal{A}(v-u), v-u\rangle_{V^*,V} > 0, \quad \forall\, v \neq u. \end{aligned}$$

So, $u = \mathcal{A}^{-1}f$ is the unique minimizer for $F(v)$.

2.2.5 General setting of the variational formulation of boundary value problems

To begin with, we recall the notation of a well-posed problem introduced by Hadamard. Let $\mathcal{P}$ be a boundary value problem and let $\mathcal{Y}$ and $\mathcal{F}$ be two Banach spaces. We say that $\mathcal{P}$ is well-posed (with respect to $\mathcal{Y}$ and $\mathcal{F}$) if the following hold:

(i) For any element $f \in \mathcal{F}$ there exists a solution $y \in \mathcal{Y}$ of $\mathcal{P}$.
(ii) The solution is unique.
(iii) The map $\mathcal{F} \ni f \mapsto y \in \mathcal{Y}$ is continuous.

Obviously, the well-posedness of a problem depends on the choice of the spaces $\mathcal{Y}$ and $\mathcal{F}$. As a matter of fact, the examples of boundary value problems with this property, which we treat in the sequel, are related to an equation of the form $\mathcal{A}y = f$, where the operator $\mathcal{A}$ is given as follows:

$$\mathcal{A} = -\mathrm{div}(\mathbf{A}(x)\nabla) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right). \tag{2.15}$$

Here, $\mathbf{A}(x) = \{a_{ij}(x)\}$ is a matrix (not necessarily symmetric) with bounded measurable elements (i.e., $\|a_{ij}\|_{L^\infty(\Omega)} \le \beta$) satisfying the ellipticity condition

$$\exists\, \alpha > 0 \text{ such that } \sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_j \ge \alpha \sum_{i=1}^{n} \lambda_i^2 \text{ a.e. on } \ \Omega\ \forall \lambda \in \mathbb{R}^n. \tag{2.16}$$

The matrix will always be associated with the bilinear form

$$a(y,\varphi) = \int_\Omega \nabla\varphi \cdot \mathbf{A}(x)\nabla y \,\mathrm{d}x. \tag{2.17}$$

The Dirichlet problem

Let $f \in H^{-1}(\Omega)$ and consider the problem

$$-\operatorname{div}(\mathbf{A}\nabla y) = f \ \text{ in } \ \Omega, \quad y = 0 \text{ on } \ \partial\Omega. \tag{2.18}$$

The corresponding variational formulation is

$$\begin{cases} \text{Find } \ y \in H_0^1(\Omega) \text{ such that} \\ a(y,\varphi) = \langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall\, \varphi \in H_0^1(\Omega). \end{cases} \tag{2.19}$$

Because of the Friedrichs inequality (2.8), the form $a(y,\varphi)$ is coercive on $H_0^1(\Omega)$. Therefore, for any $f \in H^{-1}(\Omega)$, there exists a unique element $y \in H_0^1(\Omega)$ satisfying (2.19). Moreover, $\|y\|_{H_0^1(\Omega)} \le \alpha^{-1}\|f\|_{H^{-1}(\Omega)}$. This element is called a weak solution of the Dirichlet problem (2.18). It is clear that (2.18) is a well-posed problem (in Hadamard's sense) for the choice $\mathcal{Y} = H_0^1(\Omega)$ and $\mathcal{F} = H^{-1}(\Omega)$ (or $\mathcal{F} = L^2(\Omega)$).

Assume that $\partial\Omega$ is Lipschitz continuous. Suppose we are given f in $H^{-1}(\Omega)$ and g in $H^{1/2}(\partial\Omega)$. Consider the nonhomogeneous Dirichlet problem

$$-\operatorname{div}(\mathbf{A}\nabla y) = f \ \text{ in } \ \Omega, \quad y = g \text{ on } \ \partial\Omega.$$

Using the trace notion, we say that y is a weak solution of this problem iff

$$-\operatorname{div}(\mathbf{A}\nabla y) = f \ \text{ in } \ \mathcal{D}'(\Omega), \quad \gamma(y) = g \text{ in } \ H^{1/2}(\partial\Omega). \tag{2.20}$$

Then the problem (2.20) has a unique solution y in $H^1(\Omega)$ such that

$$\|y\|_{H^1(\Omega)} \le C\left[\|f\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right],$$

where C is a positive constant depending on Ω, α, and β.

Moreover, we have the following result (see Lions and Magenes [173]).

Theorem 2.20. *Suppose that $\mathbb{N} \ni s \ge 2$. Then for any $f \in H^{s-2}(\Omega)$ and $g \in H^{s-1/2}(\partial\Omega)$, there exists a unique solution $y \in H^s(\Omega)$ of the problem*

$$-\triangle y = f \ \text{ in } \ \Omega, \quad \gamma(y) = g.$$

Moreover, in this case,

$$\|y\|^2_{H^s(\Omega)} \le C\left[\|f\|^2_{H^{s-2}(\Omega)} + \|g\|^2_{H^{s-1/2}(\partial\Omega)}\right], \tag{2.21}$$

where the constant C is independent of f and g.

The Neumann problem

Let $f \in (H^1(\Omega))^*$ and $g \in H^{-1/2}(\partial\Omega)$. Let us introduce the following notation:

$$\frac{\partial}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x) n_i \frac{\partial}{\partial x_j}, \tag{2.22}$$

where n_i denotes ith component of the unit outward normal vector to Ω. We consider the Neumann problem

$$-\operatorname{div}(\mathbf{A}\nabla y) + y = f \ \text{ in } \ \Omega, \quad \frac{\partial y}{\partial \nu_A} = 0 \ \text{ on } \ \partial\Omega. \tag{2.23}$$

The corresponding variational formulation is

$$\begin{cases} \text{Find } \ y \in H^1(\Omega) \text{ such that} \\ \widetilde{a}(y,\varphi) = \displaystyle\int_\Omega f\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in H^1(\Omega), \end{cases} \tag{2.24}$$

where now $\widetilde{a}$ is defined by

$$\widetilde{a}(y,\varphi) = \int_\Omega \nabla\varphi \cdot \mathbf{A}(x)\nabla y \,\mathrm{d}x + \int_\Omega y\varphi \,\mathrm{d}x, \quad \forall y, \varphi \in H^1(\Omega).$$

Let us observe that if y is a solution of the problem (2.24), then (2.23) holds in $\mathcal{D}'(\Omega)$. Then, due to Remark 2.17 (see (2.10)), $\mathbf{A}\nabla y$ belongs to $H^1(\Omega, \operatorname{div})$, and therefore, $\partial y/\partial \nu_A$ is well-defined as an element of $H^{1/2}(\partial\Omega)$. This is the sense to be given to the boundary condition in (2.23).

If $g = 0$ and the domain Ω is sufficiently smooth (for instance, such that the Poincaré inequality holds in Ω), then for any $f \in (H^1(\Omega))^*$, there exists a unique solution $y \in H^1(\Omega)$ of the problem (2.24). Moreover,

$$\|y\|_{H^1(\Omega)} \le \frac{1}{\min\{1,\alpha\}} \|f\|_{(H^1(\Omega))^*} \ \text{ and } \ \|y\|_{H^1(\Omega)} \le \frac{1}{\min\{1,\alpha\}} \|f\|_{L^2(\Omega)},$$

provided $f \in L^2(\Omega)$.

As for the nonhomogeneous Neumann problem in domains with a Lipschitz continuous boundary, namely

$$-\operatorname{div}(\mathbf{A}\nabla y) + y = f \ \text{ in } \ \Omega, \quad \frac{\partial y}{\partial \nu_A} = g \ \text{ on } \ \partial\Omega, \tag{2.25}$$

we have the following result: For any $f \in L^2(\Omega)$ and for any $g \in H^{-1/2}(\partial\Omega)$ there exists a unique solution $y \in H^1(\Omega)$ of the problem (2.25). Moreover, in this case,

$$\|y\|_{H^1(\Omega)} \le \frac{1}{\min\{1,\alpha\}} \left(\|f\|_{L^2(\Omega)} + C(\Omega) \|g\|_{H^{-1/2}(\partial\Omega)} \right), \tag{2.26}$$

where $C(\Omega)$ is a positive constant.

If we consider, instead of (2.25), the nonhomogeneous Neumann problem

$$-\operatorname{div}(\mathbf{A}\nabla y) = f \ \text{ in } \ \Omega, \quad \frac{\partial y}{\partial \nu_A} = g \ \text{ on } \ \partial\Omega \tag{2.27}$$

under the same hypotheses on f and g as earlier, then the corresponding bilinear form $a(y,\varphi)$ is no longer coercive on $H^1(\Omega)$, but it is coercive in the quotient space $W(\Omega) = H^1(\Omega)/\mathbb{R}$. This space is the space of classes of equivalence with respect to the relation

$$y_1 \backsimeq y_2 \Leftrightarrow y_1 - y_2 \ \text{ is a constant, } \ \forall\, y_1, y_2 \in H^1(\Omega).$$

Let us denote by $\dot{y}$ the class of equivalence represented by y. Then the natural variational formulation of (2.27) is

$$\begin{cases} \text{Find } \ \dot{y} \in W(\Omega) \text{ such that } \ \forall\, \dot{\varphi} \in W(\Omega) \\ \dot{a}(\dot{y},\dot{\varphi}) = \displaystyle\int_\Omega f\varphi \,\mathrm{d}x + \langle g, \varphi\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}, \quad \forall\, \varphi \in \dot{\varphi}, \end{cases} \tag{2.28}$$

where $\dot{a}$ is defined by

$$\dot{a}(\dot{y},\dot{\varphi}) = \int_\Omega \nabla\varphi \cdot \mathbf{A}(x)\nabla y \,\mathrm{d}x, \quad \forall y \in \dot{y}, \varphi \in \dot{\varphi}, \ \forall \dot{\varphi}, \dot{y} \in W(\Omega).$$

It is clear that this problem makes sense if the right-hand side of (2.28) is independent of $\varphi \in \dot{\varphi}$ – namely suppose that $f \in L^2(\Omega)$ and for any $g \in H^{-1/2}(\partial\Omega)$ satisfy the compatibility condition

$$\int_\Omega f \,dx + \langle g, 1\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)} = 0,$$

then there exists a unique solution $\dot{y} \in W(\Omega)$ of the problem (2.28) with the estimate

$$\|\dot{y}\|_{W(\Omega)} \le \frac{1}{\min\{1,\alpha\}} \left(\|f\|_{L^2(\Omega)} + C(\Omega)\|g\|_{H^{-1/2}(\partial\Omega)}\right). \tag{2.29}$$

We refer to Lions and Magenes [173] for the following result.

Theorem 2.21. *Suppose that $\mathbb{N} \ni s \ge 2$ and that $f \in H^{s-2}(\Omega)$ and $g \in H^{s-1-1/2}(\partial\Omega)$ satisfy the compatibility condition:*

$$\int_\Omega f(x)\,\mathrm{d}x + \int_{\partial\Omega} g(s)\,\mathrm{d}\mathcal{H}^{n-1} = 0. \tag{2.30}$$

Then, there exists a unique solution $\dot{y} \in H^s(\Omega) \setminus \mathbb{R}$ to the problem (2.27). Moreover,

$$\|\dot{y}\|^2_{H^s(\Omega)\setminus\mathbb{R}} \le C\left[\|f\|^2_{H^{s-2}(\Omega)} + \|g\|^2_{H^{s-1/2}(\partial\Omega)} + \|y\|^2_{L^2(\Omega)}\right],$$

where the constant C is independent of f and g.

The Robin problem

As in the previous case, suppose that $\partial\Omega$ is Lipschitz continuous and let $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$. The Robin boundary value problem we consider can be stated as follows:

$$-\operatorname{div}(\mathbf{A}\nabla y) + y = f \ \text{ in } \ \Omega, \quad \frac{\partial y}{\partial \nu_A} + dy = 0 \ \text{ on } \ \partial\Omega, \tag{2.31}$$

where $d \in \mathbb{R}$ is such that $d \geq 0$. The variational formulation of this problem is then

$$\begin{cases} \text{Find } \ y \in H^1(\Omega) \text{ such that} \\ \widetilde{a}(y,\varphi) = \displaystyle\int_\Omega f\varphi \,\mathrm{d}x + \langle g, \varphi\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}, \quad \forall\, \varphi \in H^1(\Omega), \end{cases} \tag{2.32}$$

where for all $y, \varphi \in H^1(\Omega)$,

$$a(y,\varphi) = \int_\Omega \nabla\varphi \cdot \mathbf{A}(x)\nabla y \,\mathrm{d}x + \int_\Omega y\varphi \,\mathrm{d}x + d\int_{\partial\Omega} y\varphi \,\mathrm{d}s. \tag{2.33}$$

In view of our suppositions, the linear form

$$F(y) = \int_\Omega fy \,\mathrm{d}x + < g, \varphi >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$$

is bounded on $H^1(\Omega)$. Hence, $F \in \left(H^1(\Omega)\right)^*$. Then, having observed that the bilinear form $a(y,\varphi)$ given by (2.33) is continuous on $H^1(\Omega) \times H^1(\Omega)$ and coercive, since d is positive, we can apply the Lax–Milgram Lemma 2.19 with $V = H^1(\Omega)$ to get a unique element $y \in H^1(\Omega)$ satisfying (2.32). Moreover, in this case we have the same estimate for $\|y\|_{H^1(\Omega)}$ as in (2.26).

To end this section, we consider the case where one has a Dirichlet condition on a part of the boundary $\partial\Omega$ and a homogeneous Robin one on the rest of the boundary. Let Ω be a bounded connected domain with a Lipschitz boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two disjoint closed sets and Γ_1 is of positive measure. Let V be the closure of $C_0^\infty(\mathbb{R}^n \setminus \Gamma_1)$ with respect to norm of $H^1(\Omega)$, that is,

$$V = \left\{ y \mid y \in H^1(\Omega), y|_{\Gamma_1} = 0 \right\}.$$

We say that $y \in V$ is a weak solution to the mixed Dirichlet–Robin problem

$$\begin{cases} -\operatorname{div}(\mathbf{A}\nabla y) = f \ \text{ in } \ \Omega, \\ y = 0 \ \text{ on } \ \Gamma_1, \quad \dfrac{\partial y}{\partial \nu_A} + dy = 0 \ \text{ on } \ \Gamma_2, \end{cases} \tag{2.34}$$

where $d \geq 0$ if

$$\int_\Omega \nabla\varphi \cdot \mathbf{A}(x)\nabla y \,\mathrm{d}x + d\int_{\Gamma_2} y\varphi \,\mathrm{d}s = \int_\Omega f\varphi \,\mathrm{d}x \quad \forall\, \varphi \in V.$$

The existence of a unique solution to this problem immediately follows from the Lax–Milgram lemma and the inequality

$$\int_\Omega \varphi^2(x)\,\mathrm{d}x \le C_0 \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^n \setminus \Gamma_1),$$

which, in analogy to (2.8), will also be called the Friedrichs inequality.

2.3 Spaces of periodic functions

In this section, we provide a notion of periodicity for functions in the Sobolev space H^1. Let us consider measurable functions defined on $\mathbb{R}^n$ and periodic in each argument $x_1, x_2, \dots, x_n$ with periods $l_1, l_2, \dots, l_n$, respectively. Let $\square$ be the parallelepiped in $\mathbb{R}^n$ defined by $\square = (0, l_1) \times \cdots \times (0, l_n)$. We will refer to $\square$ as the reference period. Then the function f is called $\square$-periodic iff

$$f(x + k l_i \mathbf{e}_i) = f(x) \quad \text{a.e. on } \mathbb{R}^n,\ \forall\, k \in \mathbb{Z},\ \forall\, i \in \{1, 2, \dots, n\},$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

The mean value of a periodic functions is essential when studying periodic oscillating functions. Let us recall that for any $\square$-periodic function f the mean value of f is the real number $\mathcal{M}_\square(f)$ given by

$$\mathcal{M}_\square(f) = \frac{1}{|\square|} \int_\square f(y)\,\mathrm{d}y,$$

where $|\square| = l_1 l_2 \cdots l_n$ is the volume of the parallelepiped $\square$. The Lebesgue space of periodic measurable functions with a finite norm $\mathcal{M}_\square^{1/\alpha}(|f|^\alpha)$ for $\alpha \ge 1$ we denote by $L^\alpha(\square)$. The following property of periodic functions is frequently used in the asymptotic analysis.

A property of the mean value. Let f be a $\square$-periodic function, $f \in L^\alpha(\square)$, $\alpha \ge 1$. Then

$$\int_\Omega f(\varepsilon^{-1}x)\varphi(x)\,\mathrm{d}x \longrightarrow \mathcal{M}_\square(f) \int_\Omega \varphi(x)\,\mathrm{d}x \quad \text{as } \varepsilon \to 0 \tag{2.35}$$

for every $\varphi \in L^{\alpha'}(\Omega)$, where $1/\alpha + 1/\alpha' = 1$, $\alpha \in (0, \infty)$, and Ω is an arbitrary bounded domain in $\mathbb{R}^n$.

Let $C^\infty_{per}(\square)$ be the subset of $C^\infty(\mathbb{R}^n)$ of $\square$-periodic functions. We denote by $H^1_{per}(\square)$ the completion of the space $C^\infty_{per}(\square)$ with respect to the H^1-norm. It should be stressed that $H^1_{per}(\square)$ does not coincide with the entire Sobolev space $H^1(\Omega)$ for $\Omega = \square$. Functions in $H^1_{per}(\square)$ as well as all other periodic functions, are assumed to be defined on $\mathbb{R}^n$ – namely if $u \in H^1_{per}(\square)$, then u is in $H^1(D)$ for any bounded open subset D of $\mathbb{R}^n$ and u satisfies the following condition:

$$u(x + kl_i\mathbf{e}_i) = u(x), \quad \forall\, k \in \mathbb{Z},\ \forall\, i \in \{1, 2, \dots, n\}.$$

Thus, we may write $H^1_{per}(\square) = \left\{u \in H^1_{loc}(\mathbb{R}^n),\ u \text{ is } \square\text{-periodic}\right\}$. The space $H^1_{per}(\square)$ possesses the following obvious property: If $u \in H^1_{per}(\square)$, then u has the same trace on the opposite site faces of $\square$.

In the sequel, we will make use of the quotient space $\mathcal{W}_{per}(\square) = H^1_{per}(\square)/\mathbb{R}$ defined as the space of equivalent classes with respect to the relation

$$u \cong v \Leftrightarrow u - v \ \text{ is a constant, } \forall\, u, v \in H^1_{per}(\square).$$

As usual, we denote by $\dot{u}$ the equivalence class represented by u. Then the quantity

$$\|\dot{u}\|_{\mathcal{W}_{per}(\square)} = \|\nabla u\|_{L^2(\square)}, \quad \forall\, u \in \dot{u},\ \dot{u} \in \mathcal{W}_{per}(\square)$$

defines a norm on $\mathcal{W}_{per}(\square)$ for which $\mathcal{W}_{per}(\square)$ is a Banach space. Moreover, the dual space $\mathcal{W}^*_{per}(\square)$ can be identified with the set

$$\left\{ F \in \mathcal{W}^*_{per}(\square) \middle|\ F(c) = 0,\ \forall\, c \in \mathbb{R} \right\},$$

with $\langle F, \dot{u}\rangle_{\mathcal{W}^*_{per}(\square),\mathcal{W}_{per}(\square)} = \langle F, u\rangle_{(H^1_{per}(\square))^*, H^1_{per}(\square)}$ for all $u \in \dot{u}$ and $\forall\, \dot{u} \in \mathcal{W}_{per}(\square)$.

We define also the space of *solenoidal periodic vector fields*, by setting

$$\mathbf{L}^2_{sol}(\square) := \left\{\mathbf{p} \in \mathbf{L}^2(\square),\ \operatorname{div}\mathbf{p} = 0 \ \text{ in } \ \mathbb{R}^n\right\}. \tag{2.36}$$

Clearly, $\mathbf{L}^2_{sol}(\square)$ is a closed subspace of $\mathbf{L}^2(\square)$, and, by definition, $\mathbf{p} \in \mathbf{L}^2_{sol}(\square)$ if $\mathbf{p} \in \mathbf{L}^2(\square)$ and the identity

$$\int_{\square} \mathbf{p} \cdot \nabla\varphi \, \mathrm{d}x = 0, \quad \forall\, \varphi \in C^\infty_{per}(\square)$$

is valid (see [261]). Moreover, in this case, the following orthogonal representation holds:

$$\mathbf{L}^2(\square) = \mathbf{L}^2_{sol}(\square) \oplus \boldsymbol{\mathcal{V}}^2_{pot}(\square), \quad \boldsymbol{\mathcal{V}}^2_{pot}(\square) = \left\{\nabla u,\ u \in H^1_{per}(\square)\right\}.$$

It should be pointed out that because of the Poincaré inequality, $\boldsymbol{\mathcal{V}}^2_{pot}(\square)$ is a closed subspace of $\mathbf{L}^2(\square)$. We also introduce the space of potential periodic vector fields

$$\mathbf{L}^2_{pot}(\square) = \mathbb{R}^n \oplus \boldsymbol{\mathcal{V}}^2_{pot}(\square).$$

Then any vector field $\mathbf{v} \in \mathbf{L}^2_{pot}(\square)$ is potential by definition and can be represented in the form $\mathbf{v} = \mathcal{M}_{\square}(\mathbf{v}) + \nabla u$.

2.4 Weak and weak-$*$ convergence in Banach spaces

In this section, we recall without proofs the basic facts from functional analysis concerning the weak and weak-$*$ convergence in Banach spaces.

Let F be a Banach space equipped with the norm $\|\cdot\|_F$ and let F^* be its dual. We set $E = F^*$. The norm of u in E is defined by the formula

$$\|u\|_E = \sup_{\|f\|_F=1} \langle u, f\rangle_{E,F},$$

where $\langle u, f\rangle_{E,F}$ denotes the value of $u \in E$ at $f \in F$.

Definition 2.22. *A sequence $\{u_k\}_{k=1}^{\infty} \subset E$ is said to be strongly convergent to $u \in E$ as $k \to \infty$ if $\|u_k - u\|_E \to 0$ as $k \to \infty$. In this case, we write*

$$u_k \to u \text{ strongly in } E \text{ as } k \to \infty.$$

Definition 2.23. *A sequence $\{u_k\}_{k=1}^{\infty} \subset E$ is said to converge weakly to u, written $u_k \rightharpoonup u$ in E, provided*

$$\langle u^*, u_k\rangle_{E^*,E} \to \langle u^*, u\rangle_{E^*,E}, \quad \forall\, u^* \in E^*.$$

For instance, let Ω be an open bounded domain in $\mathbb{R}^n$. We write $w_k \rightharpoonup w$ in $L^p(\Omega)$ if for any $\varphi \in L^{p'}(\Omega)$ $(1 \le p < \infty, p' = p/(p-1))$

$$\lim_{k\to\infty} \int_\Omega w_k \varphi \,\mathrm{d}x = \int_\Omega w\varphi \,\mathrm{d}x.$$

A weakly convergent sequence is necessarily bounded in the norm.

Theorem 2.24. *(Boundedness of weakly convergent sequences) Assume $u_k \rightharpoonup u$ in E. Then the following hold:*

(i) $\{u_k\}_{k=1}^{\infty}$ is bounded in E,
(ii) $\|u\|_E \le \liminf_{k\to\infty} \|u_k\|_E$,

that is, the norm in E is lower semicontinuous with respect to the weak convergence in E.

Moreover, a refinement of (ii) holds: If E is reflexive, $u_k \rightharpoonup u$ in E, and $\lim_{k\to\infty} \|u_k\|_E = \|u\|_E$, then

$$u_k \to u \text{ strongly in } E.$$

For the case of Lebesgue spaces, Brezis and Lieb [31] obtained the following: *Let $u_k \rightharpoonup u$ in $L^p(\Omega)$ $(1 \le p < \infty)$ and $u_k \to u$ almost everywhere in Ω. Then*

$$\lim_{k\to\infty} \Big(\|u_k\|^p_{L^p(\Omega)} - \|u_k - u\|^p_{L^p(\Omega)} \Big) = \|u\|^p_{L^p(\Omega)}.$$

The following theorem states one of the main properties of the weak convergence in reflexive Banach spaces. For the proof, which is rather technical, we refer to Yosida [251] or Kantorovich and Akilov [128].

Theorem 2.25. *(Eberlein-Ŝmuljan) Assume that E is a reflexive Banach space and let $\{u_k\}_{k=1}^{\infty}$ be a bounded sequence in E. Then the following hold:*

(i) There exist a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ *of* $\{u_k\}_{k=1}^{\infty}$ *and an element* $u \in E$ *such that*

$$u_{k_j} \rightharpoonup u \quad \text{in } E \text{ as } k \to \infty.$$

(ii) If each weakly convergent subsequence of $\{u_k\}_{k=1}^{\infty}$ *has the same limit* u*, then the whole sequence* $\{u_k\}_{k=1}^{\infty}$ *weakly converges to* u*.*

This theorem is definitely false if we take $F = L^1(\Omega)$ and $E = L^\infty(\Omega)$ as a simple example illustrates. Indeed, let us consider the following sequence in $L^1(\Omega)$ ($\Omega = (-1, 1)$):

$$x_k = \begin{cases} 0 & \text{if } t \in (-1, -1/n) \cup (1/n, 1); \\ n^2 t + n & \text{if } t \in [-1/n, 0); \\ -n^2 t + n & \text{if } t \in [0, 1/n], \end{cases} \qquad k = 1, 2, \ldots.$$

It is clear that $\sup_{k\in\mathbb{N}} \|x_k\|_{L^1(\Omega)} = 1$. However, this sequence is not compact with respect to the weak convergence in $L^1(\Omega)$. In order to prove it, we note that

$$\lim_{k\to\infty} \int_\Omega x_k(t)\varphi(t)\,\mathrm{d}t = \varphi(0), \quad \forall\, \varphi \in C_0^\infty(\Omega), \tag{2.37}$$

by the mean value theorem. Indeed, due to the continuity of φ, for any $\eta > 0$ there exists an $\varepsilon_0 > 0$ such that $|\varphi(t) - \varphi(0)| < \eta$, provided $|t| < \varepsilon_0$. Then

$$\begin{aligned} \left| \int_\Omega x_k(t)\varphi(t)\,\mathrm{d}t - \varphi(0) \right| &= \left| \int_{-1/n}^{1/n} x_k(t)\varphi(t)\,\mathrm{d}t - \frac{n}{2}\int_{-1/n}^{1/n} \varphi(0)\,\mathrm{d}t \right| \\ &\le \frac{n}{2}\int_{-1/n}^{1/n} |\varphi(t) - \varphi(0)|\,\mathrm{d}t < \frac{n}{2}\eta \int_{-1/n}^{1/n} \mathrm{d}t = \eta \end{aligned}$$

for all $\varepsilon < \varepsilon_0$.

Let us suppose that there is a function $x \in L^1(\Omega)$ such that

$$\int_\Omega x(t)\varphi(t)\,\mathrm{d}t = \varphi(0), \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{2.38}$$

Since $t\varphi(t) \in C_0^\infty(\Omega)$, from (2.37) it follows that

$$\int_\Omega x(t)t\varphi(t)\,\mathrm{d}t = t\varphi(t)|_{t=0} = 0 \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

Hence, by Raymond's lemma (see, for instance, [49]), we conclude that $tx(t) = 0$ a.e. in Ω. So, $x(t) = 0$ for almost all $t \in \Omega$, and we come into conflict with (2.38). This means that the above sequence does not converge weakly in $L^1(\Omega)$.

We will also need a more general form of the weak convergence, which is usually called the weak-$*$ convergence.

Definition 2.26. *A sequence $\{u_k\}_{k=1}^{\infty} \subset E$ is said to converge weakly-∗ to u (written $u_k \overset{*}{\rightharpoonup} u$ in $E = F^*$), provided*

$$\langle u_k, u^* \rangle_{F^*,F} \to \langle u, u^* \rangle_{F^*,F}, \quad \forall\, u^* \in F.$$

In particular, if $u_k, u \in L^1(\Omega)$ and the sequence $\{u_k\}_{k=1}^{\infty}$ is bounded in $L^1(\Omega)$, then $u_k \overset{*}{\rightharpoonup} u$, provided the relation

$$\lim_{k\to\infty} \int_\Omega u_k \varphi \, \mathrm{d}x = \int_\Omega u\varphi \, \mathrm{d}x$$

holds for any $\varphi \in C_0^\infty(\Omega)$.

The main properties of weak convergence are still valid for weak-∗ convergence. In particular, any weakly-∗ convergent sequence in E is bounded and the norm in E is lower semicontinuous with respect to the weak-∗ convergence in E. It is very important for what follows that a weak-∗ limit is uniquely defined. It is also clear that the weak convergence in E (in $L^1(\Omega)$, in particular) implies the weak-∗ convergence. However, in general, the converse statement does not hold. Indeed, let

$$\Omega = (0,1), \quad u_k = \frac{1}{\sqrt{\pi k}} \exp(-t^2 k^{-1}).$$

Then $u_k \overset{*}{\rightharpoonup} 0$ in $L^1(\Omega)$. However,

$$\int_\Omega u_k \, \mathrm{d}t = \langle 1, u_k \rangle_{L^\infty(\Omega), L^1(\Omega)} \longrightarrow \frac{1}{2} \quad \text{as} \quad k \to \infty.$$

The equivalent of Theorem 2.25 for weak-∗ convergence read as follows.

Theorem 2.27. *Let F be a separable Banach space and let $E = F^*$. If $\{u_k\}_{k=1}^{\infty}$ is a bounded sequence in E, then following hold:*

(i) There exist a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ and an element $u \in E$ such that

$$u_{k_j} \overset{*}{\rightharpoonup} u \quad \text{in} \quad E \ \text{as} \ \ k \to \infty.$$

(ii) If each weakly-∗ convergent subsequence of $\{u_k\}_{k=1}^{\infty}$ has the same limit u, then the whole sequence $\{u_k\}_{k=1}^{\infty}$ weakly-∗ converges to u.

One often has to find the limit of the product $\langle f_k, u_k \rangle_{E^*,E}$ as $k \to \infty$. In the trivial case, when either $u_k \rightharpoonup u$ in E and $f_k \to f$ in E^* or $u_k \to u$ in E and $f_k \overset{*}{\rightharpoonup} f$ in E^*, we have $\langle f_k, u_k \rangle_{E^*,E} \to \langle f, u \rangle_{E^*,E}$. However, in general, one cannot pass to the limit in the product $\langle f_k, u_k \rangle_{E^*,E}$ as $k \to \infty$ when both of these sequences are only known to be weakly convergent.

The following classical example is very significant. Let $v(t)$ be the periodic function of period 1, defined on $\mathbb{R}$ by $v(t) = \sin(2\pi t)$, and set $\square = [0,1)$, $\Omega = (a,b)$, and

$$v_k(t) := v\,(kt) = \sin\,(2\pi kt)\,, \quad t \in (a,b),$$

where $a, b \in \mathbb{R}$. Applying the mean value property (2.35), we have

$$v_k \rightharpoonup M_{\square}(v) = \int_0^1 \sin(2\pi t)\,\mathrm{d}t = 0 \quad \text{in} \quad L^2(a,b) \quad \text{as} \quad k \to \infty.$$

On the other hand, $\langle v_k, v_k \rangle_{L^2(\Omega),L^2(\Omega)}$ does not converge to 0. Indeed,

$$\begin{aligned}\langle v_k, v_k \rangle_{L^2(\Omega),L^2(\Omega)} &= \int_a^b \sin^2\,(2\pi kt)\;\mathrm{d}t = \frac{1}{2\pi k}\int_{2\pi ka}^{2\pi kb} \sin^2(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{2\pi k}\int_{2\pi ka}^{2\pi kb} \frac{1-\cos(2\tau)}{2}\,\mathrm{d}\tau \\ &= \frac{b-a}{2} + \frac{1}{8\pi k}\left[-\sin(4\pi kb) + \sin(4\pi ka)\right],\end{aligned}$$

so that, as $k \to \infty$,

$$\langle v_k, v_k \rangle_{L^2(\Omega),L^2(\Omega)} \longrightarrow \frac{b-a}{2} \neq 0.$$

To understand the ways in which a weakly convergent sequence of functions can fail to be strongly convergent, we consider a sequence $\{u_k\}_{k=1}^{\infty}$ in $L^p(\Omega)$ $(1 < p < \infty)$, where Ω is a bounded smooth open subset of $\mathbb{R}^n$, such that $u_k \rightharpoonup u$ in $L^p(\Omega)$ as $k \to \infty$. Let us observe that even if we know the functions $\{u_k\}_{k=1}^{\infty}$ to be bounded in the supremum norm, so that u_k converges weakly to u in $L^q(\Omega)$ for all $1 \leq q < \infty$, we still cannot deduce strong convergence in $L^q(\Omega)$ for any $1 \leq q < \infty$. The difficulty is with the possibility of very rapid fluctuations in the functions u_k (see the previous example). This is the problem of wild oscillations.

Second, observe that even if we know additionally that

$$u_k \to u \quad \text{a.e. in} \quad \Omega,$$

so that wild oscillations are excluded, we still cannot deduce strong convergence in $L^p(\Omega)$. The obstruction is that the mass of $|u_k - u|^p$ may somehow coalesce onto a set of zero Lebesgue measure. This is the problem of concentration. To characterize the concentration effects, let us suppose that $u_k \rightharpoonup u$ in $L^p(\Omega)$ and introduce the following measure:

$$\theta_k(E) = \int_E |u_k - u|^p\,\mathrm{d}t, \quad k = 1, 2, \ldots,$$

where E is a Borel subset of Ω. Thus, $\theta_k(E)$ controls how close the function u_k is to u in the L^p-norm restricted to the set E. Following DiPerna and Majda [95], we call the value $\theta(E) = \limsup_{k\to\infty} \theta_k(E)$ the reduced defect measure associated with the weak convergence $u_k \rightharpoonup u$ in $L^p(\Omega)$. The idea is that $\theta(E)$

encodes an information about the extent to which strong convergence fails. In particular,

$$u_k \to u \quad \text{in } L^p(E) \text{ if and only if } \theta(E) = 0 \text{ (see [105])}.$$

For instance, let $\Omega = (-1, 1)$, $u = 0$, and

$$u_k(t) = \begin{cases} k & \text{if } -k^{-1} \le t \le k^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $u_k \rightharpoonup 0$ in $L^2(\Omega)$ and θ is concentrated on $E = \{0\}$. In this case $\theta(\Omega \setminus V) = 0$ for each open set $V \supset E$.

The following theorem, taken from Valadier's work [245], gives necessary and sufficient conditions under which weak convergence implies strong convergence.

Theorem 2.28. *Suppose that $u_k \rightharpoonup u$ in $L^1(\Omega)$. Then $u_k \to u$ strongly if and only if the following criterion is satisfied: $\forall \varepsilon > 0$, $\forall A \subset \Omega$ with $\operatorname{meas}(A) > 0$, $\exists N \in \mathbb{N}$, $\exists B \subset A$ with $\operatorname{meas}(B) > 0$, such that, $\forall k \ge N$, the inequality*

$$\frac{1}{\operatorname{meas}(B)} \int_B \left| u_k(t) - \frac{1}{\operatorname{meas}(B)} \int_B u_k(t)\,\mathrm{d}t \right| \mathrm{d}t < \varepsilon$$

holds true.

2.4.1 Weak convergence of measures

Let $\mathcal{M}(\Omega)$ be the set of Borel measures on Ω. For all $\mu \in \mathcal{M}(\Omega)$, we define the functional

$$L_\mu(\varphi) = \int_\Omega \varphi \,\mathrm{d}\mu, \quad \forall\, \varphi \in L^1(\Omega, \mathrm{d}\mu). \tag{2.39}$$

It is clear that L_μ is linear and continuous on $C_0(\Omega)$. Following Riesz's theorem [28, 221], for any linear continuous functional $L_\mu : C_0(\Omega) \to \mathbb{R}$ there exists a unique measure $\mu \in \mathcal{M}(\Omega)$ such that the representation $L_\mu(\varphi) = \displaystyle\int_\Omega \varphi \,\mathrm{d}\mu$ holds true for every $\varphi \in C_0(\Omega)$. Thus, the map $\mu \mapsto L_\mu$ is a bijection between $M(\Omega)$ and $(C_0(\Omega))^*$. Moreover, in this case, we have $\|L_\mu\| = |\mu|(\Omega)$. Indeed,

$$\begin{aligned} \|L_\mu\| &= \sup \left\{ \int_\Omega \varphi \,\mathrm{d}\mu \;:\; \varphi \in C_0(\Omega),\ |\varphi| \le 1 \right\} \\ &\le \sup \left\{ \int_\Omega |\varphi| \,\mathrm{d}|\mu| \;:\; \varphi \in C_0(\Omega),\ |\varphi| \le 1 \right\} = \int_\Omega \mathrm{d}|\mu| = |\mu|(\Omega). \end{aligned}$$

Thus, measures can be identified as elements of the dual space of continuous functions vanishing on $\partial\Omega$. Hence, they inherit a notion of weak-∗ convergence which was defined earlier. In view of this, we can define a weak topology on $\mathcal{M}(\Omega)$ as follows.

Definition 2.29. *We say that a sequence* $\{\mu_k\}_{k=1}^{\infty} \subset \mathcal{M}(\Omega)$ *converges weakly to* μ *(and we write* $\mu_k \rightharpoonup \mu$*) if* $L_{\mu_k} \overset{*}{\rightharpoonup} L_\mu$ *in the weak-*$*$ *topology of* $C_0(\Omega)$*, that is,*

$$\lim_{k\to\infty} \int_\Omega \varphi \, \mathrm{d}\mu_k = \int_\Omega \varphi \, \mathrm{d}\mu, \quad \forall\, \varphi \in C_0(\Omega).$$

By the Banach–Steinhaus theorem, we have that

$$\text{if } \ \mu_k \rightharpoonup \mu, \quad \text{then } \ \sup_k |\mu_k|(\Omega) < +\infty.$$

Note, moreover, that by the lower semicontinuity of the dual norm with respect to weak-$*$ convergence, we have that $\mu \mapsto |\mu|(\Omega)$ is weakly lower semicontinuous (i.e., $|\mu|(\Omega) \le \liminf_{k\to\infty} |\mu_k|(\Omega)$, provided $\mu_k \rightharpoonup \mu$).

Example 2.30. Let $\square$ be the parallelepiped in $\mathbb{R}^n$ defined by

$$\square = (0, l_1) \times \cdots \times (0, l_n).$$

We will refer to $\square$ as the reference period. Let F be a $\square$-periodic connected domain in $\mathbb{R}^n$. Let Ω be as usual a bounded smooth open subset of $\mathbb{R}^n$. So, we may always suppose that Ω is a measurable set in the sense of Jordan. Let us introduce the following measure on Ω,

$$\mathrm{d}\mu = \rho(x)\mathrm{d}x, \quad \rho(x) = \begin{cases} |F \cap \Omega|^{-1} & \text{if } \ x \in \Omega, \\ 0 & \text{if } \ x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.40}$$

It is clear that $\mu \in \mathcal{M}(\Omega)$ as a periodic Borel positive measure with periodicity cell $\square$ and $\displaystyle\int_\square \mathrm{d}\mu = 1$. This measure is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}_n$. Let us consider the sequence $\{\mu_k\}_{k=1}^{\infty}$ in $\mathcal{M}(\Omega)$, where

$$\mu_k(B) = k^{-n}\mu(kB), \quad \forall\, k \in \mathbb{N},$$

for every Borel set $B \subset \mathbb{R}^n$. Then $\mathrm{d}\mu_k = \rho(kx)\,\mathrm{d}x$ and

$$\int_{k^{-1}\square} \mathrm{d}\mu_k = k^{-n} \int_\square \mathrm{d}\mu = k^{-n}.$$

We wish to prove that $\mu_k \rightharpoonup \mathcal{L}_n$ as $k \to \infty$. Indeed, let $\square_{\mathbf{i}} = \square + \mathbf{i}$, where $\mathbf{i}$ is a vector in $\mathbb{R}^n$ with integer components. Then $k^{-1}\square_{\mathbf{i}}$ is a partition of $\mathbb{R}^n$ for every fixed $k \in \mathbb{N}$ and

$$\int_\Omega \varphi(x)\, \mathrm{d}\mu_k = \sum \int_{k^{-1}\square_{\mathbf{i}}} \varphi(x)\, \mathrm{d}\mu_k + \sum \int_{k^{-1}\square_{\mathbf{i}} \cap \Omega} \varphi(x)\, \mathrm{d}\mu_k \tag{2.41}$$

for every $\varphi \in C_0(\Omega)$, where the first sum is taken over all $\mathbf{i}$ such that $k^{-1}\square_{\mathbf{i}}$ is inside Ω and the second sum is over all $\mathbf{i}$ such that $k^{-1}\square_{\mathbf{i}}$ and $\partial\Omega$ have

common points. Let us first consider the first sum in (2.41). Since $\varphi \in C_0(\Omega)$, there exist points $x_{\mathbf{i}} \in k^{-1}\square_{\mathbf{i}}$ such that

$$\int_{k^{-1}\square_{\mathbf{i}}} \varphi(x)\,\mathrm{d}\mu_k = \varphi(x_{\mathbf{i}}) \int_{k^{-1}\square_{\mathbf{i}}} \mathrm{d}\mu_k = \varphi(x_{\mathbf{i}})k^{-n} \int_{\square} \mathrm{d}\mu = k^{-n}\varphi(x_{\mathbf{i}}).$$

This, together with the fact that $\sum k^{-n}\varphi(x_{\mathbf{i}})$ is the construction of a Riemann sum, implies that

$$\lim_{k\to\infty} \sum k^{-n}\varphi(x_{\mathbf{i}}) \to \int_{\Omega} \varphi(x)\,\mathrm{d}x. \tag{2.42}$$

Let us consider the second sum in (2.41). We have

$$\left| \sum \int_{k^{-1}\square_{\mathbf{i}}\cap\Omega} \varphi(x)\,\mathrm{d}\mu_k \right| \le \max_{x\in\Omega} |\varphi(x)| k^{-n} M(k),$$

where $M(k)$ is the number of cubes $k^{-1}\square_{\mathbf{i}}$ containing the boundary of Ω. Since $k^{-n}M(k) \to 0$ by the Jordan measurability property of Ω, we conclude

$$\lim_{k\to\infty} \sum \int_{k^{-1}\square_{\mathbf{i}}\cap\Omega} \varphi(x)\,\mathrm{d}\mu_k = 0. \tag{2.43}$$

Thus, the desired property follows by taking (2.42)–(2.43) into account.

Theorem 2.31. *Let $\{\mu_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{M}(\Omega)$ with $\sup_k |\mu_k|(\Omega) < +\infty$. Then there exists a subsequence of $\{\mu_k\}_{k=1}^{\infty}$ weakly converging to some $\mu \in \mathcal{M}(\Omega)$.*

To characterize this result, which can be found in Federer [109] and Evans and Gariepy [106], let us consider a bounded sequence $\{u_k\}_{k=1}^{\infty}$ in $L^1(\Omega, d\mu)$, where μ is a positive Borel measure on Ω. Then Theorem 2.31 applied with $\mu_k = u_k\mu$ yields the relative compactness of $\{\mu_k\}_{k=1}^{\infty}$ only in the weak-$\mathcal{M}(\Omega)$ topology. Consequently, in general, its cluster points need not be in $L^1(\Omega)$.

Let us list some general properties of the weak convergence in the space of Radon measures $\mathcal{M}_b(\Omega)$ which we apply below. We recall that a sequence $\{\mu_k\}_{k=1}^{\infty}$ of Radon measures on $\mathbb{R}^n$ is said to be bounded if

$$\sup_{k\in\mathbb{N}} \mu_k(K) < +\infty \quad \text{for each compact set} \quad K \subset \mathbb{R}^n.$$

Lemma 2.32. *Let $\{\mu_k\}_{k=1}^{\infty}$ and μ be Radon measures on Ω such that $\mu_k \rightharpoonup \mu$ in $\mathcal{M}_b(\Omega)$ as $k \to \infty$. Then (see Zhikov [258]) the following hold:*

1. *$\eta\mu_k \rightharpoonup \eta\mu$ in $\mathcal{M}_b(\Omega)$ for every positive function $\eta \in C(\overline{\Omega})$.*
2. *$\liminf_{k\to\infty} \mu_k(A) \ge \mu(A)$ for every open set $A \subset \Omega$.*
3. *$\limsup_{k\to\infty} \mu_k(K) \le \mu(K)$ for every compact set $K \subset \Omega$.*

4. If $\lim_{k\to\infty} \mu_k(\Omega) = \mu(\Omega)$*, then the weak convergence* $\mu_k \rightharpoonup \mu$ *implies the following convergence:*

$$\lim_{k\to\infty} \int_\Omega \varphi \, \mathrm{d}\mu_k = \int_\Omega \varphi \, \mathrm{d}\mu, \quad \forall\, \varphi \in C(\overline{\Omega}).$$

Since $\mathcal{M}_b(\Omega)$ is the dual space of $C_0(\Omega)$, it follows that every Radon measure on Ω can be identified with an element of the space of distributions $\mathcal{D}'(\Omega)$ in the usual way. Therefore, we say that a Radon measure μ belongs to $\mathcal{M}_b(\Omega) \cap W^{-1,q}(\Omega)$ if and only if there exist functions $f_0, f_1, \ldots, f_n \in L^q(\Omega)$ such that

$$\int_\Omega \varphi \, \mathrm{d}\mu = \int_\Omega f_0 \varphi \, \mathrm{d}x - \sum_{k=1}^{n} \int_\Omega f_k \frac{\partial \varphi}{\partial x_k} \, \mathrm{d}x, \quad \varphi \in C_0^\infty(\Omega).$$

In other words, we say that a Radon measure μ on Ω belongs to $W^{-1,q}(\Omega)$ if there exists $f \in W^{-1,q}(\Omega)$ such that

$$\langle f, \varphi \rangle_{W^{-1,q}(\Omega), W_0^{1,p}(\Omega)} = \int_\Omega \varphi \, \mathrm{d}\mu, \quad \varphi \in C_0^\infty(\Omega).$$

In this case, we can identify $f \in W^{-1,q}(\Omega)$ and $\mu \in \mathcal{M}_b(\Omega)$. Note also that, by the Riesz theorem, every non-negative element of $W^{-1,q}(\Omega)$ is a Radon measure.

Theorem 2.33 ([105]). *Assume that a sequence of Radon measures* $\{\mu_k\}_{k=1}^\infty$ *is bounded in* $\mathcal{M}_b(\Omega)$*. Then* $\{\mu_k\}_{k=1}^\infty$ *is precompact in* $W^{-1,q}(\Omega)$ *for each* $1 \le q < 1^* = n/(n-1)$*.*

Proof. In view of Theorem 2.31, we may extract a subsequence $\{\mu_{k_j}\}_{j=1}^\infty \subset \{\mu_k\}_{k=1}^\infty$ so that $\mu_{k_j} \rightharpoonup \mu$ in $\mathcal{M}(\Omega)$ for some measure $\mu \in \mathcal{M}_b(\Omega)$. Let us set $q' = q/(q-1)$ and denote by B the closed unit ball in $W_0^{1,q'}(\Omega)$. Since $1 \le q < 1^*$, we have $q' > n$. So, by the Sobolev embedding theorem, B is a compact set in $C_0(\overline{\Omega})$. Hence, for a given $\delta > 0$, there exist functions $\{\phi_i\}_{i=1}^{N(\delta)} \subset C_0(\overline{\Omega})$ such that

$$\sup_{1\le i\le N(\delta)} \|\varphi - \phi_i\|_{C(\overline{\Omega})} < \delta, \quad \forall\, \varphi \in B.$$

Thus, if $\varphi \in B$, then

$$\left| \int_\Omega \varphi \, \mathrm{d}\mu_{k_j} - \int_\Omega \varphi \, \mathrm{d}\mu \right| \le 2\delta \sup_j |\mu_{k_j}|(\Omega) + \left| \int_\Omega \phi_i \, \mathrm{d}\mu_{k_j} - \int_\Omega \phi_i \, \mathrm{d}\mu \right|$$

for some index $1 \le i \le N(\delta)$. Consequently,

$$\lim_{j\to\infty} \sup_{\varphi\in B} \left| \int_\Omega \varphi \, \mathrm{d}\mu_{k_j} - \int_\Omega \varphi \, \mathrm{d}\mu \right| = 0,$$

and so $\mu_{k_j} \to \mu$ in $W^{-1,q}(\Omega)$.

The notion of a Radon measure can be easily extended to the case when Ω is a locally compact Hausdorff space. For more results about weak convergence of measures and its consequences, we refer to Evans and Gariepy [106], Federer [109], and Rudin [221].

2.4.2 Weak convergence in $L^1(\Omega)$

Let Ω be a bounded open domain in $\mathbb{R}^n$. Let $u_k \in L^1(\Omega)$ $(k \in \mathbb{N})$ and $u \in L^1(\Omega)$ be given functions. The sequence $\{u_k\}_{k=1}^{\infty}$ is said to be weakly convergent to u in $L^1(\Omega)$ if

$$\lim_{k\to\infty} \int_\Omega u_k \varphi \,\mathrm{d}x = \int_\Omega u\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in L^\infty(\Omega).$$

Since $L^1(\Omega)$ cannot be characterized as the dual of some Banach space, the notion of weak-∗ convergence is not interesting in this space. However, as was mentioned earlier, for any bounded sequence $\{u_k\}_{k=1}^{\infty}$ in $L^1(\Omega)$ its cluster points need not be in $L^1(\Omega)$. Indeed, having applied Theorem 2.31 with $\mathrm{d}\mu_k = u_k \,\mathrm{d}x$, we obtain the relative compactness of $\{\mu_k\}_{k=1}^{\infty}$ only in the weak-$\mathcal{M}(\Omega)$ topology. At this point, one can ask under which conditions a bounded sequence in $L^1(\Omega)$ is weakly compact. To answer this question, we need the following definitions.

Definition 2.34. *A sequence $\{u_k\}_{k=1}^{\infty}$ in $L^1(\Omega)$ is said to be equi-integrable if, for any $\eta > 0$, there exists $\delta > 0$ such that*

$$\forall\, k \in \mathbb{N}, \quad \int_E |u_k(x)| \,\mathrm{d}x < \eta, \quad \text{for any } \ E \subset \Omega \ \text{ with } \ |E| < \delta,$$

where $|E|$ stands for the Lebesgue measure of E.

Definition 2.35. *A function $h(t)$ $(t \geq 0)$ is said to be coercive, if it is non-negative, non-decreasing, and satisfies the condition*

$$\lim_{t\to\infty} t^{-1} h(t) \,\mathrm{d}t = +\infty.$$

Then the answer to the above question is as in the following proposition.

Proposition 2.36. *(Dunford–Pettis) The following statements are equivalent:*

(a) The sequence $\{u_k\}_{k=1}^{\infty}$ is weakly compact in $L^1(\Omega)$.
(b) The sequence $\{u_k\}_{k=1}^{\infty}$ is equi-integrable.
(c) There is a coercive function $h(t)$ such that $\sup_{k\in\mathbb{N}} \int_\Omega h(|u_k|) \,\mathrm{d}x < +\infty$.
(d) Given $\delta > 0$, there is $\lambda = \lambda(\delta)$ such that $\sup_{k\in\mathbb{N}} \int_{\{|u_k|>\lambda\}} |u_k| \,\mathrm{d}x < \delta$.

Theorem 2.37. *(Generalized Lebesgue's Theorem) Let* $\{u_k\}_{k=1}^{\infty}$ *be an equi-integrable sequence in* $L^1(\Omega)$ *such that* $u_k \to u$ *almost everywhere on* Ω *as* $k \to \infty$. *Then* $u \in L^1(\Omega)$ *and* $u_k \to u$ *in* $L^1(\Omega)$.

The proof of the above statements can be found in Dunford and Schwartz [99], Ekeland and Temam [104], and Natanson [198].

One of the useful implication of the condition (c) is that for any $u \in L^1(\Omega)$, there exists a coercive function φ such that $\varphi(|u|) \in L^1(\Omega)$. Of course, this result is very simple and can be proved quite easily on the basis of the following observation: For any given numerical series $\sum_{k=1}^{\infty} a_k < +\infty$, where $a_k \geq 0$, there exists a sequence $\lambda_k \to \infty$ such that $\sum_{k=1}^{\infty} \lambda_k a_k < +\infty$.

2.5 Elements of capacity theory

In this section, we give the notion of capacity as a way to study certain "small" subsets of $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $1 < p < +\infty$. The p-capacity of a subset E in Ω is

$$\operatorname{cap}_p(E, \Omega) = \inf \left\{ \int_{\Omega} |\nabla u|^p \, \mathrm{d}x \ : \ u \in \mathcal{U}_E \right\},$$

where $\mathcal{U}_E$ is the set of all functions of the Sobolev space $W_0^{1,p}(\Omega)$ such that $u \geq 1$ almost everywhere in a neighborhood of E.

We say that a property $\mathcal{P}(x)$ holds *quasi everywhere* (abbreviated as q.e.) in a set E if it holds for all $x \in E$ except for a subset N of E with $\operatorname{cap}_p(N, E) = 0$. The expression *almost everywhere* refers, as usual, to the Lebesgue measure.

A subset A of Ω is said to be p-quasi-open if for every $\varepsilon > 0$, there exists an open subset A_ε of Ω, such that $A \subseteq A_\epsilon$ and $\operatorname{cap}_p(A_\varepsilon \setminus A, \Omega) < \epsilon$. The class of all p-quasi-open subsets of Ω we denote by $\mathcal{A}(\Omega)$.

A function $f : \Omega \to \mathbb{R}$ is said to be p-quasi-continuous (resp., quasi-lower semicontinuous) if for every $\varepsilon > 0$, there exists a continuous (resp., lower semicontinuous) function $f_\varepsilon : \Omega \to \mathbb{R}$ such that $\operatorname{cap}_p(\{f \neq f_\epsilon\}, \Omega) < \varepsilon$, where $\{f \neq f_\epsilon\} = \{x \in \Omega \ : \ f(x) \neq f_\epsilon(x)\}$. It is well known that (see, e.g., Ziemer [267]) every function $u \in W_0^{1,p}(\Omega)$ has a p-quasi-continuous representative, which is uniquely defined up to a set of p-capacity 0. We will always identify the function u with its quasi-continuous representative, so that a pointwise condition can be imposed on $u(x)$ for p-quasi-every $x \in \Omega$. Note that with this convention we can write

$$\operatorname{cap}_p(E, \Omega) = \inf \left\{ \int_{\Omega} |\nabla u|^p \, \mathrm{d}x \ : \ u \in W_0^{1,p}(\Omega), u \geq 1 \text{ q.e. on } E \right\} \quad (2.44)$$

for every subset E of Ω.

Since p is fixed, the index p may be dropped when speaking about p-quasi-open sets, p-quasi-continuity, and so forth. It is clear that a set of zero capacity

has zero measure, but the converse is not true. Moreover, when $p > n$, the p-capacity of a point is strictly positive and every $W^{1,p}$-function has a continuous representative. Therefore, a property which holds p-quasi-everywhere, with $p > n$, holds in fact everywhere. We refer to [106, 120] for a review of the main properties of the p-capacity. We recall the following key results.

Theorem 2.38. *Assume $A, B \subset \Omega$. Then the following hold:*

1. $\mathrm{cap}_p(A,\Omega) = \inf\{\mathrm{cap}_p(U,\Omega) \ : \ U \text{ is open}, A \subset U \subseteq \Omega\}$.
2. $\mathrm{cap}_p(\lambda A, \mathbb{R}^n) = \lambda^{n-p}\mathrm{cap}_p(A,\mathbb{R}^n)$, $\lambda > 0$.
3. $\mathrm{cap}_p(B(x,r),\mathbb{R}^n) = r^{n-p}\mathrm{cap}_p(B(0,1),\mathbb{R}^n)$.
4. $\mathrm{cap}_p(A,\Omega) \le C\mathcal{H}^{n-p}(A)$ *for some constant C depending only on p and n.*
5. $\mathcal{L}_n(A) \le C\mathrm{cap}_p(A,\Omega)^{n/(n-p)}$ *for some constant C depending only on p and n.*
6. $\mathrm{cap}_p(A\cup B,\Omega) + \mathrm{cap}_p(A\cap B,\Omega) \le \mathrm{cap}_p(A,\Omega) + \mathrm{cap}_p(B,\Omega)$.

Theorem 2.39. *Let $u \in H^1(\mathbb{R}^n)$. Then for q.e. $x \in \mathbb{R}^n$,*

$$\lim_{\varepsilon\to 0} |B(x,\varepsilon)|^{-1} \int_{B(x,\varepsilon)} u(y)\,\mathrm{d}y = \widetilde{u}(x),$$

where $\widetilde{u}$ is a quasi-continuous representative of u.

Theorem 2.40. *Every strongly converging sequence in $H^1(\mathbb{R}^n)$ has a subsequence converging q.e. in $\mathbb{R}^n$.*

Theorem 2.41. *Let A and Ω be two bounded open subsets of $\mathbb{R}^n$ such that $A \subset \Omega$ and consider an element u of $H_0^1(\Omega)$. Then $u|_A \in H_0^1(A)$ if and only if $\widetilde{u} = 0$ quasi-everywhere on $\Omega \setminus A$, where $\widetilde{u} = 0$ is a quasi-continuous representative of u.*

In view of this, we note that the following two spaces:

$$H_o^1(A;\Omega) = \{\varphi \in H_0^1(\Omega) \ : \ \varphi = 0 \text{ a.e. in } \ \Omega\setminus A\} \text{ and}$$
$$H_0^1(A;\Omega) = \{\varphi \in H_0^1(\Omega) \ : \ \varphi = 0 \text{ q.e. in } \ \Omega\setminus A\}$$

are not equal, in general. Indeed, $\varphi \in H_0^1(A;\Omega)$ cannot be characterized by merely saying that this function and its derivatives are 0 almost everywhere in $\Omega\setminus A$. By definition, a function $\varphi \in H^1(\Omega)$ is said to be 0 quasi-everywhere in a subset E of Ω if there exists a quasi-continuous representative of φ which is 0 quasi-everywhere in E. This makes sense, since any two quasi-continuous representatives of an element φ of $H^1(\Omega)$ are equal quasi-everywhere. So, we have $H_0^1(A;\Omega) \subset H_o^1(A;\Omega)$. However, as can be seen from the following example, in general the reverse inclusion is not true.

Let $B(0,r)$ be the open ball of radius $r > 0$ in $\mathbb{R}^n$. Let us set $\Omega = B(0,3)$ and $A = B(0,2)\setminus \partial B(0,1)$. It is well known that the circular crack $\partial B(0,1)$ in A has nonzero capacity but zero Lebesgue measure $\mathcal{L}_n$. Since $\partial B(0,1)$ has

zero measure, it follows that $H^1_o(A;\Omega)$ contains functions $\psi \in H^1_0(B(0,2))$ whose restriction to $B(0,2)$ are not 0 on the sphere $\partial B(0,1)$. Hence, those functions ψ do not belong to $H^1_0(A,\Omega)$. So, $H^1_o(A;\Omega) \not\subseteq H^1_0(A;\Omega)$. However, if A has a Lipschitz boundary, then the spaces $H^1_o(A;\Omega)$ and $H^1_0(A;\Omega)$ can be identified (see Delfour and Zolésio [91]). In view of this, it is natural to introduce the following terminology (see Rauch and Taylor [217]).

Definition 2.42. *A set A is said to be stable with respect to Ω if $H^1_o(A;\Omega) = H^1_0(A;\Omega)$.*

Following Buttazzo and Dal Maso [45], let us denote by $\mathcal{M}^p_0(\Omega)$ the set of all non-negative Borel measures μ on Ω such that the following hold:

1. $\mu(B) = 0$ for every Borel set $B \subset \Omega$ with $\mathrm{cap}_p(B,\Omega) = 0$.
2. $\mu(B) = \inf\{\mu(U) \ : \ U \ \text{ quasi-open, } \ B \subseteq U\}$ for every Borel set $B \subset \Omega$.

When $p = 2$, we use the notation $\mathcal{M}_0(\Omega)$ instead of $\mathcal{M}^2_0(\Omega)$.

As examples of measures in the class $\mathcal{M}^p_0(\Omega)$ we can quote the following:

(i) $\varphi\,\mathcal{L}_n \in \mathcal{M}^p_0(\Omega)$ for every $\varphi \in L^\infty(\Omega)$, where, as usual, $\mathcal{L}_n$ is the n-dimensional Lebesgue measure.

(ii) If $n-2 < \alpha \le n$, then the α-dimensional Hausdorff measure $\mathcal{H}^\alpha$ belongs to $\mathcal{M}^p_0(\Omega)$. This is a consequence of the two following implications:

$$\mathcal{H}^{n-2}(B) < +\infty \Rightarrow \mathrm{cap}_p(B,\Omega) = 0,$$
$$\mathrm{cap}_p(B,\Omega) = 0 \Rightarrow \mathcal{H}^{n-2+\delta}(B) = 0, \quad \forall \delta > 0.$$

(iii) The measure

$$\infty_S(B) = \begin{cases} 0 & \text{if } \ \mathrm{cap}_p(B \cap S,\Omega) = 0, \\ +\infty & \text{otherwise} \end{cases} \tag{2.45}$$

belongs to $\mathcal{M}^p_0(\Omega)$ for every quasi-closed set S, and so does the measure $\mu_A = \infty_{\Omega\setminus A}$ for every open set $A \subset \Omega$, that is,

$$\mu_A(B) = \begin{cases} 0 & \text{if } \ \mathrm{cap}_p(B \setminus A,\Omega) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\mathcal{B}^*(\Omega)$ be the σ-field generated by the Borel subsets of Ω. It is well known that a subset E of Ω belongs to $\mathcal{B}^*(\Omega)$ if and only if there exists $B \in \mathcal{B}(\Omega)$ with $\mathrm{cap}_p(E \triangle B) = 0$, where $\triangle$ denotes the symmetric difference of sets. Therefore, each measure $\mu \in \mathcal{M}^p_0(\Omega)$ can be extended in a unique way to a countably additive set function, still denoted by μ, defined on the larger σ-field $\mathcal{B}^*(\Omega)$.

We say that $A(\mu)$ is a regular set for the measure $\mu \in \mathcal{M}^p_0(\Omega)$ if $A(\mu)$ is defined as the union of all open subsets A of Ω such that $\mu(A) < +\infty$. The singular set $S(\mu)$ is defined as the complement of $A(\mu)$ in Ω. It is easy to see that $A(\mu)$ is also open, and if A is a open subset of Ω which intersects $S(\mu)$, then $\mu(A) = +\infty$.

2.6 On the space $W_0^{1,p}(\Omega) \cap L^p(\Omega, \mathrm{d}\mu)$ and its properties

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $n \geq 2$. Let us fix $\mu \in \mathcal{M}_0^p(\Omega)$ and denote by $\mathbf{X}_\mu^p(\Omega)$ the vector space of all functions $u \in W_0^{1,p}(\Omega)$ such that $\int_\Omega |u|^p \, \mathrm{d}\mu < +\infty$. Note that this definition makes sense because μ vanishes on all sets of capacity 0 and every function $u \in W_0^{1,p}(\Omega)$ is defined up to a set of capacity 0. So, the integral $\int_\Omega |u|^p \, \mathrm{d}\mu$ is unambiguously defined. On $\mathbf{X}_\mu^p(\Omega)$ we consider the norm

$$\|u\|_{\mathbf{X}_\mu^p(\Omega)} = \left[\int_\Omega |Du|^p \, \mathrm{d}x + \int_\Omega |u|^p \, \mathrm{d}\mu \right]^{1/p}.$$

Theorem 2.43. *$\mathbf{X}_\mu^p(\Omega)$ is a Banach space.*

Proof. Let $\{u_i\}_{i=1}^\infty$ be a Cauchy sequence in $\mathbf{X}_\mu^p(\Omega)$. Then $\{u_i\}_{i=1}^\infty$ is a Cauchy sequence both in $W_0^{1,p}(\Omega)$ and in $L^p(\Omega, \mathrm{d}\mu)$. Therefore, $\{u_i\}_{i=1}^\infty$ converges to a function u in $W_0^{1,p}(\Omega)$ and to a function v in $L^p(\Omega, \mathrm{d}\mu)$. Taking into account the fact that every convergent sequence in $W_0^{1,p}(\Omega)$ is relatively compact with respect to the pointwise convergence q.e. on Ω, we can extract a subsequence $\{u_{i_k}\}_{k=1}^\infty$ converging to u q.e. in Ω. Since μ vanishes on all sets with capacity 0, $\{u_{i_k}\}_{k=1}^\infty$ converges to u μ-a.e. in Ω. On the other hand, a further subsequence of $\{u_{i_k}\}_{k=1}^\infty$ converges to v μ-a.e. in Ω. Hence, $u = v$ μ-a.e. in Ω and, therefore, $u \in \mathbf{X}_\mu^p(\Omega)$ and $\{u_i\}_{i=1}^\infty$ converges to u both in $W_0^{1,p}(\Omega)$ and in $L^p(\Omega, \mathrm{d}\mu)$. This implies that $\{u_i\}_{i=1}^\infty$ converges to u in $\mathbf{X}_\mu^p(\Omega)$. Thus, the normed space $\mathbf{X}_\mu^p(\Omega)$ is complete.

It is clear now that in the case when $p = 2$, $\mathbf{X}_\mu^2(\Omega) = H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu)$ is a Hilbert space with respect to the scalar product,

$$(u, v)_{\mathbf{X}_\mu^2(\Omega)} = \int_\Omega Du Dv \, \mathrm{d}x + \int_\Omega uv \, \mathrm{d}\mu. \tag{2.46}$$

Let us consider now some examples which illustrate the structure of the space $\mathbf{X}_\mu^p(\Omega)$ under some special assumptions on the measure μ.

Example 2.44. Assume that $\mu = g\mathcal{L}_n$ with $g \in L^q(\Omega)$, where

$$\begin{cases} q \in [1, +\infty) & \text{if } \ p \geq n, \\ q \in [1, p^* = pn/(n-p)] & \text{if } \ p < n. \end{cases}$$

By the Sobolev embedding theorem, we have that $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Hence, $\mathbf{X}_\mu^p(\Omega) = W_0^{1,p}(\Omega)$ with equivalent norm.

Example 2.45. Let A be an open subset of Ω and let $S = \Omega \setminus A$. If μ is equal to the measure $\mu = \infty_S + g\mathcal{L}_n$, where $g \in L^q(\Omega)$, q satisfies the conditions of the previous example, and ∞_S is defined by (2.45), then $\mathbf{X}^p_\mu(\Omega) = W^{1,p}_0(A)$, and the norms in $\mathbf{X}^p_\mu(\Omega)$ and $W^{1,p}_0(\Omega)$ are equivalent by the Poincaré inequality.

Consider now a measure $\mu \in \mathcal{M}^p_0(\Omega)$. By $\left(\mathbf{X}^p_\mu(\Omega)\right)^*$ we denote the dual space of $\mathbf{X}^p_\mu(\Omega)$, with duality pairing $\langle \cdot, \cdot \rangle_{(\mathbf{X}^p_\mu(\Omega))^*, \mathbf{X}^p_\mu(\Omega)}$. Note that even in the case when $p = 2$, the space $\mathbf{X}^2_\mu(\Omega)$ is not dense in $L^2(\Omega)$. Therefore, we do not identify the isomorphic spaces $\mathbf{X}^2_\mu(\Omega)$ and $\left(\mathbf{X}^2_\mu(\Omega)\right)^*$. Let us show that the spaces $L^{p'}(\Omega)$, $W^{-1,p'}(\Omega)$, and $L^{p'}(\Omega, \mathrm{d}\mu)$ can be viewed as linear subspaces of $(\mathbf{X}_\mu(\Omega))^*$ (here $p' = p/(p-1)$).

Let $i : \mathbf{X}^p_\mu(\Omega) \to W^{1,p}_0(\Omega)$ be the natural embedding defined by $i(u) = u$ for every $u \in \mathbf{X}^p_\mu(\Omega)$. The transpose map ${}^t i : W^{-1,p'}(\Omega) \to \left(\mathbf{X}^p_\mu(\Omega)\right)^*$ allows us to consider $W^{-1,p'}(\Omega)$ as a subspace of $(\mathbf{X}_\mu(\Omega))^*$. With a little abuse of notation, which is discussed in a moment, we write f instead ${}^t i(f)$ for every $f \in W^{-1,p'}(\Omega)$. With this convention we have

$$\langle f, v \rangle_{(\mathbf{X}^p_\mu(\Omega))^*, \mathbf{X}^p_\mu(\Omega)} = \langle f, v \rangle_{W^{-1,p'}(\Omega), W^{1,p}_0(\Omega)}, \quad \forall\, v \in \mathbf{X}^p_\mu(\Omega). \tag{2.47}$$

In particular, for $f \in L^{p'}(\Omega)$, we have

$$\langle f, v \rangle_{(\mathbf{X}^p_\mu(\Omega))^*, \mathbf{X}^p_\mu(\Omega)} = \int_\Omega f v \, \mathrm{d}x, \quad \forall\, v \in \mathbf{X}^p_\mu(\Omega).$$

The abuse in our notation consists in the fact that the map ${}^t i : W^{-1,p'}(\Omega) \to \left(\mathbf{X}^p_\mu(\Omega)\right)^*$ is, in general, not injective, because $\mathbf{X}^p_\mu(\Omega)$ is, in general, not dense in $W^{1,p}_0(\Omega)$. Therefore, there may exist two elements f and g of $W^{-1,p'}(\Omega)$ such that $f \neq g$ in $W^{-1,p'}(\Omega)$ but $f = g$ in $\left(\mathbf{X}^p_\mu(\Omega)\right)^*$, where the last equality means ${}^t i(f) = {}^t i(g)$, according to our convention (2.47).

Example 2.46. Assume that μ is the measure ∞_Ω defined in (2.45) taking $S = \Omega$. Then $\mathbf{X}^p_\mu(\Omega) = \{0\}$; hence, ${}^t i(f) = 0$ for every $f \in W^{-1,p'}(\Omega)$. Therefore, in view of (2.47), we have $f = 0$ in $\left(\mathbf{X}^p_\mu(\Omega)\right)^*$ for every $f \in W^{-1,p'}(\Omega)$.

Let $j : \mathbf{X}^p_\mu(\Omega) \to L^p(\Omega, \mathrm{d}\mu)$ be the natural embedding defined by $j(u) = u$ for every $u \in \mathbf{X}^p_\mu(\Omega)$. Then the transpose map ${}^t j : L^{p'}(\Omega, \mathrm{d}\mu) \to \left(\mathbf{X}^p_\mu(\Omega)\right)^*$ allows us to consider $L^{p'}(\Omega, \mathrm{d}\mu)$ as a subspace of $\left(\mathbf{X}^p_\mu(\Omega)\right)^*$. For every $g \in L^{p'}(\Omega, \mathrm{d}\mu)$, the image ${}^t j(g)$ is denoted by $g\mu$. With this convention we have

$$\langle g\mu, v \rangle_{(\mathbf{X}^p_\mu(\Omega))^*, \mathbf{X}^p_\mu(\Omega)} = \int_\Omega v g \, \mathrm{d}\mu, \quad \forall\, v \in \mathbf{X}^p_\mu(\Omega). \tag{2.48}$$

Since $\mathbf{X}^p_\mu(\Omega)$ is, in general, not dense in $L^p(\Omega, \mathrm{d}\mu)$, the map ${}^t j : L^{p'}(\Omega, \mathrm{d}\mu) \to \left(\mathbf{X}^p_\mu(\Omega)\right)^*$ is, in general, not injective. Therefore, there may exist two elements f and g of $L^{p'}(\Omega, \mathrm{d}\mu)$ such that $f \neq g$ in $L^{p'}(\Omega, \mathrm{d}\mu)$, that is,

$$\mu(\{x \in \Omega \ : \ f(x) \neq g(x)\}) > 0,$$

but $f\mu = g\mu$ in $\mathbf{X}^p_\mu(\Omega)$.

Example 2.47. Let E be the set of all points $x = (x_1, x_2, \dots, x_n)$ in Ω whose first coordinate x_1 is rational. Let $\mu = \infty_E + \mathcal{L}_n$. Then it is clear that $\mathbf{X}^p_\mu(\Omega) = \{0\}$. Therefore, taking $g = \chi_{\Omega\setminus E}$, where $\chi_{\Omega\setminus E}$ is the characteristic function of the set $\Omega \setminus E$, we have $g \in L^{p'}(\Omega, \mathrm{d}\mu)$ and $g \neq 0$ in $L^{p'}(\Omega, \mathrm{d}\mu)$, whereas $g\mu = {}^t j(g) = 0$ in $\left(\mathbf{X}^p_\mu(\Omega)\right)^*$.

Let us fix a measure $\mu \in \mathcal{M}^2_0(\Omega)$. Then by the Riesz–Fréchet representation theorem, for every $F \in \left(\mathbf{X}^2_\mu(\Omega)\right)^*$, there exists a unique $u \in \mathbf{X}^2_\mu(\Omega)$ such that

$$\langle F, v\rangle_{\left(\mathbf{X}^2_\mu(\Omega)\right)^*, \mathbf{X}^2_\mu(\Omega)} = (u, v)_{\mathbf{X}^2_\mu(\Omega)}, \quad \forall\, v \in \mathbf{X}^2_\mu(\Omega). \tag{2.49}$$

By definition (2.46) of the scalar product in $\mathbf{X}^2_\mu(\Omega)$, (2.49) is equivalent to

$$\int_\Omega DuDv \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}\mu = \langle F, v\rangle_{\left(\mathbf{X}^2_\mu(\Omega)\right)^*, \mathbf{X}^2_\mu(\Omega)}, \quad \forall\, v \in \mathbf{X}^2_\mu(\Omega).$$

However, according to our conventions (2.47) and (2.48), the last equality can be written in the form

$$\begin{aligned}\langle -\triangle u, v\rangle_{\left(\mathbf{X}^2_\mu(\Omega)\right)^*, \mathbf{X}^2_\mu(\Omega)} + \langle u\mu, v\rangle_{\left(\mathbf{X}^2_\mu(\Omega)\right)^*, \mathbf{X}^2_\mu(\Omega)} \\ = \langle F, v\rangle_{\left(\mathbf{X}^2_\mu(\Omega)\right)^*, \mathbf{X}^2_\mu(\Omega)}, \quad \forall\, v \in \mathbf{X}^2_\mu(\Omega).\end{aligned} \tag{2.50}$$

This shows that each element F of $\left(\mathbf{X}^2_\mu(\Omega)\right)^*$ can be represented as $F = f + g\mu$ with $f \in H^{-1}(\Omega)$ and $g \in L^2(\Omega, \mathrm{d}\mu)$. On the other hand, because of (2.50), we refer to the solution of (2.49) as the solution of the problem

$$u \in \mathbf{X}^2_\mu(\Omega), \quad -\triangle u + u\mu = F \quad \text{in} \quad \left(\mathbf{X}^2_\mu(\Omega)\right)^*.$$

2.7 Sobolev spaces with respect to a measure

Let Ω be an open domain in $\mathbb{R}^n$ and let μ be a finite positive (e.g., probability) measure of $\mathcal{M}^p_0(\Omega)$. We introduce the Sobolev space $W^{1,p}(\Omega, \mathrm{d}\mu)$ as follows.

Definition 2.48. *We say that a function u belongs to $W^{1,p}(\Omega, \mathrm{d}\mu)$ if there exist a sequence $\left\{u_k \in C^\infty(\overline{\Omega})\right\}_{k=1}^\infty$ and a vector-function $\mathbf{z} \in \mathbf{L}^p(\Omega, \mathrm{d}\mu) := [L^p(\Omega, \mathrm{d}\mu)]^n$ such that*

$$u_k \to u \ \text{ in } \ L^p(\Omega, \mathrm{d}\mu) \ \text{ and } \ \nabla u_k \to \mathbf{z} \ \text{ in } \ \mathbf{L}^p(\Omega, \mathrm{d}\mu). \tag{2.51}$$

In this case we say that $\mathbf{z}$ is a gradient or μ-gradient of u and denote it by $\nabla^\mu u$.

In other words, to define the space $W^{1,p}(\Omega, \mathrm{d}\mu)$, we construct the space $W = W(\Omega, \mathrm{d}\mu)$ as the closure in $L^p(\Omega, \mathrm{d}\mu) \times \mathbf{L}^p(\Omega, \mathrm{d}\mu)$ of the set of pairs $\big\{(u, \nabla u) \ : \ \forall\, u \in C^\infty(\overline{\Omega})\big\}$. Thus, the elements of W are pairs $(u, \mathbf{z})$, where the vector $\mathbf{z}$ is denoted by $\nabla^\mu u$, and said to be a gradient of u. As a result, the collection of the first components u is called the Sobolev space $W^{1,p}(\Omega, \mathrm{d}\mu)$. Note that if we put in (2.51) functions $u \in C_0^\infty(\Omega)$, we just obtain the definition of the Sobolev space $W_0^{1,p}(\Omega, \mathrm{d}\mu)$. Note also that we do not introduce a norm in these spaces. If $p = 2$, we usually write $H^1(\Omega, \mathrm{d}\mu) = W^{1,2}(\Omega, \mathrm{d}\mu)$.

Remark 2.49. In the above definition, the strong convergence in $L^p(\Omega, \mathrm{d}\mu)$ and $\mathbf{L}^p(\Omega, \mathrm{d}\mu)$ can be replaced by the weak convergence in the same spaces.

In general, the gradient of a $W^{1,p}(\Omega, \mathrm{d}\mu)$ function is not unique since a function u in $W^{1,p}(\Omega, \mathrm{d}\mu)$ can have many gradients. Let us denote by $\Gamma^\mu(u)$ the set of all gradients of a fixed function $u \in W^{1,p}(\Omega, \mathrm{d}\mu)$. It is clear that $\Gamma^\mu(u)$ has the structure $\Gamma^\mu(u) = \nabla^\mu u + \Gamma^\mu(0)$, where $\nabla^\mu u$ is some gradient and $\Gamma^\mu(0)$ is the set of gradients of 0. By definition, $\mathbf{z} \in \Gamma^\mu(0)$ if there exist $u_k \in C^\infty(\overline{\Omega})$ $(k = 1, 2, \dots)$ such that

$$\lim_{k\to\infty} \int_\Omega |u_k|^p \, \mathrm{d}\mu = 0 \quad \text{and} \quad \lim_{k\to\infty} \int_\Omega |\nabla u_k - \mathbf{z}|^p_{\mathbb{R}^n} \ \mathrm{d}\mu = 0.$$

Obviously, $\Gamma^\mu(0)$ is a closed subspace of the vector space $\mathbf{L}^p(\Omega, \mathrm{d}\mu)$. So, the gradient of an arbitrary $W^{1,p}(\Omega, \mathrm{d}\mu)$ function can be viewed as the corresponding equivalence class. As an illustration of nonuniqueness of the gradient, we consider the following example.

Example 2.50. **The case of a singular measure μ concentrated on the segment**. Let $I = \{x \mid a \le x_1 \le b; x_2 = 0\}$ be a segment in $\mathbb{R}^2$ and suppose that a bounded domain $\Omega \subset \mathbb{R}^2$ contains I. Let μ be a probability measure concentrated on this segment, uniformly distributed on it, and coinciding with 1D Lebesgue measure on I – namely we set

$$\mathrm{d}\mu = \frac{1}{b-a}\chi(x_1)\, \mathrm{d}x_1 \times \delta(x_2), \tag{2.52}$$

where $\chi(t)$ is the characteristic function of the segment $[a, b]$ and $\delta(t)$ is the Dirac mass concentrated at 0. It is clear that μ is a singular measure with respect to $\mathcal{L}_2$, and, by definition of the class $\mathcal{M}_0(\Omega)$, we have $\mu \in \mathcal{M}_0(\Omega)$. Note also that $\mu(\Omega \setminus I) = 0$. Therefore, any functions taking the same values on the segment I coincide as elements of $L^2(\Omega, \mathrm{d}\mu)$.

By Definition 2.48, a function u is an element of $H^1(\Omega, \mathrm{d}\mu)$ if there are a sequence of smooth functions $u_k \in C^\infty(\overline{\Omega})$ and $\mathbf{z} = (z_1, z_2) \in \mathbf{L}^2(\Omega, \mathrm{d}\mu)$ such that

$$\int_I |u - u_k|^2 \, \mathrm{d}x_1 \to 0, \ \int_I \left|\frac{\partial u_k}{\partial x_1} - z_1\right|^2 \mathrm{d}x_1 \to 0, \ \int_I \left|\frac{\partial u_k}{\partial x_2} - z_2\right|^2 \mathrm{d}x_1 \to 0.$$

Thus, due to (2.51) each element of $H^1(\Omega, \mathrm{d}\mu)$ is uniquely defined by the respective element of the 1D Sobolev space $H^1([a,b])$ and $z_1 = \partial u/\partial x_1$. Thus, $\nabla^\mu u = (\partial u/\partial x_1, z_2)$. We now show that the component z_2 can be an arbitrary element of $L^2(I)$. In other words,

$$\Gamma^\mu(0) = \{(0,\alpha)\}, \quad \alpha \in L^2(I) = L^2(\Omega, \mathrm{d}\mu).$$

Indeed, since $\Gamma^\mu(0)$ is closed in $\mathbf{L}^2(\Omega, \mathrm{d}\mu)$ and $C^\infty(\overline{I})$ is dense in $L^2(I)$, it is sufficient to verify that $(0,\alpha) \in \Gamma^\mu(0)$ for $\alpha \in C^\infty(\overline{I})$. To do so, we set $u_k(x_1,x_2) = x_2\alpha(x_1)$. Then $u_k \to 0$ strongly in $L^2(\Omega, \mathrm{d}\mu)$ as $k \to \infty$ and, moreover,

$$\left.\frac{\partial u_k}{\partial x_1}\right|_{x_2=0} = 0, \quad \left.\frac{\partial u_k}{\partial x_2}\right|_{x_2=0} = \alpha(x_1).$$

Hence, the required conclusion is obtained. To conclude this example, we note that a Borel function $u = u(x_1,x_2)$ belongs to the space $H^1(\Omega, \mathrm{d}\mu)$ if and only if $u \in H^1(\Omega)$ and the restriction (trace) of u to the segment I is an H^1 function of a single variable. Note also that the trace of a function in $H^1(\Omega)$ is defined on I and is an element of the space $H^{1/2}(I)$, in general!

Let us consider several examples of the Sobolev spaces $H^1(\Omega, \mathrm{d}\mu)$.

Example 2.51. **Network node**. Consider the segments $I_1, I_2, \dots, I_N$ starting at the origin and directed along vectors $v_1, v_2, \dots, v_N$. Suppose that $v_i/|v_i| \neq v_j/|v_j|$ for $i \neq j$, and a bounded open domain $\Omega \subset \mathbb{R}^2$ contains this star structure. Let $\mu_1, \mu_2, \dots, \mu_N$ be the 1D measures on the segments $I_1, I_2, \dots, I_N$, respectively (see, for instance, (2.52)). Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be arbitrary positive numbers. We set

$$\mu = \sum_{i=1}^{N} \lambda_i \mu_i.$$

Then it is easy to verify that a Borel function $u = u(x_1,x_2)$ belongs to the space $H^1(\Omega, \mathrm{d}\mu)$ if and only if its restriction to each segment I_j is an H^1 function of a single variable and the values of the restricted functions at the origin coincide for all segments (recall that, by the Sobolev embedding theorem, an H^1 function of a single variable is continuous).

Example 2.52. **Junction**. Let $\Omega = (-1,1)^3$, $G = \left[-\frac{1}{8}, \frac{1}{8}\right]^3$,

$$\Pi = \left\{ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^3 \ : \ x_3 = 0, (x_1,x_2) \in \left[0, \frac{3}{8}\right] \times \left[-\frac{1}{8}, \frac{1}{8}\right] \right\},$$

$$I = \left\{ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^3 \ : \ x_2 = x_3 = 0, x_1 \in \left[\frac{1}{4}, \frac{1}{2}\right] \right\},$$

μ_1 is the standard Lebesgue measure on the segment I, μ_2 is the planar Lebesgue measure on Π, and $\mathrm{d}x$ is the spatial Lebesgue measure in $\mathbb{R}^3$ restricted to the cube G. Introduce the measure on Ω as follows:

$$\mathrm{d}\mu = \mathrm{d}\mu_1 + \mathrm{d}\mu_2 + \mathrm{d}x.$$

Note that the trace of an $H^1(\Omega)$ function is defined on Π and is an element of the space $H^{1/2}(\Pi)$. The trace of an $H^1(\Pi)$ function on the segment I is also well defined. However, unlike the 2D case, in the case of dimension 3, the traces of an $H^1(\Omega)$ function on 1D segments are not defined. Therefore, we have to use the following obvious fact: The space $H^1(\Omega, \mathrm{d}\mu)$ is isomorphic to the direct sum of the spaces $H^1(\Omega, \chi_G \mathrm{d}x)$ and $H^1(\Omega, \mathrm{d}\mu_1 + \mathrm{d}\mu_2)$. Thus, the function u belongs to $H^1(\Omega, \mathrm{d}\mu)$ if $u = \widehat{u} + \widetilde{u}$, where $\widehat{u} \in H^1(G)$, $\widetilde{u} \in H^1(\Pi) \cap H^1(I, \mathrm{d}\mu_2)$, and $\widetilde{u}|_I$ is an element of $H^1(I)$. For the other properties of the space $H^1(\Omega, \mathrm{d}\mu)$, we refer to [58, 59].

We now describe some properties of the subspace $\Gamma^\mu(0)$.

(i) If $\mathbf{g} \in \Gamma^\mu(0)$ and $a \in L^\infty(\Omega, \mathrm{d}\mu)$, then $a\mathbf{g} \in \Gamma^\mu(0)$.
To see this, it suffices to consider the case $a \in C^\infty(\overline{\Omega})$. Then if follows from the definition of $\Gamma^\mu(0)$ that

$$a u_k \overset{L^2(\Omega,\mathrm{d}\mu)}{\longrightarrow} 0, \ \nabla^\mu(a u_k) = \nabla^\mu a \, u_k + a \nabla u_k \overset{\mathbf{L}^2(\Omega,\mathrm{d}\mu)}{\longrightarrow} a\mathbf{g}.$$

Hence, $a\mathbf{g} \in \Gamma^\mu(0)$.

(ii) Let Π be the orthogonal projection of $\mathbf{L}^2(\Omega, \mathrm{d}\mu)$ onto $\Gamma^\mu(0)$. Then

$$\Pi(a\mathbf{g}) = a\Pi(\mathbf{g}), \quad \forall\, \mathbf{g} \in \mathbf{L}^2(\Omega, \mathrm{d}\mu), \ a \in L^\infty(\Omega, \mathrm{d}\mu).$$

Indeed, setting $\widehat{\mathbf{g}} = \Pi(\mathbf{g})$, we obtain $\widehat{\mathbf{g}} \in \Gamma^\mu(0)$ and $\mathbf{g} - \widehat{\mathbf{g}} \perp \Gamma^\mu(0)$. Then, by property (i), we have

$$a\widehat{\mathbf{g}} \in \Gamma^\mu(0), \quad (a(\mathbf{g} - \widehat{\mathbf{g}}), h) = 0, \ \forall\, h \in \Gamma^\mu(0),$$

as required.

(iii) The set $\Gamma^\mu(0) \cap \mathbf{L}^\infty(\Omega, \mathrm{d}\mu)$ is dense in $\Gamma^\mu(0)$. In fact, if a_k is the characteristic function of the set $\{x \in \Omega \ : \ |\mathbf{g}| \le k\}$, then $a_k\mathbf{g} \in \Gamma^\mu(0)$ and $a_k\mathbf{g} \to \mathbf{g}$ as $k \to \infty$.

(iv) There exists a μ-measurable subspace $D(x) \subset \mathbb{R}^n$ such that

$$\Gamma^\mu(0) = \left\{\mathbf{g} \in \mathbf{L}^2(\Omega, \mathrm{d}\mu) \ : \ \mathbf{g}(x) \in D(x)\right\}. \tag{2.53}$$

Indeed, let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the natural basis of $\mathbb{R}^n$. We set $\boldsymbol{\xi}_i = \Pi\mathbf{e}_i$ and define $D(x)$ as the linear span of the vectors $\{\boldsymbol{\xi}_1(x), \ldots, \boldsymbol{\xi}_n(x)\}$. We denote by B the subspace of $\mathbf{L}^2(\Omega, \mathrm{d}\mu)$ defined by the right-hand side of (2.53) and show that $\Gamma^\mu(0) = B$.
Let $\mathbf{g} \in \Gamma^\mu(0) \cap \mathbf{L}^\infty(\Omega, \mathrm{d}\mu)$. Then, by property (ii), we obtain

$$\mathbf{g} = g_1\mathbf{e}_1 + \cdots + g_n\mathbf{e}_n = \Pi g = g_1\boldsymbol{\xi}_1 + \cdots + g_n\boldsymbol{\xi}_n \in B.$$

In view of property (iii) and the fact that B is closed, we have the inclusion $\Gamma^\mu(0) \subseteq B$. To verify the reverse inclusion $B \subseteq \Gamma^\mu(0)$, we note that

$$a(x)\sum_{i=1}^{n}\lambda_i\boldsymbol{\xi}_i \in \Gamma^\mu(0), \quad \forall\, a \in L^\infty(\Omega, \mathrm{d}\mu),\ \forall\, \lambda_i \in \mathbb{R}^1$$

because of properties (i) and (ii) and the fact that $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n \in \Gamma^\mu(0)$. The set of such elements is dense in B. Let us assume that there exists a $\mathbf{b} \in B$ such that $\mathbf{b} \neq 0$ and

$$\left(a\mathbf{b}, \sum_{i=1}^{n}\lambda_i\boldsymbol{\xi}_i\right)_{\mathbf{L}^2(\Omega,\mathrm{d}\mu)} = 0.$$

Since a is an arbitrary function, it follows that

$$\left(\mathbf{b}(x), \sum_{i=1}^{n}\lambda_i\boldsymbol{\xi}_i(x)\right)_{\mathbb{R}^n} = 0$$

for μ-almost all $x \in \Omega$, which means that $b(x) = 0$. The proof is complete.

Definition 2.53. *We say that a gradient $\nabla^\mu u$ is tangential for $u \in H^1(\Omega, \mathrm{d}\mu)$ if $\nabla^\mu u \perp \Gamma^\mu(0)$ (or in the equivalent form, if $\nabla^\mu u(x) \in T(x)$ for μ-almost all $x \in \Omega$, where $T(x) = (D(x))^\perp$).*

It is clear that each function in the Sobolev space $H^1(\Omega, \mathrm{d}\mu)$ has a unique tangential gradient. It is also obvious that if $\nabla^\mu u(x)$ is some gradient of u and $P(x)$ is the orthogonal projection $\mathbb{R}^n \to T(x)$, then $P(x)\nabla^\mu u(x)$ is the tangential gradient. Combining these results, we come to the following conclusion (see [256]): *There exists a μ-measurable subspace $T(x)$ such that the set of gradients of each function in $H^1(\Omega, \mathrm{d}\mu)$ has the representation $\nabla^\mu u(x) + \mathbf{g}(x)$, where $\nabla^\mu u(x) \in T(x)$ and $\mathbf{g}$ is an arbitrary vector in $\mathbf{L}^2(\Omega, \mathrm{d}\mu)$ such that $\mathbf{g}(x) \in T^\perp(x)$.* The subspace $T(x)$ is called the tangential space at the point x and $\nabla^\mu u$ is called the tangential gradient. For more results concerning the Sobolev spaces with respect to a measure and their applications, we refer to Chechkin, Zhikov, Lukkassen and Piatnitski [58], Bouchitté, Buttazzo and Seppecher [25], and Fragalà and Mantegazza [110].

2.8 Boundary value problems in Sobolev spaces with measures

The asymptotic behavior of thin and reticulated structures such as, for example, shells, plates, thin films, rod structures, skeletons, and so on is widely

discussed in the mathematical and engineering literatures. As a rule, the goal of an asymptotic analysis is to reduce dimension (i.e., to reduce the original problem to a problem on some structure of smaller dimension). For instance, the equation on a thin 3D plate is replaced with an equation on a 2D domain [257], the system describing a rod construction is reduced to a family of ordinary differential equations [265], and so on. The reduced system is usually simpler, especially from the point of view of numerical analysis.

Asymptotic methods for such problems are well known in the literature, but, as a rule, they are derived under strict restrictions on the geometry and smoothness of the structure (see, for instance, [14, 22, 66, 200, 209, 216]). There are numerous works in mechanics where asymptotics for different special models were obtained at the physical level of rigor.

From mathematical point of view, it is of great interest to construct a general approach to the asymptotic analysis of different classes of boundary value problems on thin and reticulated structures, that would be associated with the corresponding periodic measures, singular measures, partially singular measures, as well as measures converging to singular ones. A successful attempt to create such a theory was made in [25, 26, 254, 256, 257, 265]. In this section, following these works, we define the meaning of boundary value problems in spaces with measures and clarify the idea of Sobolev spaces with an arbitrary measure. The main motivation of our intention can be clarified by a simple example. Let μ be a positive finite Borel measure defined on a smooth bounded domain Ω. We have the following variational problem:

$$\inf_{\varphi\in C_0^\infty(\Omega)} \int_\Omega \Big(A(x)\nabla\varphi(x)\cdot\nabla\varphi(x) + \varphi^2(x) - 2f(x)\varphi(x)\Big)\,\mathrm{d}\mu(x),$$

where $A \in C(\overline{\Omega}, \mathbb{R}^{n\times n})$ is a coercive matrix of positive elements and $f \in C(\overline{\Omega})$ is a given function. Our aim is to describe a minimizer to this problem as an element of an appropriate Sobolev space with respect to the measure μ and associate it with a solution to the corresponding Euler equation.

Throughout this section, we assume that Ω is an open domain in $\mathbb{R}^n$ and μ is a periodic Borel (e.g., probability) measure of $\mathcal{M}_0^p(\Omega)$ such that $\int_\square \mathrm{d}\mu = 1$, where $\square = [0,1)^n$ is the cell (or the torus) of periodicity for μ. Let $H^1_{per}(\square, \mathrm{d}\mu) = W^{1,2}_{per}(\square, \mathrm{d}\mu)$ be the periodic Sobolev space with respect to the measure μ. Since μ can be identified with the corresponding periodic measure in $\mathbb{R}^n$, we will also make use the Sobolev space $H^1(\mathbb{R}^n, \mathrm{d}\mu)$.

Let $A = [a_{ij}(x)]_{i,j=1,\dots,n}$ be a continuous function with values in the space of symmetric $n\times n$ matrices satisfying the uniform ellipticity condition

$$\alpha|\xi|^2_{\mathbb{R}^n} \le (A(x)\xi,\xi)_{\mathbb{R}^n} \le \alpha^{-1}|\xi|^2_{\mathbb{R}^n}, \quad \alpha > 0,\ \xi\in\mathbb{R}^n \tag{2.54}$$

for all $x \in \mathbb{R}^n$. Let f be a given element of $L^2(\mathbb{R}^n, \mathrm{d}\mu)$ and let $\lambda > 0$.

To begin with, we define the notion of divergence with respect to the measure.

Definition 2.54. *Suppose that* $g \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$ *and* $\mathbf{v} \in \left[L^2(\mathbb{R}^n, \mathrm{d}\mu)\right]^n$. *We say that* $g(x) = \operatorname{div}^\mu \mathbf{v}(x)$ *if*

$$\int_{\mathbb{R}^n} g(x)\varphi(x)\,\mathrm{d}\mu(x) = -\int_{\mathbb{R}^n} \mathbf{v}(x)\cdot\nabla\varphi(x)\,\mathrm{d}\mu(x), \quad \forall\,\varphi \in C_0^\infty(\Omega).$$

It is easy to see that, in this definition, instead of smooth functions φ one can take functions $\varphi \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$.

Definition 2.55. *We say that a pair* $(u, \nabla^\mu u)$, *where* $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ *and* $\nabla^\mu u$ *is a* μ*-gradient of* u, *satisfies the equation*

$$-\operatorname{div}^\mu\Big(A(x)\nabla^\mu u\Big) + \lambda u = f \tag{2.55}$$

in $L^2(\mathbb{R}^n, \mathrm{d}\mu)$ *if for any* $v \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ *and any gradient* $\nabla^\mu v$ *of* v, *we have*

$$\int_{\mathbb{R}^n} A(x)\nabla^\mu u\cdot\nabla^\mu v\,\mathrm{d}\mu + \lambda\int_{\mathbb{R}^n} uv\,\mathrm{d}\mu = \int_{\mathbb{R}^n} fv\,\mathrm{d}\mu. \tag{2.56}$$

Definition 2.56. *A function* $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ *is called a solution to (2.55) if the integral identity (2.56) holds for some of the gradients of* u *and for any* $v \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ *and any gradient* $\nabla^\mu v$ *of* v.

Note that in Definitions 2.55 and 2.56, instead of functions $v \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$, one can take the test functions v in $C_0^\infty(\mathbb{R}^n)$.

Remark 2.57. In the special case when the matrix $A(x)$ in (2.55) is identity, relation (2.56) takes the form

$$\int_{\mathbb{R}^n} \nabla^\mu u\cdot\nabla^\mu v\,\mathrm{d}\mu + \lambda\int_{\mathbb{R}^n} uv\,\mathrm{d}\mu = \int_{\mathbb{R}^n} fv\,\mathrm{d}\mu.$$

Then the corresponding expression $\operatorname{div}^\mu\nabla^\mu u$ is usually denoted by $\Delta_\mu u$ and called the μ-Laplacian of u.

The main result of this section can be formulated as follows:

Lemma 2.58. *Let* $A \in C(\overline{\Omega}, \mathbb{R}^{n\times n})$ *be a symmetric matrix satisfying conditions (2.54). Then for every* $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$, *(2.55) has a unique solution* $(u, \nabla^\mu u)$, $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$. *Moreover, the choice of a* μ*-gradient of* u *is uniquely determined by the condition of orthogonality of the vector* $A(x)\nabla^\mu u$ *and the subspace* $\Gamma_\mu(0)$ *of the gradients of* 0.

Proof. Since the matrix $A = A(x)$ is positive definite, the left-hand side of (2.56) is the inner product in $W^2(\mathbb{R}^n, \mathrm{d}\mu)$ (see the definition of the space $W^2(\mathbb{R}^n, \mathrm{d}\mu)$ in Sects. 2.7 and 6.4), whereas the right-hand side of (2.56) is a continuous linear functional on $W^2(\mathbb{R}^n, \mathrm{d}\mu)$. By the Riesz representation theorem, there exists a pair $(u, \nabla^\mu u) \in W^2(\mathbb{R}^n, \mathrm{d}\mu)$ satisfying relation (2.56). Taking for a test function in (2.56) the pair $(0, \mathbf{z})$, where $\mathbf{z} \in \Gamma_\mu(0)$, we see

that the vector $A(x)\nabla^\mu u$ is orthogonal to $\Gamma^\mu(0)$. Using the Riesz theorem again, we conclude that there is a unique element $\mathbf{z}$ of the set of μ-gradients of u satisfying the orthogonality condition. Thus, the solution $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ to (2.55) is unique. □

Remark 2.59. By Lemma 2.58, the gradient $\nabla^\mu u$ chosen by a solution u to the problem (2.55) satisfies the condition $\nabla^\mu u \perp \Gamma_\mu(0)$. Hence, in view of Definition 2.53, $\nabla^\mu u$ is the tangential gradient for a function $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$. Moreover, in the case when the measure μ is defined as in Example 2.51, the orthogonality $A(x)\nabla^\mu u|_{x=0} \perp \Gamma_\mu(0)$ is equivalent to the classical Kirchoff condition.

Let us apply Lemma 2.58 to establish some additional properties of the solutions to the elliptic equation (2.55).

Lemma 2.60. *Under the assumptions of Lemma 2.58, the set of solutions to (2.55), for all $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$, is dense in the space $L^2(\mathbb{R}^n, \mathrm{d}\mu)$.*

Proof. Denote by $\mathcal{D}$ the set of solutions to the (2.55) when f runs over the entire space $L^2(\mathbb{R}^n, \mathrm{d}\mu)$. Assume that there is a nontrivial element $g \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$ which is orthogonal to $\mathcal{D}$. For a fixed $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$, we denote by $u^f \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ the corresponding solution to (2.55). Then $u^g \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ is a solution to the equation

$$-\mathrm{div}^\mu\Big(A(x)\nabla^\mu u\Big) + \lambda u = g.$$

Taking u^g as a test function in (2.56) and u^f for a test function in the last equation and taking the difference of the obtained integral identities, we come to the relation

$$\int_{\mathbb{R}^n} f u^g \,\mathrm{d}\mu = 0, \quad \forall\, f \in L^2(\mathbb{R}^n, \mathrm{d}\mu).$$

Hence, $u^g = 0$ and $g = 0$, which contradicts the assumption that g is nontrivial. The proof is complete. □

Further, we note that (2.55) can be written in the operator form $\mathcal{A}\,u + \lambda u = f$, where $\mathcal{A}$ is a self-adjoint symmetric operator. Indeed, let u^f be any element of $\mathcal{D}$ (i.e., u^f is a solution to (2.55) for some $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$). We set

$$\mathcal{A}\,u^f = f - \lambda u^f.$$

Then, in view of Lemma 2.58, the operator $(\mathcal{A} + \lambda I)^{-1}$ sends a function $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$ to the corresponding unique solution u^f of (2.55). Since this operator is nonnegative, bounded, and symmetric, we come to the required conclusion.

The following assertion can be proved in the same way as in the case of classical variational problems in the space $H^1(\mathbb{R}^n)$ (see, for instance, [169]).

Proposition 2.61. *Let $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$ and let $A \in C(\overline{\Omega}, \mathbb{R}^{n\times n})$ be a symmetric matrix satisfying conditions (2.54). Then for every $\lambda > 0$, the variational problem*

$$\inf_{\varphi\in H^1(\mathbb{R}^n,\mathrm{d}\mu)} \left\{ \int_{\mathbb{R}^n} \Big(A(x)\nabla^\mu u \cdot \nabla^\mu u + \lambda u^2 \Big)\, \mathrm{d}\mu - \int_{\mathbb{R}^n} 2fu\, \mathrm{d}\mu \right\} \tag{2.57}$$

has a unique minimum point in $H^1(\mathbb{R}^n, \mathrm{d}\mu)$ and it is a solution to (2.55).

Thus, (2.55) or the equivalent equation $\mathcal{A}\, u + \lambda u = f$ is an Euler equation for the variation problem (2.57).

The developed technique allows us to study not only problems in the entire space $\mathbb{R}^n$ but also various boundary value problems. We illustrate this by an example of a Dirichlet problem (we consider other types of boundary problems in the second part of this book).

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let μ be a positive finite Borel measure on Ω.

Definition 2.62. *We say that a function $u \in L^2(\Omega, \mathrm{d}\mu)$ belongs to the space $H^1_0(\Omega, \mathrm{d}\mu)$ and $\mathbf{z} \in \mathbf{L}^2(\Omega, \mathrm{d}\mu) = \big[L^2(\Omega, \mathrm{d}\mu)\big]^n$ is a μ-gradient of u if there exists a sequence $\{u_k \in C_0^\infty(\Omega)\}_{k=1}^\infty$ such that*

$$u_k \to u \quad \text{in} \quad L^2(\Omega, \mathrm{d}\mu) \quad \text{as} \quad k \to \infty,$$
$$\nabla u_k \to \mathbf{z} \quad \text{in} \quad \mathbf{L}^2(\Omega, \mathrm{d}\mu) \quad \text{as} \quad k \to \infty.$$

Consider the Dirichlet problem

$$-\mathrm{div}^\mu \Big(A(x)\nabla^\mu u \Big) + \lambda u = f \quad \text{in} \quad L^2(\Omega, \mathrm{d}\mu), \tag{2.58}$$
$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{2.59}$$

As before, we assume that the matrix $A \in C(\overline{\Omega}, \mathbb{R}^{n\times n})$ is symmetric and satisfies the uniform ellipticity condition in Ω like (2.54).

Definition 2.63. *We say that $u \in H^1_0(\Omega, \mathrm{d}\mu)$ is a solution to the Dirichlet problem (2.58)–(2.59) if the integral identity*

$$\int_\Omega A(x)\nabla^\mu u \cdot \nabla^\mu v\, \mathrm{d}\mu + \lambda \int_\Omega uv\, \mathrm{d}\mu = \int_\Omega fv\, \mathrm{d}\mu$$

holds true for any $v \in H^1_0(\Omega, \mathrm{d}\mu)$.

The existence and uniqueness of a solution to the problem (2.58)–(2.59) can be established in the same way as in the proof of Lemma 2.58. We recall that a gradient of the solution that satisfies the above integral identity is chosen in a unique way.

2.9 On weak compactness of a class of bounded sets in Banach spaces

Let $\mathbb{X}$ be a Banach space and $\mathbb{X}^*$ its dual. Let us recall that the weak-$*$ topology of $\mathbb{X}^*$ is the locally convex topology $\sigma(\mathbb{X}^*, \mathbb{X})$ in which the linear functionals

$$X^* \ni f \mapsto \langle f, u\rangle_{\mathbb{X}^*,\mathbb{X}}\,, \quad u \in \mathbb{X}$$

are continuous.

Due to the Banach–Alaoglu theorem, it is well known that any bounded weakly-$*$ closed subset $B \subset \mathbb{X}^*$ is compact with respect to the weak-$*$ topology of $\mathbb{X}^*$. However, it is hard to verify the property of weak-$*$ closure for sets which differ from a ball, even for relatively simple sets $B \subset \mathbb{X}^*$. Therefore, the main goal of this section is to study the compactness property for other types of subsets in $\mathbb{X}^*$ (see, for instance, [151, 152]).

Let $\mathbb{X}$ and $\{\mathbb{Z}_j\}_{j=1}^k$ be Banach spaces and let $\mathbb{X}^*$ and $\{\mathbb{Z}_j^*\}_{j=1}^k$ be their duals, respectively. Let

$$\{\Lambda_j : (D(\Lambda_j) \subset \mathbb{Z}_j) \to \mathbb{X}\}_{j=1}^k$$

be linear mappings defined on the domains $D(\Lambda_j)$ which are dense subsets of the corresponding spaces $\mathbb{Z}_j$. Let

$$\{\Lambda_j^* : (D(\Lambda_j^*) \subset \mathbb{X}^*) \to \mathbb{Z}_j^*\}_{j=1}^k$$

be their dual mappings, respectively. We set

$$\mathcal{D}^* = \bigcap_{j=1}^k D(\Lambda_j^*)$$

and endow this set with the graph norm

$$\|y^*\|_* = \|y^*\|_{\mathbb{X}^*} + \sum_{j=1}^k \|\Lambda_j^* y^*\|_{\mathbb{Z}_j^*}\,. \tag{2.60}$$

By $\mathbb{Y}^*$ we denote the normed space $\mathcal{D}^*$ equipped with the norm $\|\cdot\|_*$. Let $\mathbb{X}_\sigma^*$ and $\mathbb{Z}_{j,\sigma}^*$ be the spaces $\mathbb{X}^*$ and $\mathbb{Z}_j^*$, respectively, endowed with the topologies $\sigma(\mathbb{X}^*, \mathbb{X})$ and $\sigma(\mathbb{Z}_j^*, \mathbb{Z}_j)$. The graphs $\operatorname{gr}\Lambda_j^*$ of operators Λ_j^* are closed in $\mathbb{X}_\sigma^* \times \mathbb{Z}_{j,\sigma}^*$ for every $j = 1, 2, \ldots, k$ (see [241]). Hence, $\mathbb{Y}^*$ is a Banach space. By $\mathbb{X}_w$ we denote the Banach space X endowed with the $\sigma(\mathbb{X}, \mathbb{X}^*)$ topology. Let us consider the following class of subsets in $\mathbb{Y}^*$:

$$\mathcal{K}^* = \left\{\xi \in \mathbb{Y}^* \;:\; \|y^*\|_{\mathbb{X}^*} \le l_0,\ \|\Lambda_j^* y^*\|_{\mathbb{Z}_j^*} \le l_j,\ j = 1, 2, \ldots, k\right\}, \tag{2.61}$$

where $l_0, l_1, \ldots, l_k$ are positive numbers.

Having used the notation $\mathcal{D} = \mathbb{X} \times \prod_{j=1}^{k} D(\Lambda_j)$, we note that the set $\mathcal{D}$ can be associated with the family of linear continuous functionals $G_\phi(\cdot)$ on $\mathbb{Y}^*$. Indeed, let $\phi = (\phi_0, \phi_1, \dots, \phi_k)$ be a fixed element of $\mathcal{D}$. Then the functional $G_\phi(\cdot)$ can be defined as follows:

$$G_\phi(y^*) = \langle y^*, \phi \rangle := \langle y^*, \phi_0 \rangle_{\mathbb{X}^*, \mathbb{X}} + \sum_{j=1}^{k} \left\langle \Lambda_j^* y^*, \phi_j \right\rangle_{\mathbb{Z}_j^*, \mathbb{Z}_j} .$$

Remark 2.64. Let $\sigma(\mathbb{Y}^*, \mathcal{D})$ be the weakest topology on $\mathbb{Y}^*$ with respect to which all functionals $G_\phi(\cdot)$ are continuous. Then the pair $(\mathbb{Y}^*, \sigma(\mathbb{Y}^*, \mathcal{D}))$ is a locally convex topological space.

Definition 2.65. *We say that a sequence $\{y_n^*\}_{n=1}^{\infty} \subset \mathbb{Y}^*$ converges $\mathcal{D}$-weakly to an element $y^* \in \mathbb{Y}^*$, written $y_n^* \overset{\mathcal{D}}{\rightharpoonup} y^*$, provided*

$$\lim_{n \to \infty} \langle y_n^*, \phi_0 \rangle_{\mathbb{X}^*, \mathbb{X}} = \langle y^*, \phi_0 \rangle_{\mathbb{X}^*, \mathbb{X}} ,$$

$$\lim_{n \to \infty} \left\langle \Lambda_j^* y_n^*, \phi_j \right\rangle_{\mathbb{Z}_j^*, \mathbb{Z}_j} = \left\langle \Lambda_j^* y^*, \phi_j \right\rangle_{\mathbb{Z}_j^*, \mathbb{Z}_j}$$

for all $\phi = (\phi_0, \phi_1, \dots, \phi_k) \in \mathcal{D}$.

Note that this concept can be easy extended to the case of $\mathcal{D}$-weakly convergent nets (or generalized sequences) $\{y_\alpha^*\}_{\alpha \in A} \subset \mathbb{Y}^*$, where A is a directed set of indices.

The following result deals with the compactness property of the set $\mathcal{K}^*$.

Theorem 2.66. *Let $\mathbb{X}$ and $\{\mathbb{Z}_j\}_{j=1}^{k}$ be Banach spaces and let $\mathbb{X}^*$, $\{\mathbb{Z}_j^*\}_{j=1}^{k}$ be their duals, respectively. Let*

$$\{\Lambda_j : (D(\Lambda_j) \subset \mathbb{Z}_j) \to \mathbb{X}\}_{j=1}^{k}$$

be a given family of linear mappings with dense domains $D(\Lambda_j)$ in $\mathbb{Z}_j$ for each $j = 1, \dots, k$. Then the set $\mathcal{K}^*$, defined by (2.61), is $\mathcal{D}$-weakly compact in $\mathbb{Y}^*$ for every collection of positive numbers $l_0, l_1, \dots, l_k$.

Proof. Let $l_0, l_1, \dots, l_k$ be given positive numbers. Let $\{y_\alpha^*\}_{\alpha \in A}$ be an arbitrary net in $\mathcal{K}^*$. To prove this theorem, we have to show that there is a subnet $\{x_\beta^*\}_{\beta \in B}$ of the original net $\{y_\alpha^*\}_{\alpha \in A}$, which $\mathcal{D}$-weakly converges in $\mathbb{Y}^*$ to some element of $\mathcal{K}^*$. Taking into account the structure of the set $\mathcal{K}^*$, we see that $\{y_\alpha^*\}_{\alpha \in A}$ belongs to the close ball $B_{\mathbb{X}^*}(0, l_0)$ centered at the origin with radius l_0. Hence, by the Banach–Alaoglu theorem (see Yosida [251] or Kantorovich and Akilov [128]), there exists a subnet of $\{y_\alpha^*\}_{\alpha \in A}$, denoted by $\{x_\beta^*\}_{\beta \in B_0}$, such that $x_\beta^* \to y^*$ in $\mathbb{X}_\sigma^*$ and $\|y^*\|_{\mathbb{X}^*} \le l_0$. Setting now $j = 1$, we consider the net $\{\Lambda_1^* x_\beta^*\}_{\beta \in B_0}$. Since the elements $\Lambda_1^* x_\beta^*$ belong to the closed ball $B_{\mathbb{Z}_1^*}(0, l_1)$,

it follows that there exists a subnet $\{z_\gamma^*\}_{\gamma\in B_1}$ such that $\Lambda_1^* z_\gamma^* \to \mu_1$ in $Z_{1,\sigma}^*$ and $\|\mu_1\|_{\mathbb{Z}_1^*} \le l_1$. Repeating this iteration process under $j = 2, \dots, k$, finally, we can extract a subnet $\{d_\nu^*\}_{\nu\in B_k}$ of the origin net $\{y_\alpha^*\}_{\alpha\in A}$ such that

$$\langle d_\nu^*, \phi_0\rangle_{\mathbb{X}^*,\mathbb{X}} \overset{\nu\in B_k}{\longrightarrow} \langle y^*, \phi_0\rangle_{\mathbb{X}^*,\mathbb{X}},$$
$$\langle \Lambda_j^* d_\nu^*, \phi_j\rangle_{\mathbb{Z}_j^*,\mathbb{Z}_j} \overset{\nu\in B_k}{\longrightarrow} \langle \mu_j, \phi_j\rangle_{\mathbb{Z}_j^*,\mathbb{Z}_j},$$
$$\|y^*\|_{\mathbb{X}^*} \le l_0, \quad \|\mu_j\|_{\mathbb{Z}_j^*} \le l_j,\ j = 1, 2, \dots, k,$$

for each $\phi = (\phi_0, \phi_1, \dots, \phi_k) \in \mathcal{D}$.

In view of the properties of the domains $D(\Lambda_j)$, the operators $\Lambda_j^* : \big(D(\Lambda_j^*) \subset \mathbb{X}^*\big) \to \mathbb{Z}_j^*$ are closed in $\mathbb{X}_\sigma^* \times \mathbb{Z}_{j,\sigma}^*$. Hence, $\mu_j = \Lambda_j^* y^*$ for every $j = 1, 2, \dots, k$. As a result, we have $d_\nu^* \overset{\mathcal{D}}{\rightharpoonup} y^*$, where $\|y^*\|_{\mathbb{X}^*} \le l_0$ and $\|\Lambda_j^* y^*\|_{\mathbb{Z}_j^*} \le l_j$ for $j = 1, \dots, k$. Thus, $y^* \in \mathcal{K}^*$, which concludes the proof. □

To illustrate the possible applications of this theorem, we give the following example.

Example 2.67. Let Ω be open bounded subset of $\mathbb{R}^n$ with a Lipschitz boundary. Let $\mathbb{X} = \mathbb{Z}_j = L^1(\Omega)$ for all $j = 1, \dots, k$. We define the collection of linear mappings $\{\Lambda_j : (D(\Lambda_j) \subset \mathbb{Z}_j) \to \mathbb{X}\}_{j=1}^k$ as follows:

$$\Lambda_j y := -\partial y/\partial x_j, \quad j = 1, 2, \dots, k,$$

where $D(\Lambda_j)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|y\|_j = \|y\|_{L^1(\Omega)} + \|\partial y/\partial x_j\|_{L^1(\Omega)}.$$

Thus, if $y \in C(\Omega) \cap D(\Lambda_j)$, then

$$y \in L^1(\Omega), \quad \operatorname{supp} y(x) \ \text{ is a compact in } \ \Omega \ \text{ and } \ \partial y/\partial x_j \in L^1(\Omega),$$

where the partial derivatives $\partial y/\partial x_j$ we mean in the weak sense (see (2.6)).

It is easy to see that the following chain of embeddings holds:

$$C_0^1(\Omega) \subset W_0^{1,1}(\Omega) \subset D(\Lambda_j) \subset L^1(\Omega), \quad \forall\, j = 1, \dots, k.$$

Since $C_0^1(\Omega)$ is dense in $L^1(\Omega)$ with respect to the topology induced by the norm $\|\cdot\|_{L^1(\Omega)}$, it follows that the weak closure of the set $C_0^1(\Omega)$ coincides with the entire space $L^1(\Omega)$. Therefore, the domains $D(\Lambda_j)$ are weakly dense in $L^1(\Omega)$ for all $j = 1, \dots, k$. In view of this, the dual operators

$$\big\{\Lambda_j^* : \big(D(\Lambda_j^*) \subset L^\infty(\Omega)\big) \to L^\infty(\Omega)\big\}_{j=1}^k$$

are well defined and closed. Moreover, in this case, we have $\Lambda_j^* = \partial/\partial x_j$. As a result, we obtain

$$\mathcal{D}^* := \bigcap_{j=1}^{k} D(\Lambda_j^*) = W^{1,\infty}(\Omega),$$

where

$$W^{1,\infty}(\Omega) = \left\{ y(x) \ : \ y \in L^\infty(\Omega), \ \frac{\partial y}{\partial x_j} \in L^\infty(\Omega), \ j = 1, \dots, k \right\}$$

is the Banach space equipped with the norm

$$\|y\|_{W^{1,\infty}(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |y(x)| + \sum_{j=1}^{k} \operatorname*{ess\,sup}_{x \in \Omega} \left| \frac{\partial y(x)}{\partial x_j} \right|.$$

Hence $\mathbb{Y}^* = W^{1,\infty}(\Omega)$ and, therefore, due to the Theorem 2.66, the bounded set

$$\mathcal{K}^* = \left\{ y \in W^{1,\infty}(\Omega) \ : \ \|y\|_{L^\infty(\Omega)} \le l_0, \ \left\| \frac{\partial y}{\partial x_j} \right\|_{L^\infty(\Omega)} \le l_j, \ j = 1, \dots, k \right\}$$

is $\mathcal{D}$-weakly compact in $W^{1,\infty}(\Omega)$ for every positive numbers $l_0, l_1, \dots, l_k$. Following Definition 2.65 and using the fact that $L^1(\Omega)$ is a separable Banach space, it means that for any net $\{y_\alpha^*\}_{\alpha \in A} \subset \mathcal{K}^*$, there exists a sequence $\{x_i^*\}_{i \in \mathbb{N}}$ (which is a subnet of $\{y_\alpha^*\}_{\alpha \in A}$) such that

$$\lim_{i \to \infty} \langle x_i^*, \phi \rangle_{\mathbb{Y}^*, \mathbb{Y}} = \langle y, \phi \rangle_{\mathbb{Y}^*, \mathbb{Y}}, \quad \forall\, \phi = (\phi_0, \dots, \phi_k) \in L^1(\Omega) \times \prod_{j=1}^{k} D(\Lambda_j),$$

where $y \in \mathcal{K}^*$ and

$$\langle y, \phi \rangle_{\mathbb{Y}^*, \mathbb{Y}} = \int_\Omega y(x) \phi_0(x) \, \mathrm{d}x + \sum_{j=1}^{k} \int_\Omega \phi_j(x) \frac{\partial y(x)}{\partial x_j} \, \mathrm{d}x.$$

Thus, the set $\mathcal{K}^*$ is sequentially $\mathcal{D}$-weakly compact.

3

Variational Methods of Optimal Control Theory

Since our main interest is related to the mathematical theory of the parameterized optimal control problems (OCP_ε) for partial differential equations (PDEs), we discuss in this chapter general questions of optimal control theory, different settings of optimal control problems (OCPs) for distributed systems in variable spaces, the direct method of Calculus of Variation, the topological properties of solutions to ill-posed problems, optimality conditions for a wide class of extremal problems in the form of variational inequalities and also questions related to the construction of approximative solutions to different classes of OCPs. We refer to Alekseev, Tikchomirov and Fomin [4], Egorov [103], Fursikov [111], Ivanenko and Mel'nik [126], Lions [171], Zgurovsky and Mel'nik [253], Bonnans and Shapiro [24], etc., or the textbooks by Tröltzsch [242] and Mordukhovich [190] for more results concerning this topic. A recent account on PDE-constrained OCP's can also be found in Hinze, Pinnau, Ulbrich and Ulbrich [122].

It is well known that OCPs are extremal problems which describe the behavior of a system that can be modified by the action of an operator. Many problems in applied sciences can be modeled by means of OCPs. Two kinds of variables (or sets of variables) are then involved: One of them, called the *state variable*, describes the state of the system and cannot be modified directly by the operator; the second one, called the *control variable*, is under direct control of the operator who may choose his strategy among a given set of admissible ones.

The operator is allowed to modify the state of the system indirectly, acting directly on *admissible* control variables. A link "control–state" is normally put into the format of what is called the state equation. Typically, the size of the *admissible control* is constrained. Finally, the operator, acting directly on controls and indirectly on states through the *state equation*, must achieve a goal, usually written as a minimization of a functional, the so-called *cost functional*, which depends on the control that has been chosen as well as on the corresponding state.

The typical ingredients of an OCP are as follows:

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_3,
© Springer Science+Business Media, LLC 2011

(A_1) A space of states $\mathbb{Y}$;
(A_2) A space of controls $\mathbb{U}$;
(A_3) A cost (objective) functional $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ (CF);
(A_4) A set Ξ of admissible pairs, that is, a subset of pairs $(u, y) \in \mathbb{U} \times \mathbb{Y}$ such that $I(u, y) < +\infty$, y is linked to u through the state equation (SE) and u satisfies the control constraints $u \in U$ (CCs).

As a result, the OCP can be stated in the form of a constrained minimization problem:

$$(\text{OCP}) : \qquad \min \{I(u, y) \ : \ (u, y) \in \Xi\}. \tag{3.1}$$

3.1 The general setting

We begin this section with the following notion.

Definition 3.1. *Let (X, τ) be a topological space, let $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and let $F : X \to \overline{\mathbb{R}}$ be a functional.*

(1) We say that F is τ-lower semicontinuous *(τ-lsc) if for every $t \in \mathbb{R}$ the set $\{x \in X \ : \ F(x) \le t\}$ is τ-closed in X. We say that F is sequentially τ-lsc if*

$$F(x) \le \liminf_{n \to \infty} F(x_n)$$

for every $x \in X$ and every sequence $\{x_n\}$ converging to x in (X, τ).

(2) We say that F is τ-coercive if for every $t \in \mathbb{R}$, there exists a τ-compact subset K_t of X such that $\{x \in X \ : \ F(x) \le t\} \subset K_t$.
We say that F is sequentially τ-coercive if for every $t \in \mathbb{R}$, every sequence in $\{x \in X \ : \ F(x) \le t\}$ has a subsequence that τ-converges to a point of $\{x \in X \ : \ F(x) \le t\}$.

Let $\mathbb{Y}$ and $\mathbb{Z}$ be linear normed spaces and let $\mathbb{Y}_1$ be a reflexive separable Banach space such that $\mathbb{Y}_1$ is continuously and densely embedded in $\mathbb{Y}$. Let $\mathbb{V}$ be a separable Banach space and $\mathbb{U} = \mathbb{V}^*$ be its dual. Let U_∂ be a closed convex subset of $\mathbb{U}$. In particular, the convexity of $\mathbb{U}$ means that for any $u_1 \in U_\partial$ and $u_2 \in U_\partial$, the condition $\alpha u_1 + (1 - \alpha) u_2 \in U_\partial$ holds true for all $\alpha \in (0, 1)$.

We consider the following OCP: Find a pair $(u^0, y^0) \in U_\partial \times Y_1$ such that a functional

$$I(u, y) \to \inf \tag{3.2}$$

subject to the restrictions

$$L(u, y) + F(y) = 0, \ \text{and} \ \ u \in U_\partial, \tag{3.3}$$

where $L : \mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z}$ is a linear continuous mapping, $F : \mathbb{Y}_1 \to \mathbb{Z}$ is a nonlinear mapping and $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is a proper cost functional which is always supposed to be bounded below and sequentially lower semicontinuous

with respect to the product of the weak-$*$ topology for $\mathbb{U}$ and of the weak topology for $\mathbb{Y}$.

Hereinafter, we associate $u \in U_\partial$ with an admissible control and $y \in \mathbb{Y}_1$ with a state of the control process. Let us note also that in the applications, the state space $\mathbb{Y}_1$ endowed with its weak topology and the control space $\mathbb{U}$ endowed with its weak-$*$ topology are not metrizable, in general. Nevertheless, due to some growth assumptions on the cost functional I, we may often restrict ourselves to work on a bounded subset of $\mathbb{Y}_1 \times \mathbb{U}$ which is, thanks to the initial assumptions, metrizable.

However, the current point of view leaves room for ill-posed boundary value problems. Indeed, we suppose that

$$\exists u \in \mathbb{U},\ \exists\, y \in \mathbb{Y}_1\ :\ L(u,y) + F(y) = 0.$$

Thus, the corresponding state y may be not unique, in general. So, we admit the situation when there exist controls $u \in \mathbb{U}$ such that $\nexists\, y \in \mathbb{Y}_1$ satisfying the equation $L(u,y) + F(y) = 0$. It is clear that in this case, it is not possible to write $y = y(u)$. To avoid this situation, it is convenient to introduce the set of admissible pairs for the problem (3.2)–(3.3):

$$\Xi = \{(u,y) \in \mathbb{U} \times \mathbb{Y}_1\ :\ L(u,y) + F(y) = 0,\ u \in U_\partial,\ I(u,y) < +\infty\}\,. \quad (3.4)$$

We say that a pair $(u^0, y^0) \in \mathbb{U} \times \mathbb{Y}_1$ is optimal for the problem (3.2)–(3.3) if

$$(u^0, y^0) \in \Xi \ \text{ and } \ I(u^0, y^0) = \inf_{(u,y) \in \Xi} I(u,y).$$

In what follows, we assume that the following hypotheses hold true:

(H1) (Regularity condition) $\Xi \neq \emptyset$.

(H2) (Coerciveness condition) For any $\lambda > 0$, the set

$$\Xi_\lambda = \{(u,y) \in \Xi\ :\ I(u,y) \leq \lambda\}$$

is bounded in $\mathbb{U} \times \mathbb{Y}_1$.

(H3) (Compactness condition) There exists a normed space $\mathbb{Y}_{-1}$ containing $\mathbb{Y}$ such that the embedding $\mathbb{Y}_1 \hookrightarrow \mathbb{Y}_{-1}$ is compact and there exists a dense subset S of $\mathbb{Z}^*$ such that for any $z^* \in S$, the mapping $\mathbb{Y}_1 \ni y \mapsto \langle z^*, F(y)\rangle_{\mathbb{Z}^*,\mathbb{Z}}$ can be extended to a continuous functional defined on the whole space $\mathbb{Y}_{-1}$.

In order to explain the role of these assumptions, we give the following results (see Fursikov [111]).

Theorem 3.2. *Assume that the following conditions hold true:*

(i) The embedding $\mathbb{Y}_1 \hookrightarrow \mathbb{Y}$ is continuous.

(ii) There is an element $F_0 \in \mathbb{Z}$ such that $F(y) = F_0$ for all $y \in \mathbb{Y}_1$.

(iii) $L : \mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z}$ *is a linear continuous mapping with respect to the weak topology of* $\mathbb{Z}$ *and the product of the weak-* topology for* $\mathbb{U}$ *and of the weak topology for* $\mathbb{Y}$.

Then, under hypotheses (H1)–(H2), the OCP (3.2)–(3.3) admits at least one solution. In addition, if I *is a strictly convex functional, that is, the inequality*

$$I(\alpha u_1 + (1-\alpha)u_2, \alpha y_1 + (1-\alpha)y_2) < \alpha I(u_1, y_1) + (1-\alpha) I(u_2, y_2)$$

is valid whenever $(u_1, y_1) \neq (u_2, y_2)$ *and* $\alpha \in (0,1)$*, then the OCP (3.2)–(3.3) has a unique solution.*

Proof. Since the original problem is regular, it follows that there exists a minimizing sequence $\{(u_k, y_k)\}_{k \in \mathbb{N}} \subset \Xi$ for the original problem, that is,

$$\lim_{k \to \infty} I(u_k, y_k) = \inf_{(u,y) \in \Xi} I(u, y) > -\infty.$$

Hence, $\sup_{k \in \mathbb{N}} I(u_k, y_k) \leq C$, where the constant C is independent of k. Then in view of (H2), we obtain that $\sup_{n \in \mathbb{N}} [\|u_k\|_{\mathbb{U}} + \|y_k\|_{\mathbb{Y}_1}] \leq C$. Hence, using the Banach–Alaoglu theorem and passing to a subsequence, we may assume that

$$u_k \overset{*}{\rightharpoonup} u^0 \ \text{ in } \ \mathbb{U}, \quad y_k \rightharpoonup y^0 \ \text{ in } \ \mathbb{Y}_1.$$

Since the set U_∂ is convex and closed, it follows that U_∂ is sequentially closed with respect to the weak-$*$ topology. Therefore, $u^0 \in U_\partial$. In addition, for any $z^* \in \mathbb{Z}^*$, we have

$$\begin{aligned} \langle z^*, L(u_k, y_k) \rangle_{\mathbb{Z}^*, \mathbb{Z}} &= \langle L^* z^*, (u_k, y_k) \rangle_{\mathbb{U}^* \times \mathbb{Y}_1^*, \mathbb{U} \times \mathbb{Y}_1} \\ &\xrightarrow[k \to \infty]{} \langle L^* z^*, (u^0, y^0) \rangle_{\mathbb{U}^* \times \mathbb{Y}_1^*, \mathbb{U} \times \mathbb{Y}_1} = \langle z^*, L(u^0, y^0) \rangle_{\mathbb{Z}^*, \mathbb{Z}}, \end{aligned}$$

where $L^* z^* \in \mathbb{U}^* \times \mathbb{Y}^*$.

Thus, summing up the results obtained above – namely, $L(u^0, y^0) + F_0 = 0$, $u^0 \in U_\partial$ and $I(u^0, y^0) < +\infty$ – we conclude that (u^0, y^0) is an admissible pair to the problem (3.2)–(3.3). Since the cost functional I is lower semicontinuous with respect to the product of the weak-$*$ topology for $\mathbb{U}$ and of the weak topology for $\mathbb{Y}$ convergence, we get

$$-\infty < I(u^0, y^0) \leq \lim_{n \to \infty} I(u_n, y_n) = \inf_{(u,y) \in \Xi} I(u, y).$$

Thus, the pair (u^0, y^0) is optimal for the problem (3.2)–(3.3). This pair is unique in the case when the cost functional is strictly convex. Indeed, let (u_1^0, y_1^0) and (u_2^0, y_2^0) be two different solutions to the original problem. Since the set U_∂ is convex and the state equation $L(u, y) + F_0 = 0$ is linear, it follows that $\big((u_1^0 + u_2^0)/2, (y_1^0 + y_2^0)/2\big) \in \Xi$. Hence,

$$I\left((u_1^0+u_2^0)/2,(y_1^0+y_2^0)/2\right) < \frac{1}{2}\left[I(u_1^0,y_1^0)+I(u_2^0,y_2^0)\right] = \inf_{(u,y)\in\,\Xi} I(u,y),$$

and we come to a contradiction. Hence, our assumption was wrong, and this concludes the proof. □

Theorem 3.3. *Under hypotheses (H1)–(H3), OCP (3.2)–(3.3) admits at least one solution.*

Proof. Since $\Xi \neq \emptyset$ and the cost functional is bounded below on Ξ, it follows that there exists a minimizing sequence $\{(u_k,y_k)\}_{k\in\mathbb{N}} \subset \Xi$ for the (3.2)–(3.3), that is,

$$I(u_k,y_k) \xrightarrow[n\to\infty]{} I_{\min} \equiv \inf_{(u,y)\in\,\Xi} I(u,y) > -\infty.$$

Using the arguments of the previous theorem, we may extract a subsequence of $\{(u_k,y_k)\}_{k\in\mathbb{N}} \subset \Xi$ (still indexed by k) such that

$$\begin{gathered} u_k \overset{*}{\rightharpoonup} u^0 \ \text{ in } \ \mathbb{U}, \quad y_k \rightharpoonup y^0 \ \text{ in } \ \mathbb{Y}_1, \\ \langle z^*, L(u_k,y_k)\rangle_{\mathbb{Z}^*,\mathbb{Z}} \xrightarrow[k\to\infty]{} \left\langle z^*, L(u^0,y^0)\right\rangle_{\mathbb{Z}^*,\mathbb{Z}} \quad \forall\, z^* \in \mathbb{Z}^*. \end{gathered} \tag{3.5}$$

Hence, $u^0 \in U_\partial$.

By (H3), the embedding $\mathbb{Y}_1 \hookrightarrow \mathbb{Y}_{-1}$ is compact and, hence, $y_n \to y^0$ strongly in $\mathbb{Y}_{-1}$. Therefore,

$$\langle z^*, F(y_k)\rangle_{\mathbb{Z}^*,\mathbb{Z}} \xrightarrow[k\to\infty]{} \left\langle z^*, F(y^0)\right\rangle_{\mathbb{Z}^*,\mathbb{Z}}, \quad \forall\, z^* \in S \subset \mathbb{Z}^*. \tag{3.6}$$

Then, taking relations (3.5) and (3.6) and the density of the embedding $S \subset \mathbb{Z}^*$ into account and passing to the limit in $L(u_k,y_k)+F(y_k)=0$ as $k\to\infty$, we obtain

$$L(u^0,y^0)+F(y^0)=0.$$

As a result, $(u^0,y^0) \in \Xi$. Since the embedding $\mathbb{Y}_1 \subset \mathbb{Y}$ is continuous, it follows that $y_n \rightharpoonup y^0$ in $\mathbb{Y}$. Using the property of lower semicontinuity for I with respect to the product of the weak-$*$ topology for $\mathbb{U}$ and of the weak topology for $\mathbb{Y}$ convergence, we get

$$-\infty < I(u^0,y^0) \le \lim_{n\to\infty} I(u_n,y_n) = I_{\min}.$$

Thus, the pair (u^0,y^0) is optimal for the problem (3.2)–(3.3). □

Remark 3.4. It is well known (see, for instance, Ekeland and Temam [104]) that a convex lower semicontinuous functional is also lower semicontinuous with respect to the weak-$*$ topology. So, instead of the suppositions concerning the cost functional I given by Theorem 3.2, we can take the following ones:

(a) I is lower semicontinuous on $\Xi \subset \mathbb{U}\times\mathbb{Y}$ with respect to the strong topology of $\mathbb{U} \times \mathbb{Y}$.
(b) $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is a convex functional, that is, for any $(u_1, y_1) \in \Xi$ and $(u_2, y_2) \in \Xi$, the inequality

$$I(\alpha u_1 + (1-\alpha)u_2, \alpha y_1 + (1-\alpha)y_2) \le \alpha I(u_1, y_1) + (1-\alpha) I(u_2, y_2) \quad (3.7)$$

holds true for every $\alpha \in [0, 1]$.

In addition, if I is a strictly convex functional (i.e., the inequality (3.7) is strict whenever $(u_1, y_1) \neq (u_2, y_2)$ and $\alpha \in (0, 1)$) then the OCP (3.2)–(3.3) with a nonlinear state equation has a unique solution (see Lions [169]).

Definition 3.5. *A sequence of pairs $\{(u_k, y_k)\}_{k=1}^{\infty} \subset \mathbb{U} \times \mathbb{Y}$ is said to be a w-convergent to a pair $(u, y) \in \mathbb{U} \times \mathbb{Y}$ if $u_k \overset{*}{\rightharpoonup} u$ in $\mathbb{U}$ and $y_k \rightharpoonup y$ in $\mathbb{Y}$.*

As a consequence of the assumptions given above, we have the following result.

Lemma 3.6. *Under the hypotheses of Theorem 3.2 and the supposition that the cost functional I is bounded on any bounded subset of $U_\partial \times \mathbb{Y}$, the set of admissible solutions Ξ to the problem (3.2)–(3.3) is sequentially closed with respect to the product of the weak-$*$ topology for $\mathbb{U}$ and of the weak topology for $\mathbb{Y}$.*

Proof. Let $\{(u_k, y_k)\}_{k=1}^{\infty}$ be any w-convergent sequence of admissible pairs, that is, $(u_k, y_k) \in \Xi$ for all $k \in \mathbb{N}$ and $u_k \overset{*}{\rightharpoonup} \widehat{u}$ in $\mathbb{U}$ and $y_k \rightharpoonup \widehat{y}$ in $\mathbb{Y}$. Hence,

$$\sup_{k\in N} \left[\|u_k\|_{\mathbb{U}} + \|y\|_{\mathbb{Y}}\right] < +\infty.$$

Then, by the initial supposition, we have $\sup_{k\in\mathbb{N}} I(u_k, y_k) < +\infty$. By virtue of the coerciveness property (H2), we obtain

$$u_k \overset{*}{\rightharpoonup} \widehat{u} \ \text{ in } \ \mathbb{U} \ \text{ and } \ y_k \rightharpoonup \widehat{y} \ \text{ in } \ \mathbb{Y}_1.$$

Thus, having repeated all arguments of the proof of Theorem 3.3, we conclude that $L(\widehat{u}, \widehat{y}) + F(\widehat{y}) = 0$ and $u \in U_\partial$, that is, $(\widehat{u}, \widehat{y}) \in \Xi$. □

As an application of Theorem 3.3, take an OCP as follows:

$$I(u, y) = \int_\Omega |y(x) - z_\partial(x)|^3 \, \mathrm{d}x + N \int_\Omega |u(x)|^2 \, \mathrm{d}x \to \inf, \quad (3.8)$$

$$-\Delta y(x) - y\sqrt{|y(x)|} = f(x) + u(x), \quad x \in \Omega, \ y|_{\partial\Omega} = 0, \quad (3.9)$$

$$u \in U_\partial, \quad U_\partial = \left\{u \in L^2(\Omega) \ : \ \alpha_1(x) \le u(x) \le \alpha_2(x) \text{ a.e. in } \ \Omega\right\}, \quad (3.10)$$

where Ω is a bounded open domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$,

$$z_\partial \in L^3(\Omega), \quad f \in H^{-1}(\Omega), \quad N > 0, \quad \alpha_i \in C(\Omega), \ i = 1, 2,$$

and $\alpha_1(x) \le \alpha_2(x) - \delta$ in Ω with some $\delta > 0$.

To give the variational formulation of the boundary value problem (3.9), we say that y is a weak solution to that problem with a fixed right-hand side $f(x) + u(x)$ if $y \in H_0^1(\Omega)$ and the integral identity

$$\int_\Omega \Big[\nabla y(x) \cdot \nabla \varphi(x) - y(x)\sqrt{|y(x)|}\varphi(x)\Big]\, \mathrm{d}x = \langle f + u, \varphi\rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)}$$

holds true for every $\varphi \in C_0^\infty(\Omega)$.

However, there is a good reason to believe that we do not, in general, have the existence of a weak solution $y(x)$ to the above boundary value problem for any functions $f \in H^{-1}(\Omega)$ and $u \in U_\partial$ (see Lions [172]). So, we have to begin with the following assumption: *There exists a pair* $(u, y) \in L^2(\Omega) \times \big(H_0^1(\Omega) \cap L^3(\Omega)\big)$ *satisfying relations (3.9) and (3.10).* Evidently, this supposition is fulfilled if $f \in L^2(\Omega)$ and $-f \in U_\partial$. In this case, the pair $(-f, 0)$ is admissible and we have the following result.

Theorem 3.7. *Assume that the set of admissible pairs for the problem (3.8)–(3.10) is nonempty. Then this problem admits at least one solution* $(u^0, y^0) \in L^2(\Omega) \times \big(H_0^1(\Omega) \cap L^3(\Omega)\big)$.

Proof. In order to prove the desired result, we make use of Theorem 3.3. To do so, we set

$$\mathbb{Y} = L^3(\Omega), \quad \mathbb{Y}_1 = L^3(\Omega) \cap H_0^1(\Omega), \quad \mathbb{U} = L^2(\Omega),$$
$$\mathbb{Z} = H^{-1}(\Omega),\ L(u, y) = -\Delta y - u,\ F(y) = -|y|^{3/2} - f.$$

Now we observe that the cost functional $I : L^2(\Omega) \times L^3(\Omega) \to \overline{\mathbb{R}}$ is convex and lower semicontinuous with respect to the weak topology of $L^2(\Omega) \times L^3(\Omega)$. Since $H^{-1}(\Omega) = \big(H_0^1(\Omega)\big)^*$ and the operator $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is continuous [168], it follows that the mappings

$$L(u, y) = -\Delta y - u \ :\ L^2(\Omega) \times \big(H_0^1(\Omega) \cap L^3(\Omega)\big) \to H^{-1}(\Omega),$$
$$F(y) = -|y|^{3/2} - f \ :\ H_0^1(\Omega) \cap L^3(\Omega) \to H^{-1}(\Omega)$$

are continuous as well.

Let us show that the coerciveness property (H2) is valid. Indeed, for any $\lambda > 0$, we have

$$\begin{aligned}\Xi_\lambda = \Big\{&(u, y) \in L^2(\Omega) \times \big(H_0^1(\Omega) \cap L^3(\Omega)\big) \ :\ u \in U_\partial,\\ &-\Delta y = y\sqrt{|y|} + f + u,\ I(u, y) \le \lambda\Big\}\\ \subset \Big\{&(u, y) \in L^2(\Omega) \times \big(H_0^1(\Omega) \cap L^3(\Omega)\big) \ :\ -\Delta y = y\sqrt{|y|} + f + u,\\ &\|u\|_{L^2(\Omega)} \le \sqrt{\lambda/N},\ \|y\|_{L^3(\Omega)} \le \sqrt[3]{\lambda} + \|z_\partial\|_{L^3(\Omega)},\ u \in U_\partial\Big\}.\end{aligned}$$

Since

$$\|y\|_{H_0^1(\Omega)} \le \|f\|_{H^{-1}(\Omega)} + C\|u\|_{L^2(\Omega)} + C\|y\|_{L^3(\Omega)},$$

we have

$$\Xi_\lambda \subset \Big\{(u,y) \in L^2(\Omega) \times \big(H_0^1(\Omega) \cap L^3(\Omega)\big) \ : \ \|u\|_{L^2(\Omega)} \le \sqrt{\lambda/N},$$
$$\|y\|_{L^3(\Omega)} \le \sqrt[3]{\lambda} + \|z_\partial\|_{L^3(\Omega)},$$
$$\|y\|_{H_0^1(\Omega)} \le \|f\|_{H^{-1}(\Omega)} + C\left(\sqrt[3]{\lambda} + \|z_\partial\|_{L^3(\Omega)}\right) + C\sqrt{\lambda/N}\Big\}.$$

Hence, Ξ_λ is a bounded subset of $L^2(\Omega) \times \big(H_0^1(\Omega) \cap L^3(\Omega)\big)$.

It remains only to verify the compactness property (H3). With this aim, we set $\mathbb{Y}_{-1} = L^{3/2}(\Omega)$. Since Ω is a bounded domain, it follows that $L^3(\Omega) \subset L^{3/2}(\Omega)$, that is, the embedding $\mathbb{Y}_1 \hookrightarrow \mathbb{Y}_{-1}$ is continuous. On the other hand, using the compactness of the embedding $H_0^1(\Omega) \cap L^3(\Omega) \hookrightarrow L^2(\Omega)$ and the estimate $\|y\|^{3/2}_{L^{3/2}(\Omega)} \le C\|y\|_{L^2(\Omega)}$ that is fulfilled for every $y \in L^2(\Omega)$, we obtain that

$$H_0^1(\Omega) \cap L^3(\Omega) \hookrightarrow L^{3/2}(\Omega)$$

is compact.

It is easy to see that the functional

$$y \mapsto \langle F(y), \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = -\int_\Omega |y(x)|^{3/2}\varphi(x)\,\mathrm{d}x - < f, \varphi >_{H^{-1}(\Omega),H_0^1(\Omega)}$$

is continuous on $L^{3/2}(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega) = \big(H^{-1}(\Omega)\big)^*$, we obtain the desired property (H3). Thus, all suppositions of Theorem 3.3 hold true. Hence, the OCP (3.8)–(3.10) is solvable, and we obtain the required conclusion. □

To illustrate the significance of the main hypotheses to Theorem 3.3, let us consider the following OCP for which the compactness condition (H3) fails. As we will see later, it is an explicit example of an OCP for which an optimal solution does not exist:

$$I(u,y) = \int_0^1 \big(y^2(t) + u^2(t)\big)\,\mathrm{d}t \to \inf, \tag{3.11}$$

$$\dot{y}^4(t) + u(t) - 1 = 0, \quad y(0) = y(1) = 0, \tag{3.12}$$

where $\dot{y}(t) = \mathrm{d}y/\mathrm{d}t$. Let us set

$$\mathbb{Y} = \mathbb{U} = \mathbb{Z} = L^2(0,1), \quad U_\partial = L^2(0,1),$$
$$\mathbb{Y}_1 = W_0^{1,8}(0,1) = \big\{y \in W^{1,8}(0,1) \ : \ y(0) = y(1) = 0\big\},$$
$$L(u,y) = u(t), \quad F(y) = \dot{y}^4(t) - 1.$$

Then under such a choice of the operators and spaces, all assumptions of Theorem 3.3, with the exception of the compactness property (H3), hold true.

Indeed, since $(u,y) = \left(1-\pi^4\cos^4\pi t, \sin\pi t\right) \in L^2(0,1)\times W_0^{1,8}(0,1)$, it follows that the set of admissible pairs

$$\Xi = \left\{(u,y)\in L^2(0,1)\times W_0^{1,8}(0,1) \ : \ u = 1-\dot y^4(t)\right\}$$

is nonempty. Let us show the coerciveness property (H2). To do so, we make use of the Hölder inequality

$$\int_0^1 \dot y^4(t)\,\mathrm{d}t \le \left(\int_0^1 \dot y^8(t)\,\mathrm{d}t\right)^{1/2}.$$

Then

$$\begin{aligned}
&\left\{y\in W_0^{1,8}(0,1) \ : \ \int_0^1 \left(y^2(t)+\left(1-\dot y^4(t)\right)^2\right)\,\mathrm{d}t \le \lambda\right\}\\
&\quad\subset \left\{y\in W_0^{1,8}(0,1) \ : \ \int_0^1 \dot y^8(t)\,\mathrm{d}t \le \lambda - 1 + 2\int_0^1 \dot y^4(t)\,\mathrm{d}t\right\}\\
&\quad\subset \left\{y\in W_0^{1,8}(0,1) \ : \ \|\dot y\|^8_{L^8(0,1)} - 2\|\dot y\|^4_{L^8(0,1)} + 1 - \lambda \le 0\right\}\\
&\quad= \left\{y\in W_0^{1,8}(0,1) \ : \ \|\dot y\|^8_{L^8(0,1)} \le \left(1+\sqrt{\lambda}\right)^2\right\}.
\end{aligned}$$

As a result, for any $\lambda > 0$, we have

$$\begin{aligned}
&\{(u,y)\in\Xi \ : \ I(u,y)\le\lambda\}\\
&\quad= \left\{(u,y)\ \middle|\ \begin{array}{c}\displaystyle\int_0^1 \left(y^2(t)+\left(1-\dot y^4(t)\right)^2\right)\,\mathrm{d}t \le \lambda,\\ u = 1-\dot y^4, \quad y(0)=y(1)=0\end{array}\right\}\\
&\quad\subset \left\{(u,y)\ \middle|\ \begin{array}{c}\|\dot y\|^8_{L^8(0,1)} \le \left(1+\sqrt{\lambda}\right)^2,\\ \|u\|^2_{L^2(0,1)} = \displaystyle\int_0^1 \left(1-\dot y^4(t)\right)^2\,\mathrm{d}t \le \lambda\end{array}\right\}.
\end{aligned}$$

Since the set on the right-hand side of this inequality is bounded in $L^2(0,1)\times W_0^{1,8}(0,1)$, it follows that the OCP (3.11)–(3.12) is coercive.

It remains to prove that the compactness property (H3) fails to hold. To do so, we show that no pair $(u,y)\in\Xi$ can solve the OCP (3.11)–(3.12). Then, by Theorem 3.3, it means that (H3) cannot be valid. Indeed, let us rewrite the original problem as follows:

$$I(y) = \int_0^1 \left(y^2(t)+\left(1-\dot y^4(t)\right)^2\right)\,\mathrm{d}t \longrightarrow \inf, \quad y(0)=y(1)=0. \tag{3.13}$$

Let $\{y_k\}_{k=1}^\infty$ be the following sequence in $W_0^{1,8}(0,1)$:

$$y_k(t) = \int_0^t \operatorname{sign}\sin(2\pi k t)\,\mathrm{d}t, \quad \forall\, k\in\mathbb{N}.$$

It is clear that $y_k(0) = y_k(1) = 0$, $|\dot{y}_k(t)|^4 = 1$ for almost all $t \in (0,1)$ and $y_k(t) \to y^*(t) \equiv 0$ as $k \to \infty$ uniformly with respect to $t \in [0,1]$. Then $I(y_k) \to 0$ as $k \to \infty$. Hence,

$$\inf_{y \in W_0^{1,8}(0,1)} I(y) = \inf_{y \in W_0^{1,8}(0,1)} \int_0^1 \left(y^2(t) + \left(1 - \dot{y}^4(t)\right)^2 \right) \mathrm{d}t = 0.$$

On the other hand, as immediately follows from (3.13), if $y(t) = y^*(t) \equiv 0$ then $I(y^*) = 1$. Moreover, in this case, we have

$$I(y) \geq \int_0^1 y^2(t)\,\mathrm{d}t > 0 \quad \forall\, y(\cdot) \not\equiv y^*(\cdot).$$

Thus, the OCP (3.11)–(3.12) has no optimal solutions.

3.2 Abstract extremal problems

In this section, we study the OCPs in Banach spaces for essentially nonlinear operator equations with additional control and state constraints. This topic has been widely studied by many authors. We only mention Barbu [19], Fattorini [108], Fursikov [111], Ivanenko and Mel'nik [126], Lions [172], Zgurovsky and Mel'nik [253]. Our prime interest is to study these problems in the case when the control and state constraints take the form of some inequalities and inclusions in Banach spaces.

Let $\mathbb{Y}$ be a reflexive Banach space, $\mathbb{Y}^*$ be its dual, and $\mathbb{Z}$ be a Banach space partially ordered by closed reproducing pointed cone $\Lambda \subset \mathbb{Z}$. Let $\mathbb{U}$ be a control space which is assumed to be dual to some separable Banach space $\mathbb{V}$ ($\mathbb{U} = \mathbb{V}^*$). Let U_∂ be a subset of admissible controls in $\mathbb{U}$ and let K be a subset of $\mathbb{Y}$. The OCPs we consider can be described in a general manner as follows:

$$I(u,y) \longrightarrow \inf \tag{3.14}$$

subject to

$$A(u,y) = f, \tag{3.15}$$

$$F(u,y) \geq_\Lambda 0, \tag{3.16}$$

$$y \in K, \quad u \in U_\partial \subset \mathbb{U}, \tag{3.17}$$

where

$$A : \mathbb{U} \times \mathbb{Y} \to \mathbb{Y}^*, \quad F : \mathbb{U} \times \mathbb{Y} \to \mathbb{Z}$$

are nonlinear mappings, $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is a cost functional and f is a given element of $\mathbb{Y}^*$.

Definition 3.8. *We say that the problem (3.14)–(3.17) is regular if for every $f \in \mathbb{Y}^*$, there exists a pair $(u,y) \in \mathbb{U} \times \mathbb{Y}$, where $y = y(u)$ is the corresponding solution of (3.15), such that (u,y) satisfies the restrictions (3.16), (3.17) and $I(u,y) < +\infty$. In this case, the pair $(u, y(u))$ is said to be admissible.*

As usual, we denote by Ξ the set of all admissible pairs to the problem (3.14)–(3.17). Let us note that, in general, the mapping $u \to y(u)$ can be multivalued.

Remark 3.9. As an example of the situation, when the mapping $u \to y(u)$ is multivalued, we have the following OCP

$$\text{Minimize} \quad \left\{ I(u,y) = \int_\Omega |y(x) - y_d(x)|^2 \, \mathrm{d}x + \int_\Omega |\nabla y(x)|^2_{\mathbb{R}^N} u \, \mathrm{d}x \right\} \tag{3.18}$$

subject to the constraints

$$-\text{div}\ (u(x)\nabla y) + y = f \quad \text{in} \quad \Omega, \quad y = 0 \ \text{on} \ \ \partial\Omega, \tag{3.19}$$

$$u \in L^1(\Omega), \quad \int_\Omega u \, \mathrm{d}x = m, \quad \xi_1(x) \le u(x) \le \xi_2(x) \ \ \text{a.e. in} \ \ \Omega, \tag{3.20}$$

where $f \in L^2(\Omega)$ is a given function and ξ_1 and ξ_2 are given elements of $L^1(\Omega)$ satisfying the condition

$$\xi_1(x) \le \xi_2(x) \ \ \text{for a.e.} \ \ x \in \Omega, \quad \xi_1^{-1} \in L^1(\Omega). \tag{3.21}$$

We can associate with each admissible control u (i.e., u satisfies conditions (3.20)–(3.21)), the weighted Sobolev space $W_u = W(\Omega, u\,\mathrm{d}x)$, where

$$W_u = \left\{ y \in W_0^{1,1}(\Omega) \ : \ \|y\|_u = \left(\int_\Omega \left(y^2 + u\,|\nabla y|^2 \right) \, \mathrm{d}x \right)^{1/2} < +\infty \right\}.$$

It is reasonable to say that a function $y = y(\rho, f) \in W_u$ is a weak solution to the boundary value problem (3.19) for a fixed control $u \in \mathcal{R}_{ad}$ if the integral identity

$$\int_\Omega \Big((\nabla y, \nabla \varphi)_{\mathbb{R}^N} u + y\varphi \Big) \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x \tag{3.22}$$

holds for any test function $\varphi \in C_0^\infty(\Omega)$.

It is clear that the question of uniqueness of a weak solution leads us to the question of whether $C_0^\infty(\Omega)$ is dense in W_u. However, as was indicated in [266], for a "typical" admissible control u, the space $C_0^\infty(\Omega)$ is not dense in W_u; hence, there is no uniqueness of weak solutions (for more details and other types of solutions, we refer to [23, 159, 255, 266]). Thus, for OCP (3.18)–(3.21) the mapping $u \mapsto y(u)$ is multivalued, in general.

We begin with the following suppositions:

(A1) U_∂ is a bounded sequentially weakly-$*$ closed subset of $\mathbb{U}$.
(A2) The operator $A : \mathbb{U} \times \mathbb{Y} \to \mathbb{Y}^*$ is coercive, that is,

$$\inf_{u \in G} \frac{\langle A(u,y), y \rangle_{\mathbb{Y}^*;\mathbb{Y}}}{\|y\|_{\mathbb{Y}}} \to +\infty \quad \text{as} \quad \|y\|_{\mathbb{Y}} \to +\infty$$

for any bounded subset $G \subset \mathbb{U}$.

(A3) The operator $A : \mathbb{U} \times \mathbb{Y} \to \mathbb{Y}^*$ possesses the property $(\mathfrak{M})$, that is, for any w-convergent sequence $\{(u_k, y_k)\}_{k=1}^{\infty}$, the conditions

$$A(u_k, y_k) \rightharpoonup d \ \text{ in } \ \mathbb{Y}^*, \quad \limsup_{k\to\infty} \langle A(u_k, y_k), y_k\rangle_{\mathbb{Y}^*;\mathbb{Y}} \le \langle d, y\rangle_{\mathbb{Y}^*;\mathbb{Y}}$$

imply the relation $d = A(u, y)$.

(A4) The operator $F : \mathbb{U} \times \mathbb{Y} \to \mathbb{Z}$ is sequentially continuous in the following sense: $F(u_k, y_k) \rightharpoonup F(u, y)$ in $\mathbb{Z}$ whenever $(u_k, y_k) \overset{w}{\to} (u, y)$.

(A5) K is a weakly closed subset of $\mathbb{Y}$.

(A6) The cost functional $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is sequentially lower semi-continuous with respect to the w-convergence in $\mathbb{U} \times \mathbb{Y}$, that is,

$$\liminf_{i\to\infty} I(u_k, y_k) \ge I(u, y)$$

whenever $(u_k, y_k) \overset{w}{\to} (u, y)$.

Before proceeding further, we note that the cone $\Lambda \subset \mathbb{Z}$ is uniquely defined by its conjugate semigroup Λ^*, that is,

$$\Lambda = \left\{\xi \in \mathbb{Z} \ : \ \langle\varphi, \xi\rangle_{\mathbb{Z}^*;\mathbb{Z}} \ge 0 \quad \forall\, \varphi \in \Lambda^*\right\}. \tag{3.23}$$

Let us denote by $\mathbb{P}_+(\Lambda^*)$ the set of all equivalence classes with respect to the following binary relation:

$$\varphi_1 \backsim \varphi_2 \Longleftrightarrow \exists\ t \in \mathbb{R} \setminus \{0\} \ : \ \varphi_1 = t\varphi_2, \ \varphi_1, \varphi_2 \in \Lambda^*.$$

Let $\Pi^* : \Lambda^* \setminus 0 \to \mathbb{P}_+(\Lambda^*)$ be the corresponding canonical quotient mapping. We assume that $\mathbb{P}_+(\Lambda^*)$ is endowed with the quotient topology. It is clear that, in this case, the mapping

$$\Pi^*|_{S_1^*\cap\Lambda^*} : S_1^* \cap \Lambda^* \to \mathbb{P}_+(\Lambda^*)$$

is a continuous surjection, that is, every equivalence class can be interpreted as the image of some element of Λ^* belonging to the unit sphere S_1^* in $\mathbb{Z}^*$. Hence, if $F(v, \xi) \notin \Lambda$ for some pair $(v, \xi) \in U_\partial \times \mathbb{Y}$, then there is an element $\psi \in S_1^* \cap \Lambda^*$ such that $\langle\psi, F(v, \xi)\rangle_{\mathbb{Z}^*;\mathbb{Z}} < 0$. Taking into account these observations, we introduce the following penalized version of the original problem (3.14)–(3.17):

$$I_\varepsilon(u, y) = I(u, y) + \varepsilon^{-1} \sup_{\psi\in S_1^*\cap\Lambda^*} \left[\mu\left(\langle\psi, F(u, y)\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right)\right] \to \inf, \tag{3.24}$$

$$A(u, y) = f, \tag{3.25}$$

$$y \in K, \quad u \in U_\partial \subset \mathbb{U}, \tag{3.26}$$

where $\mu : \mathbb{R} \to \mathbb{R}_+$ is a lower semicontinuous monotone decreasing function such that $\mu(0) = 0$ and μ is strictly monotone on $\mathbb{R}_-$. We denote by $\mathcal{F}(\mathbb{R}, \mathbb{R}_+)$ the set of all functions μ with the properties mentioned above.

To begin with, we assert the following results.

Lemma 3.10. *For every $\varepsilon > 0$ and every fixed $\mu \in \mathcal{F}(\mathbb{R}, \mathbb{R}_+)$, the problem (3.24)–(3.26) has a nonempty set of solutions.*

Proof. Let $\overline{\Xi}$ be a set of admissible solutions to the penalized problem (3.24)–(3.26). It is clear that $\Xi \subset \overline{\Xi}$ for every $\varepsilon > 0$. First, we show that the cost functional $I_\varepsilon : U_\partial \times \mathbb{Y} \to \overline{\mathbb{R}}$ is bounded below on the set $\overline{\Xi}$. Let us assume the converse. Then there exists a sequence $\{(u_k, y_k)\}_{k=1}^{\infty} \subset \overline{\Xi}$ such that $I_\varepsilon(u_k, y_k) < -k$ for all $k \in \mathbb{N}$. Due to the initial assumptions, we have $\{u_k\}_{k=1}^{\infty} \subset U_\partial$; hence, the sequence $\{u_k\}_{k=1}^{\infty}$ is bounded in $\mathbb{U}$. Thus, we may assume that $u_k \overset{*}{\rightharpoonup} u$ in $\mathbb{U}$ and $u \in U_\partial$. Since

$$\langle A(u_k, y_k), y_k \rangle_{\mathbb{Y}^*;\mathbb{Y}} = \langle f, y_k \rangle_{\mathbb{Y}^*;\mathbb{Y}} \le \|f\|_{\mathbb{Y}^*} \|y_k\|_{\mathbb{Y}}, \quad \forall\, k \in \mathbb{N}$$

and the operator A is coercive (see (A2)), it follows that

$$\sup_{k \in \mathbb{N}} \frac{\langle A(u_k, y_k), y_k \rangle_{\mathbb{Y}^*;\mathbb{Y}}}{\|y_k\|_{\mathbb{Y}}} \le \|f\|_{\mathbb{Y}^*}.$$

Hence, $\{y_k\}_{k=1}^{\infty} \subset K$ is a bounded sequence in $\mathbb{Y}$ and there exists an element $y \in K$ such that, passing to a subsequence if necessary, we obtain $y_k \rightharpoonup y$ in $\mathbb{Y}$. Then, having used the sequential lower semicontinuity of I with respect to the w-convergence and the non-negativeness of the term

$$\sup_{\psi \in S_1^* \cap \Lambda^*} \left[\mu \left(\langle \psi, F(u, y) \rangle_{\mathbb{Z}^*;\mathbb{Z}} \right) \right],$$

we come to the contradiction

$$I(u, y) \le \liminf_{k \to \infty} I(u_k, y_k) \le \liminf_{k \to \infty} I_\varepsilon(u_k, y_k) < -\infty.$$

Thus, the cost functional $I_\varepsilon : U_\partial \times \mathbb{Y} \to \overline{\mathbb{R}}$ is bounded below on the set $\overline{\Xi}$.

Let $\{(u_k, y_k)\}_{k=1}^{\infty} \subset \overline{\Xi}$ be a minimizing sequence of admissible pairs to the problem (3.24)–(3.26). By the previous arguments, this sequence is bounded in $\mathbb{U} \times \mathbb{Y}$. Since $\overline{\Xi} \subset U_\partial \times K$ and the set $U_\partial \times K$ is sequentially closed with respect to the w-convergence, we may assume that there exists a pair $(u_\varepsilon^0, y_\varepsilon^0) \in U_\partial \times K$ such that $(u_k, y_k) \overset{w}{\to} (u_\varepsilon^0, y_\varepsilon^0)$. Then, taking into account that $A(u_k, y_k) = f$ for all $k \in \mathbb{N}$ and passing to the limit in the equality

$$\langle A(u_k, y_k), y_k \rangle_{\mathbb{Y}^*;\mathbb{Y}} = \langle f, y_k \rangle_{\mathbb{Y}^*;\mathbb{Y}}$$

as $k \to \infty$, we obtain

$$\lim_{k \to \infty} \langle A(u_k, y_k), y_k \rangle_{\mathbb{Y}^*;\mathbb{Y}} = \left\langle f, y_\varepsilon^0 \right\rangle_{\mathbb{Y}^*;\mathbb{Y}}.$$

Hence, $A(u_\varepsilon^0, y_\varepsilon^0) = f$ by the $(\mathfrak{M})$ property of the operator $A : \mathbb{U} \times \mathbb{Y} \to \mathbb{Y}^*$. Thus, the limit pair $(u_\varepsilon^0, y_\varepsilon^0)$ is admissible for the problem (3.24)–(3.26) (i.e., $(u_\varepsilon^0, y_\varepsilon^0) \in \overline{\Xi}$).

Let us show that $(u_\varepsilon^0, y_\varepsilon^0) \in \overline{\Xi}$ is an optimal pair to this problem. Indeed, in view of the continuity property of the operator $F : \mathbb{U} \times \mathbb{Y} \to \mathbb{Z}$, we have $F(u_k, y_k) \rightharpoonup F(u_\varepsilon^0, y_\varepsilon^0)$ in $\mathbb{Z}$. Since the cone Λ is reproducing, it follows that the elements of conjugate semigroup Λ^* are linear continuous functionals on $\mathbb{Z}$. Hence,

$$\langle \psi, F(u_k, y_k)\rangle_{\mathbb{Z}^*;\mathbb{Z}} \to \langle \psi, F(u_\varepsilon^0, y_\varepsilon^0)\rangle_{\mathbb{Z}^*;\mathbb{Z}} \quad \text{as} \quad k \to \infty \quad \forall\, \psi \in \lambda^*.$$

By our assumptions, $\mathbb{Z} \ni \xi \mapsto \mu\left(\langle \psi, \xi\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right) \in \mathbb{R}$ is a convex lower semicontinuous function. Hence, this function is semi-continuous with respect to the weak topology of $\mathbb{Z}$. Therefore,

$$\begin{aligned}\inf_{(u,y)\in\overline{\Xi}} I_\varepsilon(u, y) &= \liminf_{k\to\infty} I_\varepsilon(u_k, y_k)\\ &\geq I(u_\varepsilon^0, y_\varepsilon^0) + \varepsilon^{-1} \liminf_{k\to\infty} \sup_{\psi\in S_1^*\cap\Lambda^*} \left[\mu\left(\langle \psi, F(u_k, y_k)\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right)\right]\\ &\geq I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0),\end{aligned}$$

and we obtain the required conclusion: $(u_\varepsilon^0, y_\varepsilon^0)$ is an optimal pair to the penalized problem (3.24)–(3.26). □

Lemma 3.11. *Let $\left\{(u_\varepsilon^0, y_\varepsilon^0) \in \overline{\Xi}\right\}_{\varepsilon>0}$ be a sequence of optimal pairs to the problem (3.24)–(3.26) when the parameter $\varepsilon > 0$ varies in a strictly decreasing sequence of positive numbers which converge to 0. Then there is a subsequence of $\left\{(u_\varepsilon^0, y_\varepsilon^0)\right\}_{\varepsilon>0}$, still denoted by ε, such that*

$$(u_\varepsilon^0, y_\varepsilon^0) \overset{w}{\to} (u^0, y^0) \quad \text{in} \quad \mathbb{U} \times \mathbb{Y} \quad \text{as} \quad \varepsilon \to 0, \tag{3.27}$$

$$A(u^0, y^0) = f, \quad F(u^0, y^0) \geq_\Lambda 0, \quad y^0 \in K, \quad u^0 \in U_\partial \subset \mathbb{U}, \tag{3.28}$$

$$I(u^0, y^0) = \inf_{(u,y)\in\Xi} I(u, y). \tag{3.29}$$

Proof. In the same way as in the proof of Lemma 3.10, we can conclude that the sequence $\left\{(u_\varepsilon^0, y_\varepsilon^0) \in \overline{\Xi}\right\}_{\varepsilon>0}$ is relatively w-compact in $\mathbb{U} \times \mathbb{Y}$ and, passing to a subsequence when the occasion requires, we get

$$u_\varepsilon^0 \overset{*}{\rightharpoonup} u^0 \quad \text{in} \quad \mathbb{U}, \quad y_\varepsilon^0 \rightharpoonup y^0 \quad \text{in} \quad \mathbb{Y}, \quad (u^0, y^0) \in \overline{\Xi}.$$

Let us prove that $F(u^0, y^0) \geq 0$. Let (u, y) be any admissible pair to the original problem, that is, $(u, y) \in \Xi$. Then, by our assumptions,

$$\mu\left(\langle \psi, F(u, y)\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right) = 0.$$

Therefore,

$$I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) \leq I_\varepsilon(u, y) \equiv I(u, y),$$

from which it follows that

$$\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\left\langle\psi,F(u_\varepsilon^0,y_\varepsilon^0)\right\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right)\right]\le\varepsilon C,$$

where a constant C is independent of both ε and $\psi\in S_1^*\cap\Lambda^*$. Then, using the w-lower semicontinuity property, we have

$$\liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^0,y_\varepsilon^0)\ge\liminf_{\varepsilon\to 0} I(u_\varepsilon^0,y_\varepsilon^0)\ge I(u^0,y^0),$$

and

$$\begin{aligned}0&\le\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\left\langle\psi,F(u^0,y^0)\right\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right)\right]\\&\le\liminf_{\varepsilon\to 0}\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\left\langle\psi,F(u_\varepsilon^0,y_\varepsilon^0)\right\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right)\right]=0.\end{aligned}$$

Since this is equivalent to the inequality $F(u^0,y^0)\ge 0$, it follows that the limit pair (u^0,y^0) is admissible for the original problem (3.14)–(3.17).

It remains to prove that this pair is optimal. Let us assume the converse – namely, suppose there is a pair $(\widehat{u},\widehat{y})\in\Xi$ such that $I(\widehat{u},\widehat{y})<I(u^0,y^0)$. Then this pair is admissible to the penalized problem (i.e., $(\widehat{u},\widehat{y})\in\overline{\Xi}$). Hence,

$$\begin{aligned}I(\widehat{u},\widehat{y})\equiv I_\varepsilon(\widehat{u},\widehat{y})&\ge\inf_{(u,y)\in\overline{\Xi}} I_\varepsilon(u,y)=I_\varepsilon(u_\varepsilon^0,y_\varepsilon^0)\quad\forall\,\varepsilon>0,\\I(\widehat{u},\widehat{y})&\ge\liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^0,y_\varepsilon^0)\ge I(u^0,y^0),\end{aligned}$$

and it leads us to the contradiction. The proof is complete. □

As a direct consequence of the previous lemmas, we have the following result.

Theorem 3.12. *Assume that the properties (A1)–(A6) hold true. Then the set of optimal solutions to the problem (3.14)–(3.17) is nonempty if and only if this problem is regular.*

Remark 3.13. Note that Theorem 3.12 is still true if instead of assumption (A3), we require that the operator $A:\mathbb{U}\times\mathbb{Y}\to\mathbb{Y}^*$ is quasi-monotone. We recall that an operator $A:\mathbb{U}\times\mathbb{Y}\to\mathbb{Y}^*$ is said to be quasi-monotone if for any sequence $\{(u_k,y_k)\}_{k=1}^\infty$ which is w-convergent to some pair (u,y), the condition

$$\limsup_{k\to\infty}\langle A(u_k,y_k),y_k-y\rangle_{\mathbb{Y}^*,\mathbb{Y}}\le 0 \tag{3.30}$$

implies the relation

$$\liminf_{k\to\infty}\langle A(u_k,y_k),y_k-\xi\rangle_{\mathbb{Y}^*,\mathbb{Y}}\ge\langle A(u,y),y-\xi\rangle_{\mathbb{Y}^*,\mathbb{Y}}\quad\forall\,\xi\in\mathbb{Y}. \tag{3.31}$$

Indeed, let us prove the implication: "A is quasi-monotone" $\Longrightarrow$ "A possesses the property $(\mathfrak{M})$." Let $\{(u_k,y_k)\}_{k=1}^\infty$ be a sequence such that

$$(u_k, y_k) \overset{w}{\rightharpoonup} (u, y),\ A(u_k, y_k) \rightharpoonup \zeta \ \text{ in } \ \mathbb{Y}^*,$$
$$\limsup_{k\to\infty} \langle A(u_k, y_k), y_k\rangle_{\mathbb{Y}^*,\mathbb{Y}} \le \langle \zeta, y\rangle_{\mathbb{Y}^*,\mathbb{Y}}\,.$$

This immediately leads us to the inequality (3.30). Hence, by quasi-monotonicity, we have

$$\begin{aligned}\langle A(u,y), y-\xi\rangle_{\mathbb{Y}^*,\mathbb{Y}} &\le \liminf_{k\to\infty} \langle A(u_k,y_k), y_k-\xi\rangle_{\mathbb{Y}^*,\mathbb{Y}}\\ &\le \limsup_{k\to\infty} \langle A(u_k,y_k), y_k-\xi\rangle_{\mathbb{Y}^*,\mathbb{Y}}\\ &\le \langle \zeta, y-\xi\rangle_{\mathbb{Y}^*,\mathbb{Y}} \quad \forall\, \xi\in\mathbb{Y}.\end{aligned}$$

Thus, $A(u,y)=\zeta$, and we come to the conclusion that quasi-monotone operators A possess the property $(\mathfrak{M})$.

3.3 Extremal problems for steady-state processes

In this section, we consider the application of the previous results to OCPs governed by elliptic PDEs. We show that there is a principle difference between the theory of OCPs for systems with distributed parameters and the theory of PDEs.

Before we give the formal settings of various OCPs, we note that, for both physical and mathematical reasons, the size of the admissible controls should be constrained and that the cost of effecting controls should be accounted for in the optimization process. Limits on the size of the control are also needed in order to obtain a mathematically meaningful problem (e.g., to guarantee the existence of an optimal solution in a certain function class). There are two common ways of constraining the control. The first one is to impose an explicit bound on the control — $u \in U_\partial$ in our case. The second way of constraining the control is to add some norm of the control to the cost functional. Both ways of constraining the control allow one to use different norms to measure the control. The physical problem does not always tell us which norm to use. The choice of a norm is also influenced by the need to establish the well-posedness of the problem (i.e., the existence of an optimal solution in some functional class, the regularity of the optimal solution, etc.).

3.3.1 Dirichlet and Neumann boundary control problems

We begin this subsection with the following optimal boundary control problem. Let Ω be an open bounded domain in $\mathbb{R}^n$ with a Lipschitz continuous boundary $\partial\Omega$. Let $z_\partial \in L^2(\Omega)$, $f \in L^2(\Omega)$ and $N > 0$ be given data. Let $U_\partial \subset L^2(\partial\Omega)$ be a convex closed set of admissible Neumann controls. The OCP is to minimize the cost functional

$$I(u,y) = \int_\Omega |y(x) - z_\partial(x)|^2\,\mathrm{d}x + N\int_{\partial\Omega} |u(s)|^2\,\mathrm{d}\mathcal{H}^{n-1} \tag{3.32}$$

subject to

$$\triangle y(x) = f(x), \quad x \in \Omega, \tag{3.33}$$

$$\partial y/\partial \nu = u(s), \quad s \in \partial\Omega, \quad u \in U_\partial, \tag{3.34}$$

where $\triangle = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplace operator and ν is the unit outward normal vector to Ω. From a physical point of view, we want to find a boundary control $u \in U_\partial$ such that the corresponding state y is as close as possible to a desired state z_∂ and with minimal cost.

The typical examples of the sets of admissible controls U_∂ with the above properties are

$$U_\partial^1 = \left\{ u \in L^2(\partial\Omega) \ : \ \|u\|^2_{L^2(\partial\Omega)} \le R^2 \right\},$$

$$U_\partial^2 = \left\{ u \in L^2(\partial\Omega) \ : \ u \ge 0 \text{ almost everywhere on } \partial\Omega \right\},$$

$$U_\partial^3 = \left\{ u \in L^2(\partial\Omega) \ : \ \alpha(x) \le u \le \beta(x) \text{ almost everywhere on } \partial\Omega \right\}.$$

Note that if we choose the control variable u in the form

$$u(x) = \frac{\alpha(x) + \beta(x)}{2} + \frac{\beta(x) - \alpha(x)}{2} \sin v(x), \tag{3.35}$$

where v is a "free" control mapping and replace (3.35) in the boundary value problem (3.33)–(3.34), we obtain a new minimization problem where the control constraints expressed by U_∂^3 are automatically satisfied. Such techniques (which are basically due to Miele; see [239]) may be easily implemented in a large class of control constraints. A thorough exposition along these lines may be found in the book by Banichuk [18].

In order to apply the direct method of the calculus of variations (see Theorem 3.2) to the optimal boundary control problem (3.32)–(3.34), we make use of the space

$$H_\Delta^s(\Omega) = \left\{ y \in H^s(\Omega) \ : \ \triangle y \in L^2(\Omega) \right\}, \quad s \in [0, 2), \tag{3.36}$$

and endow $H_\Delta^s(\Omega)$ with the norm

$$\|y\|_{H_\Delta^s(\Omega)} = \left(\|y\|^2_{H^s(\Omega)} + \|\triangle y\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Remark 3.14. It is worth noticing that, in this case (see Lions and Magenes [173]), the trace operator $y \mapsto (y|_{\partial\Omega}, \partial y/\partial\nu|_{\partial\Omega})$ is well defined and continuous as the mapping

$$\Pi : H_\Delta^s(\Omega) \to H^{s-1/2}(\partial\Omega) \times H^{s-3/2}(\partial\Omega). \tag{3.37}$$

Moreover, assume that $f \in L^2(\Omega)$ and $g \in H^{s-1-1/2}(\partial\Omega)$ satisfy the compatibility condition (2.30), where the second integral in (2.30) should be interpreted as $\langle g, 1 \rangle_{H^{s-1-1/2}(\partial\Omega), H^{-s+1+1/2}(\partial\Omega)}$ in the case when $s - 1 - 1/2 < 0$.

Then the Neumann problem (2.28) has a unique solution y in the quotient space $H^s_\Delta(\Omega)/\mathbb{R}$ with the estimate

$$\|\widehat{y}\|^2_{H^s_\Delta(\Omega)} \le C\left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{H^{s-1-1/2}(\partial\Omega)} + \|\widehat{y}\|^2_{L^2(\Omega)}\right) \tag{3.38}$$

for all $\widehat{y}$ such that $\widehat{y} - y$ is a constant.

Taking the compatibility condition to the boundary value problem (3.33)–(3.34) into account, we set

$$L_f = \left\{ u \in L^2(\partial\Omega) \ : \ \int_{\partial\Omega} u(s)\,\mathrm{d}\mathcal{H}^{n-1} = \int_\Omega f(x)\,\mathrm{d}x \right\} \tag{3.39}$$

and prove the following result.

Theorem 3.15. *Assume that $U_\partial \cap L_f \neq \emptyset$. Then the optimal Neumann boundary control problem (3.32)–(3.34) admits a unique solution $(u^0, y^0) \in L^2(\partial\Omega) \times H^{3/2}_\Delta(\Omega)$.*

Proof. To prove this assertion, we make use of Theorem 3.2. With this aim, we set

$$\mathbb{Y} = L^2(\Omega), \quad \mathbb{Y}_1 = H^{3/2}_\Delta(\Omega), \quad \mathbb{U} = L^2(\partial\Omega),$$
$$\mathbb{Z} = (L^2(\Omega), L^2(\partial\Omega)), \quad L(u,y) = (\triangle y, \partial y/\partial\nu - u), \quad \text{and} \quad F(y) = (-f, 0).$$

Then $\mathbb{Y}_1$ is continuously and densely embedded in $\mathbb{Y}$ and, as follows from Remark 3.14, the operator $L : \mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z}$ is continuous. Let us show that, in this case, the hypotheses (H1)–(H2) are valid.

Indeed, as was indicated in Remark 3.14, for every fixed $f \in L^2(\Omega)$ and $u \in L^2(\partial\Omega)$ the boundary value problem (3.33)–(3.34) has a solution $y \in H^{3/2}_\Delta(\Omega)$, provided

$$\int_{\partial\Omega} u(s)\,\mathrm{d}\mathcal{H}^{n-1} - \int_\Omega f(x)\,\mathrm{d}x = 0.$$

Moreover, this solution satisfies the estimate (see (3.38) with $s = 3/2$)

$$\|y\|^2_{H^{3/2}_\Delta(\Omega)} + \|y\|^2_{L^2(\Omega)} \le C\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)} + \|y\|^2_{L^2(\Omega)}\right), \tag{3.40}$$

where a constant C is independent of f and u.

Since $U_\partial \cap L_f \neq \emptyset$, it follows that there is an admissible control $u^* \in U_\partial$ such that $u^* \in L_f$. As a result, the boundary value problem (3.33)–(3.34) is solvable under $u = u^*$. Let $y^* \in H^{3/2}_\Delta(\Omega)$ be a solution. Then the pair (u^*, y^*) is admissible to the problem (3.32)–(3.34). Thus, hypothesis (H1) is satisfied. To verify the coerciveness condition (H2), we make use of the estimate (3.40). Then, for any admissible pair $(u, y) \in \Xi \subset L^2(\partial\Omega) \times H^{3/2}_\Delta(\Omega)$, we have

$$\begin{aligned}\|(u,y)\|^2_{\mathbb{U}\times\mathbb{Y}_1} &:= \|y\|^2_{H^s(\Omega)} + \|\triangle y\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}\\ &\le (1+C)\left[\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)} + \|y\|^2_{L^2(\Omega)}\right]\\ &\le C_1 I(u,y) + C_2,\end{aligned}$$

where the constants C_1 and C_2 are independent of (u,y). Therefore, the property (H2) holds true. So, to conclude the proof, it remains to apply Theorem 3.2. □

In contrast to the above problem, for given functions $z_\partial \in L^2(\partial\Omega)$ and $f \in L^2(\Omega)$, let us consider the following Dirichlet boundary control problem

$$I(u,y) = \int_{\partial\Omega}\left|\frac{\partial y(s)}{\partial\nu} - z_\partial(s)\right|^2 \mathrm{d}\mathcal{H}^{n-1} + N\int_{\partial\Omega}|u(s)|^2\,\mathrm{d}\mathcal{H}^{n-1} \longrightarrow \inf, \tag{3.41}$$

$$\triangle y(x) = f(x), \quad x \in \Omega, \tag{3.42}$$

$$y|_{\partial\Omega} = u, \quad u \in U_\partial, \tag{3.43}$$

where $N > 0$, $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative to $\partial\Omega$ and U_∂ is a convex closed subset of $L^2(\partial\Omega)$. The distinguishing feature of this problem is the fact that the cost functional (3.41) includes the boundary observations of the outward normal derivative of the state system.

Theorem 3.16. *Assume that $U_\partial \cap H^1(\partial\Omega) \neq \emptyset$. Then the optimal boundary control problem (3.41)–(3.43) admits a unique solution $(u^0, y^0) \in L^2(\partial\Omega) \times H^{3/2}_\Delta(\Omega)$.*

Proof. Let us set

$$\mathbb{Y} = \mathbb{Y}_1 = H^{3/2}_\Delta(\Omega), \quad \mathbb{U} = L^2(\partial\Omega),$$
$$\mathbb{Z} = (L^2(\Omega), L^2(\partial\Omega)),\ L(u,y) = (\triangle y, y|_{\partial\Omega} - u),\ F(y) = (-f, 0).$$

It is clear that the operator $L : \mathbb{U}\times\mathbb{Y}_1 \to \mathbb{Z}$ is continuous. In addition, due to Remark 3.14, the operator $\partial_\nu : H^{3/2}_\Delta(\Omega) \to L^2(\partial\Omega)$ is continuous. Hence, the cost functional $I : L^2(\partial\Omega)\times H^{3/2}_\Delta(\Omega) \to \mathbb{R}$ is continuous as well.

To verify hypothesis (H1), we note that the Dirichlet boundary value problem (3.42)–(3.43) has a unique solution $y \in H^{3/2}_\Delta(\Omega)$ for every $u \in H^1(\partial\Omega)$ (see [173]). Since $U_\partial \cap H^1(\partial\Omega) \neq \emptyset$ by our assumptions, it follows that the set of admissible pairs Ξ is nonempty in this case. Thus, regularity property (H1) is valid.

Let $(u,y) \in L^2(\partial\Omega)\times H^{3/2}_\Delta(\Omega)$ be any admissible pair. Then, using the estimate (3.40), we can write

$$\|y\|^2_{H^{3/2}_\Delta(\Omega)} \le C_1\left(\|f\|^2_{L^2(\Omega)} + \|\partial_\nu y\|^2_{L^2(\partial\Omega)} + \|y\|^2_{L^2(\Omega)}\right). \tag{3.44}$$

On the other hand, for a given pair (u, y), related to the Dirichlet boundary value problem (3.42)–(3.43), we have the estimate (see [173])

$$\|y\|^2_{H^{1/2}(\Omega)} + \|\triangle y\|^2_{L^2(\Omega)} \le C_2 \left[\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}\right].$$

Since the embedding $H^{1/2}(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, we apply the last estimate to the term $\|y\|^2_{L^2(\Omega)}$ on the right-hand side of inequality (3.44). As a result, we get

$$\begin{aligned}\|y\|^2_{H^{3/2}_{\Delta}(\Omega)} + \|u\|^2_{L^2(\partial\Omega)} &\le C\left[\|f\|^2_{L^2(\Omega)} + \|\partial_\nu y\|^2_{L^2(\partial\Omega)} + \|u\|^2_{L^2(\partial\Omega)}\right] \\ &\le C_3 I(u, y) + C_4.\end{aligned}$$

Thus, the coerciveness property (H2) is also valid. So, by Theorem 3.2, the optimal boundary control problem (3.41)–(3.43) has a nonempty set of solutions. However, the cost functional (3.41) is not strictly convex on $L^2(\partial\Omega)\times H^{3/2}_{\Delta}(\Omega)$. Therefore, to prove the uniqueness of an optimal solution, we assume the converse – namely, suppose the original problem admits two different solutions: $(u^0_1, y^0_1) \in \Xi$ and $(u^0_2, y^0_2) \in \Xi$, respectively. Then

$$\partial_\nu y^0_1\big|_{\partial\Omega} = \partial_\nu y^0_2\big|_{\partial\Omega}, \quad \text{and} \quad u^0_1 = u^0_2.$$

Since $u^0_i = y^0_i\big|_{\partial\Omega}$, it implies that $y^0_1\big|_{\partial\Omega} = y^0_2\big|_{\partial\Omega}$. As a result, for the function $z = y^0_1 - y^0_2 \in H^{3/2}_{\Delta}(\Omega)$, we have

$$\triangle z(x) = 0 \ \text{ in } \ \Omega, \ \ z|_{\partial\Omega} = 0, \ \ \partial_\nu z|_{\partial\Omega} = 0. \tag{3.45}$$

Hence, $z \in C^\infty(\overline{\Omega})$ (see [173]) and $z(x) \equiv 0$ as a unique solution of the Cauchy problem (3.45) (see [189]). Thus, $y^0_1 \equiv y^0_2$ and $u^0_1 \equiv u^0_2$. This concludes the proof. □

The last OCP that we are going to consider in this subsection can be stated as follows. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain with a Lipschitz boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two disjoint closed subsets and Γ_1 has a positive Hausdorff measure $\mathcal{H}^{n-1}$. For given functions $z_\partial \in L^2(\Omega)$ and $f \in L^2(\Omega)$, we consider the following problem

$$I(u, y) = \int_\Omega |y(x) - z_\partial(x)|^2 \,\mathrm{d}x + N \int_{\Gamma_0} |u(s)|^2 \,\mathrm{d}\mathcal{H}^{n-1} \longrightarrow \inf, \tag{3.46}$$

$$\triangle y(x) = f(x), \quad x \in \Omega, \tag{3.47}$$

$$y|_{\Gamma_0} = u, \quad y|_{\Gamma_1} = 0, \quad u \in U_\partial, \tag{3.48}$$

where $N > 0$ and U_∂ is a convex closed subset of $L^2(\Gamma_0)$.

Having taken

$$\mathbb{Y} = L^2(\Omega), \quad \mathbb{Y}_1 = \left\{ y \in H_\Delta^{1/2}(\Omega) \ : \ y|_{\Gamma_1} = 0 \right\},$$
$$\mathbb{U} = L^2(\Gamma_0), \quad \mathbb{Z} = L^2(\Omega) \times L^2(\Gamma_0),$$
$$L(u, y) = (\triangle y, y|_{\Gamma_0} - u), \quad F = (-f, 0)$$

and applying Theorem 3.2 (see also Remark 3.14), we come to the following conclusion.

Theorem 3.17. *There exists a unique solution* $(u^0, y^0) \in L^2(\Gamma_0) \times H_\Delta^{1/2}(\Omega)$ *to the optimal boundary control problem (3.46)–(3.48).*

3.3.2 Ill-posed control objects

This subsection deals with the OCPs for incorrect (ill-posed) boundary value problems. However, as we will see later, there are extremal problems for such objects which have a particular sense and they are well-posed. To begin with, we consider the following OCP with distributed controls:

$$I(u, y) = \int_\Omega |y(x) - z_\partial(x)|^2 \, \mathrm{d}x + N \int_\Omega |u(x)|^2 \, \mathrm{d}x \longrightarrow \inf, \tag{3.49}$$
$$\triangle y(x) = u(x) \ \text{ in } \ \Omega, \quad u \in U_\partial, \tag{3.50}$$

where $z_\partial \in L^2(\Omega)$ is a given function, $N > 0$, U_∂ is a nonempty convex closed subset of $L^2(\Omega)$. Since there are no boundary conditions for the control system (3.50), it means that this is an underdetermined problem and hence, this problem is ill-posed. Nevertheless, we have the following result concerning the correctness of the corresponding OCP.

Theorem 3.18. *The OCP (3.49)–(3.50) admits a unique solution* $(u^0, y^0) \in L^2(\Omega) \times L^2(\Omega)$.

Proof. To make use of Theorem 3.2, we set $\mathbb{Y} = \mathbb{Y}_1 = \mathbb{U} = L^2(\Omega)$, $\mathbb{Z} = H^{-2}(\Omega)$, $F = 0$, and $L(u, y) = \triangle y - u$. It is easy to see that, in this case, all suppositions of Theorem 3.2 hold true with the exception of hypothesis (H1). However, if we take u as an arbitrary admissible control (i.e., $u \in U_\partial$) and $y = y(u)$ as the corresponding solution of the Dirichlet problem

$$\triangle y(x) = u(x) \ \text{ in } \ \Omega, \quad y|_{\partial\Omega} = 0,$$

then $y \in H^2(\Omega) \cap H_0^1(\Omega)$ (see Theorem 2.20). Hence, $(u, y) \in \Xi$, and we obtain the required conclusion. □

The another example of an ill-posed extremal problem can be stated as follows:

$$I(u, y) = \int_\Omega |y(x) - z_\partial(x)|^2 \, \mathrm{d}x + N \int_\Omega |u(x)|^2 \, \mathrm{d}x \longrightarrow \inf, \tag{3.51}$$
$$\triangle y(x) = u(x) \ \text{ in } \ \Omega, \ y|_{\partial\Omega} = 0, \ \partial_\nu y|_{\partial\Omega} = 0 \quad u \in U_\partial, \tag{3.52}$$

where $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative to $\partial\Omega$. As is obvious, the boundary value problem (3.52) is an overdetermined problem; hence, this problem is ill-posed as well.

Following Fursikov (see [111]), we introduce the following set:

$$\Delta H_0^2(\Omega) = \left\{\triangle z \ : \ z \in H_0^2(\Omega)\right\}.$$

Theorem 3.19. *Under the assumption that $U_\partial \cap \Delta H_0^2(\Omega) \neq \emptyset$, the OCP (3.51)–(3.52) has a unique solution $(u^0, y^0) \in L^2(\Omega) \times H_0^2(\Omega)$.*

Proof. Let us set $\mathbb{Y} = \mathbb{Z} = \mathbb{U} = L^2(\Omega)$, $\mathbb{Y}_1 = H_0^2(\Omega)$, $F = 0$ and $L(u, y) = \triangle y - u$. Then the condition $U_\partial \cap \Delta H_0^2(\Omega) \neq \emptyset$ immediately implies the fulfillment of the regularity assumption (H1). To apply Theorem 3.2, it remains only to verify the coerciveness property (H2). In order to do it, we note that the set of admissible pairs Ξ to the problem (3.51)–(3.52) can be represented in the form

$$\begin{aligned}\Xi &:= \left\{(u, y) \in L^2(\Omega) \times H_0^2(\Omega) \ : \ \triangle y = u \text{ in } \ \Omega, \ u \in U_\partial\right\} \\ &\subset \left\{(u, y) \in L^2(\Omega) \times \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \ : \ \triangle y = u, \ u \in U_\partial\right\} =: \Xi_1. \quad (3.53)\end{aligned}$$

However, in view of the a priori estimate (2.21), we have

$$\|y\|^2_{H^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \leq (C + 1)\|u\|^2_{L^2(\Omega)} \leq C_1 I(u, y) + C_2.$$

Hence, the set $\{(u, y) \in \Xi_1 \ : \ I(u, y) \leq R\}$ is bounded in $L^2(\Omega) \times H^2(\Omega)$ for every $R > 0$. So, taking into account inclusion (3.53), we come to the same conclusion with respect to the set Ξ. This concludes the proof. □

Thus, the examples of OCPs cited above illustrate the fundamental difference between the notion of well-posedness of the boundary value problems and the corresponding OCPs.

3.3.3 Optimal control of the Cauchy problem for an elliptic equation

Let Ω be a bounded connected open domain with a smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two disjoint closed sets, both of positive measure and $\Gamma_1 \cap \Gamma_2 = \emptyset$. We consider the following control problem:

$$\begin{aligned}I(u_1, u_2, y) = \|y|_{\partial\Omega} - z_\partial\|^2_{L^2(\Gamma_2)} \\ + \frac{N_1}{2}\|u_1 - f_1\|^2_{L^2(\Gamma_1)} + \frac{N_2}{2}\|u_2 - f_2\|^2_{L^2(\Gamma_1)} \longrightarrow \inf, \quad (3.54)\end{aligned}$$

$$\triangle y(x) = 0 \ \text{ in } \ \Omega, \ y|_{\Gamma_1} = u_1, \ \partial_\nu y|_{\Gamma_1} = u_2, \quad (3.55)$$

$$u = (u_1, u_2) \in U_\partial, \quad (3.56)$$

where $z_\partial \in L^2(\Gamma_2)$, $f_i \in L^2(\Gamma_1)$, $i = 1, 2$, are given functions, $N_1 > 0$, $N_2 > 0$, U_∂ is a convex closed subset of $L^2(\Gamma_1) \times L^2(\Gamma_1)$.

By analogy with [111], we define the space

$$\mathbb{Y}_1 = \left\{ y \in H^{1/2}_\Delta(\Omega) \ : \ \partial_\nu y|_{\Gamma_1} \in L^2(\Gamma_1) \right\} \tag{3.57}$$

endowed with the norm

$$\|y\|^2_{\mathbb{Y}_1} = \|y\|^2_{H^{1/2}_\Delta(\Omega)} + \| \partial_\nu y|_{\Gamma_1} \|_{L^2(\Gamma_1)}.$$

Let us show that

$$y \in \mathbb{Y}_1 \quad \text{implies} \quad y|_{\Gamma_1} \in H^1(\Gamma_1). \tag{3.58}$$

Indeed, let φ be a smooth function of $C^\infty(\overline{\Omega})$ such that $\varphi(x) = 1$ in a neighborhood of the manifold Γ_1 and $\varphi(x) = 0$ in a neighborhood of Γ_2. Assume that a function $y \in \mathbb{Y}_1$ is such that $\triangle y(x) = 0$ in Ω. Then $\triangle(\varphi y(x)) \in L^2(\Omega)$ and $y \in H^2(\Omega_1)$ for any subset Ω_1 of Ω satisfying the condition $\overline{\Omega_1} \subset \Omega$. In addition, in view of the properties of φ, we have

$$\partial_\nu(\varphi y)|_{\Gamma_2} = 0 \quad \text{and} \quad \partial_\nu(\varphi y)|_{\Gamma_1} = \partial_\nu y|_{\Gamma_1} \in L^2(\Gamma_1).$$

Then, in view of Remark 3.14 (see also [173]), we can conclude $\varphi y \in H^{3/2}_\Delta(\Omega)$. Thereby, by the trace theorem, we just obtain the desired property:

$$y|_{\Gamma_1} \in H^1(\Gamma_1).$$

We are now in a position to prove the main result of this subsection.

Theorem 3.20. *Assume that there is a triplet (u_1, u_2, y) such that $(u_1, u_2) \in U_\partial$ and $y \in \mathbb{Y}_1$ satisfies the conditions (3.55). Then the OCP (3.54)–(3.56) has a unique solution $(u_1^0, u_2^0, y^0) \in H^1(\Gamma_1) \times L^2(\Gamma_2) \times \mathbb{Y}_1$.*

Proof. To make use of Theorem 3.2, we set

$$\mathbb{Y} = \mathbb{Y}_1, \quad \mathbb{U} = L^2(\Gamma_1) \times L^2(\Gamma_1), \quad \mathbb{Z} = L^2(\Gamma_1) \times L^2(\Gamma_1) \times L^2(\Omega),$$
$$L(u, y) = (\triangle y, \, y|_{\Gamma_1} - u_1, \, \partial_\nu y|_{\Gamma_1} - u_2), \quad \text{and} \quad F(y) = (0, 0, 0), \ \forall\, y \in \mathbb{Y}_1.$$

In view of the initial suppositions, the problem (3.54)–(3.56) is regular.

Let us prove the coerciveness property (H2). To do so, we note that for any $u_1 \in L^2(\Gamma_1)$ and $v \in L^2(\Gamma_2)$, the Dirichlet boundary value problem

$$\triangle y(x) = 0 \ \text{ in } \ \Omega, \ \ y|_{\Gamma_1} = u_1, \ \ y|_{\Gamma_2} = v$$

has a unique solution $y \in H^{1/2}_\Delta(\Omega)$ with the estimate (see [173])

$$\|y\|^2_{H^{1/2}_\Delta(\Omega)} \le C \left(\|u_1\|^2_{L^2(\Gamma_1)} + \|v\|^2_{L^2(\Gamma_2)} \right). \tag{3.59}$$

It now follows that

$$\begin{aligned}
&\|y\|^2_{\mathbb{Y}_1} + \|u_1\|^2_{L^2(\Gamma_1)} + \|u_2\|^2_{L^2(\Gamma_1)} \\
&\quad \le \|y\|^2_{H^{1/2}_{\Delta}(\Omega)} + \|u_1\|^2_{L^2(\Gamma_1)} + 2\|u_2\|^2_{L^2(\Gamma_1)} \\
&\quad \le C\| \, y|_{\Gamma_2} \, \|^2_{L^2(\Gamma_2)} + (C+1)\|u_1\|^2_{L^2(\Gamma_1)} + 2\|u_2\|^2_{L^2(\Gamma_1)} \\
&\quad \le C_1 I(u_1, u_2, y) + C_2.
\end{aligned}$$

Hence, the set $\{(u_1, u_2, y) \in \Xi \ : \ I(u, y) \le R\}$ is bounded in $L^2(\Gamma_1) \times L^2(\Gamma_1) \times \mathbb{Y}_1$ for every $R > 0$. Now, by Theorem 3.2, the problem (3.54)–(3.56) has a solution $(u_1^0, u_2^0, y^0) \in L^2(\Gamma_1) \times L^2(\Gamma_2) \times \mathbb{Y}_1$. Combining this with the property (3.58), we get

$$(u_1^0, u_2^0, y^0) \in H^1(\Gamma_1) \times L^2(\Gamma_2) \times \mathbb{Y}_1.$$

To conclude the proof, we should repeat the arguments of Theorem 3.16 concerning the proof of the uniqueness of the optimal solution. □

3.3.4 Controls with hard constraints

Before we consider the OCPs with the so-called rigid controls, we note that, typically, the size of the admissible controls should be constrained. There are two common ways of constraining the control. The first one is to impose an explicit bound on the control — $u \in U_\partial$ in our case. Such constraints are called "hard". The second way of constraining the control is to add some norm of the control to the cost functional. Those constraints are termed "soft".

To study the correctness of the extremal problems for elliptic equations with hard constrained controls, we begin with the following OCP

$$I(u, y) = \| \, y|_{\partial\Omega} - z_\partial\|^2_{H^1(\partial\Omega)} \longrightarrow \inf, \tag{3.60}$$

$$\triangle y(x) = 0 \ \text{ in } \ \Omega, \quad \partial_\nu y|_{\partial\Omega} = u \ \text{ on } \ \partial\Omega, \tag{3.61}$$

$$u \in U_\partial, \tag{3.62}$$

where $z_\partial \in H^1(\partial\Omega)$ is a given function and U_∂ is a convex closed subset of $L^2(\partial\Omega)$ such that

$$U_\partial \cap \left\{ u \in L^2(\partial\Omega) \ : \ \int_{\partial\Omega} u \, \mathrm{d}\mathcal{H}^{n-1} = 0 \right\} \neq \emptyset. \tag{3.63}$$

It should be stressed here that the set of admissible controls with hard constraints to the problem (3.60)–(3.63) can be unbounded with respect to the $L^2(\partial\Omega)$-norm.

Theorem 3.21. *For any $z_\partial \in H^1(\partial\Omega)$, the OCP (3.60)–(3.63) admits a unique solution $(u^0, y^0) \in L^2(\partial\Omega) \times H^{3/2}_{\Delta}(\Omega)$.*

Proof. In order to apply Theorem 3.2, we set $\mathbb{Y} = \mathbb{Y}_1 = H^{3/2}_{\Delta}(\Omega)$, $\mathbb{U} = L^2(\partial\Omega)$, $\mathbb{Z} = L^2(\Omega) \times L^2(\partial\Omega)$, $L(u,y) = (\triangle y, \partial_\nu y|_{\partial\Omega} - u)$ and $F(y) = (0,0)$ for all $y \in \mathbb{Y}_1$. Since the trace operator $y \mapsto \partial y/\partial\nu|_{\partial\Omega}$ is well defined and continuous as the mapping

$$\Pi : H^{3/2}_{\Delta}(\Omega) \to L^2(\partial\Omega),$$

it follows that the mappings $L : \mathbb{U}\times\mathbb{Y}_1 \to \mathbb{Z}$ and $I : \mathbb{U}\times\mathbb{Y}_1 \to \mathbb{R}$ are continuous as well. Moreover, as indicated in Remark 3.14, the Neumann problem (3.61) has a unique solution $\dot{y}$ in the quotient space $H^{3/2}_{\Delta}(\Omega)/\mathbb{R}$ with the estimate

$$\|y\|^2_{H^{3/2}_{\Delta}(\Omega)} \le C\left(\|u\|^2_{L^2(\partial\Omega)} + \|y\|^2_{L^2(\Omega)}\right). \tag{3.64}$$

Thus, the regularity assumption (H1) holds true.

To prove the coerciveness property (H2), we note that due to the properties of the Dirichlet problem for the Laplace operator

$$\triangle y = 0 \ \text{ in } \ \Omega, \quad y|_{\partial\Omega} = g \in H^1(\partial\Omega),$$

we have [173]

$$\|y\|^2_{H^{3/2}_{\Delta}(\Omega)} \le C\| y|_{\partial\Omega} \|^2_{H^1(\partial\Omega)} \le C_1 I(u,y) + C_2. \tag{3.65}$$

On the other hand,

$$\|u\|^2_{L^2(\partial\Omega)} := \| \partial_\nu y|_{\partial\Omega} \|^2_{L^2(\partial\Omega)} \le C_3 \|y\|^2_{H^{3/2}_{\Delta}(\Omega)}. \tag{3.66}$$

So, combining (3.65) and (3.66), we obtain

$$\|u\|^2_{L^2(\partial\Omega)} + \|y\|^2_{H^{3/2}_{\Delta}(\Omega)} \le C_1 I(u,y) + C_2 + C_3\left[C_1 I(u,y) + C_2\right]. \tag{3.67}$$

This implies the fulfilment of the hypothesis (H2).

As for the uniqueness of the optimal solution, we assume the converse. Let $(u^0_1, y^0_1) \in \Xi$ and $(u^0_2, y^0_2) \in \Xi$ be two different solutions to the problem (3.60)–(3.63). Since the cost functional (3.60) is strictly convex with respect to $y|_{\partial\Omega}\, y$ and the set of admissible pairs Ξ is convex as well, it follows that

$$I\left(\frac{u^0_1 + u^0_2}{2}, \frac{y^0_1 + y^0_2}{2}\right) < \frac{1}{2}\left[I(u^0_1, y^0_1) + I(u^0_2, y^0_2)\right] = \inf_{(u,y)\in\,\Xi} I(u,y). \tag{3.68}$$

This contradiction implies $y^0_1|_{\partial\Omega} = y^0_2|_{\partial\Omega}$. To conclude the proof, it remains to note that the corresponding Dirichlet problem has a unique solution. So, $y^0_1(x) \equiv y^0_2(x)$ in Ω, and thereby $u^0_1 = u^0_2$. □

In the next example, we carry out the analysis of an OCP with hard control constraints for the case when the determining norm in the cost functional is replaced by a weaker one, namely we consider the following problem

$$I(u,y) = \| y|_{\partial\Omega} - z_\partial\|^2_{L^2(\partial\Omega)} \longrightarrow \inf, \tag{3.69}$$

$$\triangle y(x) = 0 \ \text{ in } \ \Omega, \quad \partial_\nu y|_{\partial\Omega} = u \ \text{ on } \ \partial\Omega, \tag{3.70}$$

$$u \in U_\partial, \tag{3.71}$$

where $z_\partial \in L^2(\partial\Omega)$ is a given function and U_∂ is a convex closed subset of $L^2(\partial\Omega)$ satisfying the condition (3.63).

Theorem 3.22. *Assume that the set of admissible controls U_∂ is bounded in $L^2(\partial\Omega)$. Then for any $z_\partial \in L^2(\partial\Omega)$, the OCP (3.69)–(3.71) admits a unique solution $(u^0, y^0) \in L^2(\partial\Omega) \times H^{3/2}_\Delta(\Omega)$.*

Proof. Indeed, having defined the spaces $\mathbb{Y}$, $\mathbb{Y}_1$, $\mathbb{Z}$ and $\mathbb{U}$, and the operators L and F as in Theorem 3.21, in other respects it remains only to repeat all arguments of the previous proof with the exception of the verification of hypothesis (H2). In this case, instead of (3.67), it is enough to use another estimate:

$$\begin{aligned}
\|y\|^2_{H^{3/2}_\Delta(\Omega)} &\le C\left(\| \partial_\nu y|_{\partial\Omega} \|^2_{L^2(\partial\Omega)} + \|y\|^2_{L^2(\Omega)}\right)\\
&\le C_1\left(\|u\|^2_{L^2(\partial\Omega)} + \|y\|^2_{H^{1/2}(\Omega)}\right)\\
&\le C_1\left(\sup_{u\in U_\partial} \|u\|^2_{L^2(\partial\Omega)} + C_2\| y|_{\partial\Omega} \|^2_{L^2(\partial\Omega)}\right)\\
&\le C_3\left(I(u,y) + C_4\right),
\end{aligned}$$

which immediately follows from the estimates (3.38) and (3.65). □

3.4 Optimal control problems for parabolic equations

In this section, we study the existence of solutions and their uniqueness to some classes of OCPs associated with linear evolution equations of the parabolic type. We carry out an analysis of such problems for both well-posed and ill-posed initial boundary value problems.

3.4.1 Distributed control

Let Ω be an open bounded subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let $T > 0$, $N > 0$, $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \partial\Omega$. Let $z_\partial \in L^2(Q)$ and $y_0 \in L^2(\Omega)$ be given functions. We begin with the following OCP

$$I(u,y) = \frac{1}{2}\|y - z_\partial\|^2_{L^2(Q)} + \frac{N}{2}\|u\|^2_{L^2(Q)} \longrightarrow \inf, \tag{3.72}$$

$$\dot y(t,x) - \eta\triangle y(t,x) = u(t,x), \ \text{ in } \ Q, \quad y|_\Sigma = 0, \quad y|_{t=0} = y_0, \tag{3.73}$$

$$u \in U_\partial, \tag{3.74}$$

where U_∂ is a nonempty convex closed subset of $L^2(Q)$ and $\dot{y}(t,x) = \partial y(t,x)/\partial t$, $\eta \neq 0$. It is well known that for a fixed $u \in U_\partial$ and a positive $\eta > 0$, the initial boundary value problem (3.73) is well-posed, whereas the condition $\eta < 0$ corresponds to the so-called inverse problem for the heat equation, which is always an ill-posed problem from the point of view of PDEs.

Typical examples of the sets of admissible controls U_∂ to the problem (3.72)–(3.74) are

$$U_\partial = \left\{ u \in L^2(Q) \ : \ \|u - u_0\|^2_{L^2(Q)} \leq R \right\}, \tag{3.75}$$

$$U_\partial = \left\{ u \in L^2(Q) \ : \ u(t,x) \geq 0 \text{ a.e. on } Q \right\}, \tag{3.76}$$

$$U_\partial = \left\{ u \in L^2(Q) \ : \ \alpha(t,x) \leq u(t,x) \leq \beta(t,x) \text{ a.e. on } Q \right\}, \tag{3.77}$$

where $\alpha \in L^\infty(Q)$, $\alpha(t,x) < \beta(t,x)$ for almost all $(t,x) \in Q$.

To begin, we make use of the space

$$\mathbb{V} = \left\{ v \in L^2(Q) \ : \ \dot{v} - \eta \triangle v \in L^2(Q) \right\} \tag{3.78}$$

endowed with the norm

$$\|v\|^2_{\mathbb{V}} = \|v\|^2_{L^2(Q)} + \|\dot{v} - \eta \triangle v\|^2_{L^2(Q)}. \tag{3.79}$$

Let us define the mappings

$$\gamma_\tau : v \mapsto v|_{t=\tau}, \quad \gamma^0_\Sigma : v \mapsto v|_\Sigma, \quad \gamma^1_\Sigma : v \mapsto \partial_\nu v|_\Sigma,$$

where $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative to $\partial\Omega$. In this case, the following result takes a place.

Lemma 3.23 ([111]). *The trace operators*

$$\gamma_\tau : \mathbb{V} \to H^{-1}(\Omega), \tag{3.80}$$

$$\gamma^0_\Sigma : \mathbb{V} \to H^{-1/2}(\partial\Omega; H^{-1}(0,T)), \tag{3.81}$$

$$\gamma^1_\Sigma : \mathbb{V} \to H^{-3/2}(\partial\Omega; H^{-1}(0,T)). \tag{3.82}$$

are well defined and continuous.

We are now in a position to prove the first result of this section.

Theorem 3.24. *Assume that the parameter η in (3.73) is positive. Then there exists a unique solution $(u^0, y^0) \in L^2(Q) \times \mathbb{V}_1$ of OCP (3.72)–(3.74), where*

$$\mathbb{V}_1 = \{ v \in \mathbb{V} \ : \ v|_\Sigma = 0 \}, \quad \|v\|_{\mathbb{V}} = \|v\|_{\mathbb{V}_1}. \tag{3.83}$$

Proof. We set $\mathbb{Y} = \mathbb{U} = L^2(Q)$, $\mathbb{Y}_1 = \mathbb{V}_1$, $\mathbb{Z} = L^2(Q) \times H^{-1}(\Omega)$, $L(u,y) = (\dot{y} - \eta\triangle y - u, y|_{t=0})$, and $F = (0, -y_0)$. Then, in view of the definition of the space $\mathbb{V}_1$ and Lemma 3.23 (see (3.80)), the operator $L : \mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z}$ is continuous. To verify hypothesis (H1), we note that the initial-boundary

value problem (3.72)–(3.74) has a unique solution $y \in L^2(0,T;H_0^1(\Omega))$ for every $u \in L^2(Q)$ and $y_0 \in L^2(\Omega)$ (see [169]). So, $y \in L^2(Q)$. On the other hand, $\dot{y} - \eta\triangle y = u \in U_\partial \subset L^2(Q)$. Then $y \in \mathbb{Y}_1$, and hence $\Xi \neq \emptyset$. Thus, hypothesis (H1) is valid.

Let us verify the coercivity condition (H2). To do so, we fix an arbitrary admissible pair $(u,y) \in \Xi \subset \mathbb{U} \times \mathbb{Y}_1$ and estimate its norm in $\mathbb{U} \times \mathbb{Y}_1$. One gets

$$\begin{aligned}\|u\|_{\mathbb{U}}^2 + \|y\|_{\mathbb{Y}_1}^2 &= \|y\|_{L^2(Q)}^2 + \|\dot{y} - \eta\triangle y\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 \\ &= \|y\|_{L^2(Q)}^2 + 2\|u\|_{L^2(Q)}^2 \leq CI(u,y) = C_1.\end{aligned}$$

Thereby, the set $\{(u,y) \in \Xi \ : \ I(u,y) \leq R\}$ is bounded in $\mathbb{U} \times \mathbb{Y}_1$ for every $R > 0$. So, applying Theorem 3.2 and taking into account the strict convexity of the cost functional (3.72), we come to the required conclusion. □

Regarding the case when $\eta < 0$ (i.e., when the corresponding initial boundary value problem is ill-posed), we have the following result.

Theorem 3.25. *Assume that $y_0 \in H_0^1(\Omega)$ and the L^2-interior of the set of admissible controls U_∂ is nonempty (i.e., $\operatorname{int} U_\partial \neq \emptyset$). Then, for any $\eta < 0$, there exists a unique solution $(u^0, y^0) \in L^2(Q) \times \mathbb{Y}_1$ of OCP (3.72)–(3.74).*

Remark 3.26. In particular, the main assumption of this theorem will be satisfied if we take the set U_∂ in the form (3.75).

Proof. Let us define the spaces $\mathbb{Y}$, $\mathbb{Y}_1$, $\mathbb{Z}$, and $\mathbb{U}$, and the operators L and F as in Theorem 3.24. We have to prove only the regularity property (H1) because in other respects we should repeat all the arguments of the previous proof.

As indicated by Lions and Magenes in [173], if $y_0 \in H_0^1(\Omega)$ then there exists a function $y \in H^{1,2}(Q)$ such that $y|_\Sigma = 0$ and $y|_{t=0} = y_0$. Let us set $g = \dot{y} - \eta\triangle y$. Then $g \in L^2(Q)$. As a result, we have that for a fixed $u \in U_\partial$, the initial-boundary value problem (3.73) is solvable if and only if there exists a solution to the modified problem

$$\dot{v}(t,x) - \eta\triangle v(t,x) = p(t,x) \ \text{ in } \ Q, \quad v|_\Sigma = 0, \quad v|_{t=0} = 0, \tag{3.84}$$

where $p = u - g$.

Assume that $u^* \in \operatorname{int} U_\partial$. Then $p^* = u^* - g \in \operatorname{int} P_\partial$, where

$$P_\partial = \{p = u - g \ : \ u \in U_\partial\} \subset L^2(Q).$$

By $\{e_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ let us denote the eigenfunctions and the corresponding eigenvalues to the spectral problem

$$-\triangle e = \lambda e, \quad e|_{\partial\Omega} = 0.$$

Then we may suppose that the sequence $\{\lambda_k\}_{k=1}^\infty$ is nondecreasing. Following the main idea of the Fourier expansion method, we set

$$p_k(t,x) = \sum_{i=1}^{k} p^i(t) e_i(x), \qquad p^i(t) = \int_\Omega p^*(t,x) e_i(x)\,\mathrm{d}x,$$
$$v_k(t,x) = \sum_{i=1}^{k} v^i(t) e_i(x), \qquad v^i(t) = \int_0^t \exp\left[\eta\lambda_i(\tau - t)\right] p^i(t)\,\mathrm{d}t.$$

It is clear that

$$\dot{v}_k(t,x) - \eta\triangle v_k(t,x) = p_k(t,x) \ \text{ in } \ Q, \quad v_k|_\Sigma = 0, \quad v_k|_{t=0} = 0$$

for any $k \in \mathbb{N}$. Since $\|p^* - p_k\|_{L^2(Q)} \to 0$ as $k \to \infty$, it follows that the pair (p_k, v_k) satisfies relations (3.84) with $p = p^* := u^* - g$ for sufficiently large $k \in \mathbb{N}$. As a result, we can assert that there is a pair $(u^*, y) \in U_\partial \times \mathbb{V}_1$ satisfying the initial-boundary value problem (3.73). Thus, the regularity property (H1) holds true, and this concludes the proof. □

3.4.2 Control in the initial conditions

In this subsection, we consider an OCP for which a control action appears in the initial condition. Following Lions' terminology [169], such controls are called "starting". Let $z_\partial \in L^2(\Omega)$, $f \in L^2(Q)$, $y_0 \in L^2(\Omega)$, $N > 0$, $\eta \neq 0$ and $Q = (0,T) \times \Omega$. The corresponding control problem can be stated as follows:

$$I(u,y) = \frac{1}{2}\|y(T,\cdot) - z_\partial\|^2_{L^2(\Omega)} + \frac{N}{2}\|u - y_0\|^2_{L^2(\Omega)} \longrightarrow \inf, \tag{3.85}$$
$$\dot{y}(t,x) - \eta\triangle y(t,x) = f(t,x) \ \text{ in } \ Q, \quad y|_\Sigma = 0, \quad y|_{t=0} = u, \tag{3.86}$$
$$u \in U_\partial, \tag{3.87}$$

where U_∂ is nonempty convex closed subset of $L^2(\Omega)$.

In what follows, we will need the following result concerning the solvability of the initial-boundary value problem (3.86) (see Lions and Magenes [173]).

Theorem 3.27. *Let $\mathbb{Y}_1$ be the vector space defined by*

$$\mathbb{Y}_1 = \left\{ y \in L^2(0,T; H_0^1(\Omega)) \ : \ \dot{y} \in L^2(0,T; H^{-1}(\Omega)) \right\} \tag{3.88}$$

equipped with the norm of the graph. Then the following hold:

(a) For any $\tau \in [0,T)$ the mapping $\gamma_\tau : \mathbb{Y}_1 \to L^2(\Omega)$ is a continuous epimorphism, where $\gamma_\tau : v \mapsto v|_{t=\tau}$ is the trace operator.
(b) The mapping $(\partial/\partial t - \eta\triangle, \gamma_0) : \mathbb{Y}_1 \to L^2(Q) \times L^2(\Omega)$ is an isomorphism.

We are now in a position to carry out the analysis of the original starting control problem.

Theorem 3.28. *If $\eta > 0$ then there exists a unique solution $(u^0, y^0) \in L^2(\Omega) \times \mathbb{Y}_1$ of the OCP (3.85)–(3.87).*

Proof. Having set

$$\mathbb{Y} = \mathbb{Y}_1, \quad \mathbb{U} = L^2(\Omega), \quad \mathbb{Z} = L^2(0,T;H^{-1}(\Omega)) \times L^2(\Omega),$$
$$L(u,y) = (\dot{y} - \eta\triangle y,\, y|_{t=0} - u), \quad F = (-f, 0),$$

we note that the operator $L : \mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z}$ is continuous. Moreover, for any fixed control $u \in U_\partial$, there exists a unique solution $y \in \mathbb{Y}_1$ to the problem (3.86). Hence, hypothesis (H1) is obviously true in this case. Let us verify condition (H2). To do so, we multiply the state equation (3.86) by $y(t,x)$ and integrate with respect to (t,x) over Q. One gets

$$\frac{1}{2}\|y(T,\cdot)\|^2_{L^2(\Omega)} - \frac{1}{2}\|y(0,\cdot)\|^2_{L^2(\Omega)} + \eta\int_0^T \|\nabla y(t,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}t = \int_0^T (f,y)_{L^2(\Omega)}\,\mathrm{d}t. \tag{3.89}$$

Then an application of the Cauchy–Buniakowsky and Friedrichs inequalities to the right-hand side of this equality yields

$$\begin{aligned}\eta\int_0^T \|\nabla y(t,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}t &\le \frac{1}{2}\|y(0,\cdot)\|^2_{L^2(\Omega)} + \|f\|_{L^2(Q)}\|y\|_{L^2(Q)}\\ &\le \frac{1}{2}\|y(0,\cdot)\|^2_{L^2(\Omega)} + C_\eta\|f\|^2_{L^2(\Omega)}\\ &\quad + \frac{\eta}{2}\int_0^T \|\nabla y(t,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}t.\end{aligned} \tag{3.90}$$

Using the fact that $-|\eta|\triangle \in \mathcal{L}\left(L^2(0,T;H_0^1(\Omega)), L^2(0,T;H^{-1}(\Omega))\right)$, we obtain

$$\begin{aligned}\|\dot{y}\|_{L^2(0,T;H^{-1}(\Omega))} &\le \|f\|_{L^2(Q)} + |\eta|\|\triangle y\|_{L^2(0,T;H^{-1}(\Omega))}\\ &\le \|f\|_{L^2(Q)} + |\eta|\|y\|_{L^2(0,T;H_0^1(\Omega))}.\end{aligned} \tag{3.91}$$

As a result, there is a constant $C > 0$ depending on η such that

$$\|y\|^2_{\mathbb{Y}_1} \le C\left(\|f\|^2_{L^2(Q)} + \|y(0,\cdot)\|^2_{L^2(\Omega)}\right). \tag{3.92}$$

Therefore, for any admissible pair $(u,y) \in L^2(\Omega) \times \mathbb{Y}_1$, we have

$$\begin{aligned}\|y\|^2_{\mathbb{Y}_1} + \|u\|^2_{L^2(\Omega)} &\le C\left(\|y(T,\cdot)\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q)} + \|u\|^2_{L^2(\Omega)}\right)\\ &\le C_1 I(u,y) + C_2,\end{aligned} \tag{3.93}$$

which implies the fulfilment of hypothesis (H2).

As for the uniqueness of the optimal solution, we may use the standard approach. Let $(u_1^0, y_1^0) \in \Xi$ and $(u_2^0, y_2^0) \in \Xi$ be two different solutions to the

problem (3.85)–(3.87). Having set $z = y_1^0 - y_2^0$ and $v = u_1^0 - u_2^0$ and applied inequality (3.68), it can be shown that $y_1^0(T,x) = y_2^0(T,x)$ almost everywhere in Ω. Then

$$\dot{z} - \eta\triangle z = 0 \ \text{ in } \ Q, \quad z|_{\Sigma} = 0, \quad z|_{t=0} = v = 0, \quad z|_{t=T} = 0, \tag{3.94}$$

and hence $z(t,x) \equiv 0$. This completes the proof. □

We pass now to the case when $\eta < 0$.

Theorem 3.29. *Assume that $\eta < 0$ and $\operatorname{int} U_\partial \neq \emptyset$. Then there exists a unique solution $(u^0, y^0) \in L^2(\Omega) \times \mathbb{Y}_1$ of the OCP (3.85)–(3.87).*

Proof. Let us define the spaces $\mathbb{Y}$, $\mathbb{Y}_1$, $\mathbb{Z}$, and $\mathbb{U}$, and the operators L and F as in Theorem 3.28. First, we show that the regularity assumption (H1) holds true. To do so, we use the arguments of the proof of Theorem 3.25. With this aim, we introduce the problem

$$\dot{v}(t,x) - \eta\triangle v(t,x) = f(t,x) \ \text{ in } \ Q, \quad v|_{\Sigma} = 0, \quad v|_{t=T} = 0. \tag{3.95}$$

Let $v \in \mathbb{Y}_1$ be a solution to (3.95). Let us set $P_\partial = \{p = u - v(0,\cdot) \ : \ u \in U_\partial\}$. Then, in view of the initial assumptions, we have $\operatorname{int} P_\partial \neq \emptyset$. Let p be any element of $\operatorname{int} P_\partial$ and let $\{e_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ be the collections defined as in the proof of Theorem 3.25. For every $k \in \mathbb{N}$, we set

$$v_k(t,x) = \sum_{i=1}^{k} \exp\left(-\eta\lambda_i t\right) p_i e_i(x),$$

where

$$p_i = \int_\Omega p(x) e_i(x)\,\mathrm{d}x = \int_\Omega (u(x) - v(0,x)) e_i(x)\,\mathrm{d}x.$$

It is clear that, in this case, we have $\lim_{k\to\infty} \|v_k(0,\cdot) - p\|_{L^2(\Omega)} = 0$. Hence, $v_k(0,\cdot) \in P_\partial$ for all sufficiently large $k \in \mathbb{N}$. Therefore, the pair $(u(x), y(t,x)) = (p(x) + v_k(0,x), v(t,x) + v_k(t,x))$ satisfies relations (3.86), that is, this is an admissible pair to the original problem (3.85)–(3.87). Thus, $\Xi \neq \emptyset$, and the regularity property is established.

In order to prove hypothesis (H2), we take into account the relation (3.89). Since $\eta < 0$, it follows that instead of the estimates (3.90) and (3.92), we will have

$$\begin{aligned} |\eta| \int_0^T \|\nabla y(t,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ \leq \frac{1}{2}\|y(T,\cdot)\|^2_{L^2(\Omega)} + C_\eta \|f\|^2_{L^2(\Omega)} + \frac{|\eta|}{2}\int_0^T \|\nabla y(t,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}t, \\ \|y\|^2_{\mathbb{Y}_1} \leq C\left(\|f\|_{L^2(Q)} + \|y(T,\cdot)\|^2_{L^2(\Omega)}\right). \end{aligned}$$

As a result, supposition (H2) is the direct consequence of the following inequality

$$\|y\|^2_{\mathbb{Y}_1} + \|u\|^2_{L^2(\Omega)} \le C\left(\|y(T,\cdot)\|^2_{L^2(\Omega)} + \|f\|_{L^2(Q)} + \|u\|^2_{L^2(\Omega)}\right)$$
$$\le C_1 I(u,y) + C_2, \quad \forall\,(u,y) \in L^2(\Omega)\times\mathbb{Y}_1. \tag{3.96}$$

To conclude the proof, it remains to note that the restriction $z|_{t=T} = 0$ in (3.94) can be seen as the initial condition to the problem (3.94) (the so-called reverse problem). In this case, the function $z(t,x) = y_1^0 - y_2^0 \equiv 0$ is the unique solution to (3.94). Therefore, $z|_{t=0} = u_1^0 - u_2^0 \equiv 0$, and this implies the uniqueness of the solution to the original OCP (3.85)–(3.87). The proof is complete. □

3.4.3 Neumann boundary control

Consider the optimal boundary control problem

$$I(u,y) = \frac{1}{2}\|y(T,\cdot) - z_\partial\|^2_{L^2(\Omega)} + \frac{N}{2}\|u\|^2_{L^2(\Sigma)} \longrightarrow \inf, \tag{3.97}$$

$$\dot y(t,x) - \eta\triangle y(t,x) = f(t,x) \ \text{ in } \ Q, \quad \partial_\nu y|_\Sigma = u, \quad y|_{t=0} = y_0, \tag{3.98}$$

$$u \in U_\partial, \tag{3.99}$$

where $z_\partial \in L^2(\Omega)$, $f \in L^2(Q)$, $y_0 \in H^{1/2}(\Omega)$, $N > 0$, $\eta \ne 0$, $\Sigma = (0,T)\times\partial\Omega$, $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative to $\partial\Omega$, and U_∂ is a nonempty convex closed subset of $L^2(\Sigma)$.

Due to a result of Lions and Magenes [173], it is well known that for every $u \in L^2(\Sigma)$ and $\eta > 0$, there exists a unique solution y to the initial-boundary value problem (3.98) in the space $L^2(0,T;H^{3/2}(\Omega)) \cap H^{3/4}(0,T;L^2(\Omega))$. So, following the line of the proof of the theorems cited above, the reader can easily show that the OCP (3.97)–(3.99) has a unique solution (u^0,y^0) in the class $L^2(\Sigma)\times\mathbb{Y}_1 = L^2(\Sigma)\times\big\{y \in L^2(0,T;H^{3/2}(\Omega))$ $\cap H^{3/4}(0,T;L^2(\Omega)) \ : \ \dot y(t,x) - \eta\triangle y(t,x) \in L^2(Q)\big\}$, provided this problem is regular. However, as we will see later, the same result remains valid in a less regular space of admissible pairs – namely, in the case when we take $L^2(\Sigma)\times L^2(0,T;H^1(\Omega))$ instead of $L^2(\Sigma)\times\mathbb{Y}_1$. To show this, we begin with the following notion.

Definition 3.30. *We say that a pair $(u,y) \in L^2(\Sigma)\times L^2(0,T;H^1(\Omega))$ satisfies relations (3.98) in the weakened sense if the integral identity*

$$\int_0^\tau\int_\Omega \left[y(t,x)\left(\dot\varphi(t,x) + \eta\triangle\varphi(t,x)\right) + f(t,x)\varphi(t,x)\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_\Omega \left(y(\tau,x)\varphi(\tau,x) - y_0(x)\varphi(0,x)\right)\,\mathrm{d}x$$
$$-\eta\int_0^\tau\int_{\partial\Omega} u(t,s)\varphi(t,s)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \tag{3.100}$$

holds true for all $\tau \in [0,T]$ *and* $\varphi \in H^{1,2}(Q)$ *such that* $\partial_\nu \varphi|_\Sigma = 0$.

The main question is: In which sense does the trace $y(t,x)|_{t=\tau}$ exist? Having put $\varphi(t,x) \equiv \varphi(x)$, where $\varphi \in H_0^2(\Omega)$, in (3.100), we can rewrite this relation in the form

$$\int_\Omega y(\tau,x)\varphi(x)\,\mathrm{d}x = \int_\Omega y_0(x)\varphi(x)\,\mathrm{d}x + \eta\int_0^\tau\int_{\partial\Omega} u(t,s)\varphi(s)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t + \int_0^\tau\int_\Omega \left[f(t,x)\varphi(x) - \eta\nabla y(t,x)\cdot\nabla\varphi(x)\right]\,\mathrm{d}x\,\mathrm{d}t. \tag{3.101}$$

It is clear that the right-hand side of (3.101) can be continuously extended by φ from $H_0^2(\Omega)$ onto $H_0^1(\Omega)$. Therefore, the left-hand side of (3.101) can be associated with a linear continuous functional on $H_0^1(\Omega)$, that is,

$$\int_\Omega y(\tau,x)\varphi(x)\,\mathrm{d}x = \langle y(\tau,\cdot),\varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}.$$

Using the fact that the right-hand side of (3.101) is a continuous function with respect to the τ, we can conclude that any function $y \in L^2(0,T;H^1(\Omega))$ satisfying (3.101) has a trace $y(\tau,\cdot)|_{t=\tau} \in H^{-1}(\Omega)$ for every $\tau \in [0,T]$. In fact, as immediately follows from Lemma 3.31, we have more precise result $y(\tau,\cdot)|_{t=\tau} \in L^2(\Omega)$.

Lemma 3.31 ([111]). *If a pair* (u,y) *satisfies relations (3.98) in the weakened sense, then*

$$\frac{1}{2}\int_\Omega \left(y^2(\tau,x) - y_0^2(x)\right)\,\mathrm{d}x + \eta\int_0^\tau\int_\Omega |\nabla y(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau\int_\Omega f(t,x)y(t,x)\,\mathrm{d}x\,\mathrm{d}t + \eta\int_0^\tau\int_{\partial\Omega} u(t,s)y(t,s)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t. \tag{3.102}$$

The set of all pairs $(u,y) \in U_\partial \times L^2(0,T;H^1(\Omega))$ satisfying (3.98) in the weakened sense is called the set of admissible solutions to the problem (3.97)–(3.99), and we denote it by Ξ. Due to the well-known solvability result [173], it is clear that for given $f \in L^2(Q)$, $y_0 \in H^{1/2}(\Omega)$ and $\eta > 0$, the set Ξ is nonempty. As for the case $\eta > 0$, the verification of the condition $\Xi \neq \emptyset$ is not trivial, in general. For instance, if $f = 0$, $y_0 = 0$ and $0 \in U_\partial$, then $\Xi \neq \emptyset$ because the pair $(0,0)$ is admissible to (3.98).

Theorem 3.32. *Assume that there exists a pair* $(u,y) \in U_\partial \times L^2(0,T;H^1(\Omega))$ *satisfying* (3.98) *in the weakened sense. Then the OCP (3.97)–(3.99) has a unique solution* $(u^0,y^0) \in U_\partial \times L^2(0,T;H^1(\Omega))$.

Proof. Let us fix an arbitrary pair $(u,y) \in \Xi$ and suppose that $\eta < 0$. Then, taking into account Lemma 3.31, we have

$$\frac{1}{2}\int_\Omega \left(y^2(\tau_2,x)-y^2(\tau_1,x)\right)\,\mathrm{d}x+\eta\int_{\tau_1}^{\tau_2}\int_\Omega |\nabla y(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\tau_1}^{\tau_2}\int_\Omega f(t,x)y(t,x)\,\mathrm{d}x\,\mathrm{d}t+\eta\int_{\tau_1}^{\tau_2}\int_{\partial\Omega} u(t,s)y(t,s)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \quad (3.103)$$

for any $0\le\tau_1<\tau_2\le T$. Then, replacing τ_1 by τ and τ_2 by T in (3.103) and using the the Cauchy–Buniakowsky and Friedrichs inequalities, we get

$$\begin{aligned}
&\int_\Omega y^2(\tau,x)\,\mathrm{d}x+2|\eta|\int_\tau^T\int_\Omega |\nabla y(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\int_\Omega y^2(T,x)\,\mathrm{d}x+2\int_\tau^T \|f(t,\cdot)\|_{L^2(\Omega)}\|y(t,\cdot)\|_{L^2(\Omega)}\,\mathrm{d}t\\
&\qquad+2|\eta|\int_\tau^T \|u(t,\cdot)\|_{L^2(\partial\Omega)}\|y(t,\cdot)\|_{L^2(\partial\Omega)}\,\mathrm{d}t\\
&\quad\le\int_\Omega y^2(T,x)\,\mathrm{d}x+C_\delta\left(\int_\tau^T \|f(t,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\int_\tau^T \|u(t,\cdot)\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t\right)\\
&\qquad+\delta\int_\tau^T \|y(t,\cdot)\|^2_{H^1(\Omega)}\,\mathrm{d}t=:J_0. \qquad (3.104)
\end{aligned}$$

Hence,

$$\int_0^T\int_\Omega y^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\le TJ_0,\quad \int_0^T\int_\Omega |\nabla y(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\le\frac{1}{2|\eta|}J_0.$$

As a result, we obtain

$$\begin{aligned}
\int_\tau^T \|y(t,\cdot)\|^2_{H^1(\Omega)}\,\mathrm{d}t&\le\delta\left(T+\frac{1}{2|\eta|}\right)\int_\tau^T \|y(t,\cdot)\|^2_{H^1(\Omega)}\,\mathrm{d}t\\
&\quad+C_{1\delta}\left(\int_\tau^T \|f(t,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\int_\tau^T \|u(t,\cdot)\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t\right)\\
&\quad+\left(T+\frac{1}{2|\eta|}\right)\int_\Omega \|y(T,x)\|^2\,\mathrm{d}x. \qquad (3.105)
\end{aligned}$$

If we take $\delta\le 1/2\,(T+1/(2|\eta|))^{-1}$, then it follows that

$$\|y\|^2_{L^2(0,T;H^1(\Omega))}\le C\left(\|f\|^2_{L^2(Q)}+\|u\|^2_{L^2(\Sigma)}+\|y(T,\cdot)\|^2_{L^2(\Omega)}\right), \quad (3.106)$$

where the constant C is independent of u and y. By analogy, it can be shown that in the case when $\eta>0$, we have a similar estimate

$$\|y\|^2_{L^2(0,T;H^1(\Omega))}\le C\left(\|f\|^2_{L^2(Q)}+\|u\|^2_{L^2(\Sigma)}+\|y_0\|^2_{L^2(\Omega)}\right). \quad (3.107)$$

Combining (3.106), (3.107) and (3.97), we obtain

$$\|y\|^2_{L^2(0,T;H^1(\Omega))} + \|u\|^2_{L^2(\Sigma)} \leq C_1 I(u,y) + C_2 \left(\|f\|^2_{L^2(Q)} + \|y_0\|^2_{L^2(\Omega)} \right) + C_3. \quad (3.108)$$

Let $\{(u_k, y_k)\}_{k=1}^{\infty} \subset \Xi$ be a minimizing sequence of admissible pairs to the problem (3.97)–(3.99), that is,

$$\lim_{k\to\infty} I(u_k, y_k) = \inf_{(u,y)\in\Xi} I(u,y). \quad (3.109)$$

Then, in view of (3.109) and (3.108), we conclude that there is a constant $C > 0$ independent of $k \in \mathbb{N}$ such that

$$\|y_k\|^2_{L^2(0,T;H^1(\Omega))} + \|u_k\|^2_{L^2(\Sigma)} \leq C, \quad \forall\, k \in \mathbb{N}. \quad (3.110)$$

So, passing to subsequences, we may suppose that

$$y_k \rightharpoonup y^0 \ \text{ in } \ L^2(0,T;H^1(\Omega)), \quad u_k \rightharpoonup u^0 \ \text{ in } \ L^2(\Sigma). \quad (3.111)$$

Moreover, substituting y for y_k in (3.101), it is easy to see that the function $t \mapsto (y_k(t,\cdot,\varphi)_{L^2(\Omega)}$ belongs to $H^1(0,T)$ and satisfies the estimate

$$\| \, (y_k(t,\cdot,\varphi)_{L^2(\Omega)} \, \|_{H^1(0,T)} \leq C,$$

where the constant C is independent of k. Therefore, due to the Sobolev embedding theorem, we may suppose that

$$(y_k(t,\cdot),\varphi)_{L^2(\Omega)} \longrightarrow \left(y^0(t,\cdot),\varphi\right)_{L^2(\Omega)} \ \text{ in } \ C[0,T]. \quad (3.112)$$

Thus, plugging y_k into the integral identity (3.100), passing there to the limit as $k \to \infty$ and taking into account (3.111) and (3.112), we see that the limit pair (u^0, y^0) satisfies relations (3.98) in the weakened sense. Moreover, since U_∂ is a convex closed subset of $L^2(\Sigma)$, it follows that U_∂ is closed with respect to the weak convergence in $L^2(\Sigma)$. Hence, (3.111) implies that $u^0 \in U_\partial$. Thus, $(u^0, y^0) \in \Xi$.

It remains to show that this pair is optimal for (3.97)–(3.99). Indeed, as follows from (3.110) and (3.102) with $\tau = T$, we have the estimate $\|y_k(T,\cdot)\|_{L^2(\Omega)} \leq C$, where C does not depend on k. So, in view of (3.112), we may suppose that

$$y_k(T,\cdot) \rightharpoonup y^0(T,\cdot) \ \text{ in } \ L^2(\Omega). \quad (3.113)$$

Finally, using the lower semicontinuity for I with respect to the product of the weak topologies for $L^2(\Sigma)$ and $L^2(\Omega)$, we get

$$\inf_{(u,y)\in\Xi} I(u,y) = \liminf_{k\to\infty} I(u_k, y_k) \geq I(u^0, y^0).$$

Thus, (u^0, y^0) is an optimal pair to the original problem (3.97)–(3.99). To conclude the proof, it remains only to note that the uniqueness of the optimal solution (u^0, y^0) can be easily proved by the standard approach (see Theorems 3.28 and 3.29). □

3.5 Optimal control problems for hyperbolic equations

Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. Let, as usual, $T > 0$, $N > 0$, $Q = (0,T) \times \Omega$, and $\Sigma = (0,T) \times \partial\Omega$. For given functions $z_\partial \in L^2(Q)$, $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$, we consider the following OCP:

$$I(u,y) = \frac{1}{2}\|y - z_\partial\|^2_{L^2(Q)} + \frac{N}{2}\|u\|^2_{L^2(Q)} \longrightarrow \inf, \tag{3.114}$$

$$\ddot{y}(t,x) - \triangle y(t,x) + \eta y(t,x) = u(t,x) \ \text{ in } \ Q, \tag{3.115}$$

$$y|_\Sigma = 0, \quad y|_{t=0} = y_0, \quad \dot{y}|_{t=0} = y_1, \tag{3.116}$$

$$u \in U_\partial, \tag{3.117}$$

where U_∂ is nonempty convex closed subset of $L^2(Q)$, $\eta \in \mathbb{R}$ and $\ddot{y}(t,x) = \partial^2 y(t,x)/\partial t^2$.

We make use of the spaces

$$\mathbb{V} = \left\{v \in L^2(Q) \ : \ \ddot{v} - \triangle v \in L^2(Q), v|_\Sigma = 0\right\}, \tag{3.118}$$

$$\mathbb{V}_1 = \left\{v \in L^2(Q) \ : \ \ddot{v} - \triangle v \in L^2(Q)\right\} \tag{3.119}$$

endowed with the graph norm

$$\|v\|^2_{\mathbb{V}} = \|v\|^2_{\mathbb{V}_1} = \|v\|^2_{L^2(Q)} + \|\ddot{v} - \triangle v\|^2_{L^2(Q)}. \tag{3.120}$$

In order to show that the space $\mathbb{V}$ is well defined, we recall that a function v belongs to the space $H^{-1/2}(\partial\Omega; H^{-2}(0,T))$ if for any $\varphi \in H_0^2(0,T)$ the distribution

$$v_\varphi(x) = \int_0^T v(t,x)\varphi(t)\,\mathrm{d}t := \langle v(\cdot,x), \varphi\rangle_{H^{-2}(0,T), H_0^2(0,T)}, \quad x \in \partial\Omega,$$

is an element of $H^{-1/2}(\partial\Omega)$, and we cite the following result.

Proposition 3.33 ([111]). *For the trace operator $\gamma^0_\Sigma : v \mapsto v|_\Sigma$, the mapping*

$$\gamma^0_\Sigma : \mathbb{V}_1 \to H^{-1/2}(\partial\Omega; H^{-2}(0,T)) \tag{3.121}$$

is well defined and continuous.

By this proposition, the equality $y|_\Sigma = 0$ has a meaning. So, the definition of the space $\mathbb{V}$ is correct. As a result, the boundary condition $y|_\Sigma = 0$ in (3.116) should be interpreted as an equality in $H^{-1/2}(\partial\Omega; H^{-2}(0,T))$. As for the correctness of the initial conditions $y|_{t=0} = y_0$ and $\dot{y}|_{t=0} = y_1$, we note that $\ddot{y} = h + \triangle y \in L^2(0,T; H^{-2}(\Omega))$, provided $y \in L^2(Q)$ and $h := \ddot{y} - \triangle y \in L^2(Q)$. Hence, the mappings

$$\gamma_0 \frac{\partial}{\partial t} : \mathbb{V}_1 \to H^{-2}(\Omega), \quad \gamma_0 : \mathbb{V}_1 \to H^{-2}(\Omega)$$

are well defined and continuous. Thus, in the case when $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$, the conditions $y|_{t=0} = y_0,\ \dot{y}|_{t=0} = y_1$ are meaningful.

Let us define the space

$$\mathbb{Y}_1 = \left\{ y \in L^2(0,T; H_0^1(\Omega)) \cap \mathbb{V} \ :\ \dot{y} \in L^2(0,T; L^2(\Omega)) \right\} \tag{3.122}$$

and make use of the following result (see Lions [172]).

Lemma 3.34. *Under the conditions that $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$ and $f \in L^2(Q)$, there exists a unique solution $y \in \mathbb{Y}_1$ to the initial-boundary value problem*

$$\ddot{y} - \triangle y + \eta y = f,\ \ y|_\Sigma = 0, \quad y|_{t=0} = y_0, \quad \dot{y}|_{t=0} = y_1,$$

with the estimate

$$\|y\|_{L^\infty(0,T;H_0^1(\Omega))} + \|y\|_{L^\infty(0,T;L^2(\Omega))} \\ \leq C \left(\|y_0\|_{H_0^1(\Omega)} + \|y_1\|_{L^2(\Omega)} + \|f\|_{L^2(Q)} \right), \tag{3.123}$$

where the constant C is independent of y_0, y_1 and f.

We are now in a position to state the main result of this section.

Theorem 3.35. *Assume that the set of admissible controls U_∂ possesses the property that there exists a pair $(u, y) \in U_\partial \times \mathbb{Y}_1$ satisfying relations (3.115) and (3.116). Then the OCP (3.114)–(3.117) has a unique solution $(u^0, y^0) \in U_\partial \times \mathbb{Y}_1$.*

Remark 3.36. It is clear that the regularity assumption of this theorem holds true in the case when $U_\partial = L^2(Q)$ and $y_0, y_1 \in C^\infty(\Omega)$. Indeed, if we take a function $y \in C^\infty(Q)$ satisfying the boundary condition $y|_\Sigma = 0$, then the pair $(\ddot{y} - \triangle y + \eta y, y)$ is admissible for the problem (3.114)–(3.117). So, $\Xi \neq \emptyset$. As for the case $U_\partial \neq L^2(Q)$, the fulfilment of the condition $\Xi \neq \emptyset$, in general, is an open question. However, if $y_0 = y_1 \equiv 0$ and $0 \in U_\partial$, then $\Xi \neq \emptyset$ since $(u, y) = (0, 0)$ satisfies relations (3.115) and (3.116).

Proof. In order to apply Theorem 3.2, we set $\mathbb{Y} = L^2(Q)$, $\mathbb{U} = L^2(Q)$, $\mathbb{Z} = L^2(Q) \times H^{-2}(\Omega) \times H^{-2}(\Omega)$, $\mathbb{Y}_1$ as in (3.122), $L(u, y) = (\ddot{y} - \triangle y + \eta y - u,\ y|_{t=0},\ \dot{y}|_{t=0})$ and $F(y) = (0, -y_0, -y_1)$ for all $y \in \mathbb{Y}_1$. Since

$$\|y\|_{\mathbb{Y}_1}^2 = \|y\|_{L^2(Q)}^2 + \|\ddot{y} - \triangle y\|_{L^2(Q)}^2 + \|y\|_{L^2(0,T;H_0^1(\Omega))}^2 + \|\dot{y}\|_{L^2(Q)}^2, \tag{3.124}$$

it follows that the embedding $\mathbb{Y}_1 \hookrightarrow \mathbb{Y}$ is continuous and $L : \mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z}$ is a linear continuous operator. By our assumptions, the original problem is regular. So, it remains to verify hypothesis (H2). To do so, we make use of Lemma 3.34. We obtain

$$
\begin{aligned}
&\|y\|^2_{\mathbb{Y}_1} + \|u\|^2_{\mathbb{U}} \\
&= \|y\|^2_{L^2(Q)} + \|\ddot{y} - \triangle y\|^2_{L^2(Q)} \\
&+\|y\|^2_{L^2(0,T;H^1_0(\Omega))} + \|\dot{y}\|^2_{L^2(Q)} + \|u\|^2_{\mathbb{U}} \\
&\leq \|y\|^2_{L^2(0,T;H^1_0(\Omega))} + \|\dot{y}\|^2_{L^2(Q)} + \|u - \eta y\|^2_{L^2(Q)} + \|y\|^2_{L^2(Q)} + \|u\|^2_{\mathbb{U}} \\
&\leq C\left(\|y_0\|_{H^1_0(\Omega)} + \|y_1\|_{L^2(\Omega)} + \|u\|_{L^2(Q)}\right) \\
&+C_1\left(\|y\|^2_{L^2(Q)} + \|u\|^2_{\mathbb{U}}\right) \leq C_2 I(u,y) + C_3, \qquad (3.125)
\end{aligned}
$$

which implies the fulfillment of hypothesis (H2).

As for the uniqueness of the optimal solution, this fact immediately follows from the strict convexity of the cost functional I and Theorem 3.2. The proof is complete. □

3.6 Optimality system to optimal control problems

Since the optimality system serves as the basis for computing numerical approximations to optimal solutions, our prime interest in this section is to discuss different ways and procedures for the derivation of such systems to a wide class of OCPs. The main attention will be paid to the Lagrange multiplier principle adapted even to the control systems that are described by ill-posed boundary value problems.

3.6.1 The general setting of the Lagrange multiplier principle

Let $\mathbb{Y}$ and $\mathbb{Z}$ be linear normed spaces and let $\mathbb{Y}_1$ be a reflexive, separable Banach space such that $\mathbb{Y}_1$ is continuously and densely embedded in $\mathbb{Y}$. Let $\mathbb{V}$ be a separable Banach space and $\mathbb{U} = \mathbb{V}^*$ be its dual. Let U_∂ be a closed convex subset of $\mathbb{U}$. Now we discuss the Lagrange multiplier principle with regard to the following OCP:

$$\text{Minimize } I(u,y) \qquad (3.126)$$

over all $(u,y) \in \mathbb{U} \times \mathbb{Y}_1$ subject to the state equation and control constraint

$$L(u,y) + F_0 = 0 \text{ and } u \in U_\partial, \qquad (3.127)$$

where $L : \mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z}$ is a linear continuous mapping, $F_0 \in \mathbb{Z}$ and $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is a proper cost functional. As usual, by Ξ we denote the set of admissible pairs for the problem (3.126)–(3.127), that is,

$$\Xi = \{(u,y) \in \mathbb{U} \times \mathbb{Y}_1 \ : \ L(u,y) + F_0 = 0,\ u \in U_\partial,\ I(u,y) < +\infty\}. \qquad (3.128)$$

In this subsection, we assume that all suppositions of Theorem 3.2 hold true. Let $(u^0, y^0) \in \Xi$ be an optimal pair to the problem (3.126)–(3.127),

which obviously exists due to Theorem 3.2. We associate with this problem the Lagrange function

$$\mathcal{L}(u, y, \lambda, \psi) = \lambda I(u, y) + \langle \psi, L(u, y) + F_0 \rangle_{\mathbb{Z}^*, \mathbb{Z}}, \tag{3.129}$$

where $\lambda \in \mathbb{R}^1_+$ and $\psi \in \mathbb{Z}^*$. In accordance with the Lagrange multiplier principle, there is a pair $(\lambda, \psi) \neq 0$ such that $(u^0, y^0) \in \Xi$ is a minimizer to the extremal problem

$$\mathcal{L}(u, y, \lambda, \psi) \longrightarrow \inf, \quad u \in U_\partial. \tag{3.130}$$

In order to give a rigorous substantiation of this principle, we begin with the case when the set of admissible pairs Ξ has a nonempty interior. Obviously, this assumption immediately implies a very restrictive condition from the practical point of view: $\operatorname{int} U_\partial \neq \emptyset$. Let us introduce the set

$$\operatorname{Ker} L = \{(u, y) \in \mathbb{U} \times \mathbb{Y}_1 \ : \ L(u, y) = 0\}.$$

As a result, the standard Lagrange multiplier principle can be quoted in the following particular form (see [24, 111, 125])

Theorem 3.37. *Let $(u^0, y^0) \in \Xi$ be a solution of the OCP (3.126)–(3.127). Assume that the following conditions hold true: The cost functional I is Gâteaux differentiable at (u^0, y^0), $\operatorname{int} \Xi \neq \emptyset$ and the set $\operatorname{Im} L \equiv L(\mathbb{U} \times \mathbb{Y}_1)$ is closed in $\mathbb{Z}$. Then there exists a pair $(\lambda, \psi) \in \mathbb{R}_+ \times \mathbb{Z}^* \setminus \{0\}$ such that*

$$\left\langle \mathcal{D}_y \mathcal{L}(u^0, y^0, \lambda, \psi), y - y^0 \right\rangle_{\mathbb{Y}_1^*, \mathbb{Y}_1} + \left\langle \mathcal{D}_u \mathcal{L}(u^0, y^0, \lambda, \psi), u - u^0 \right\rangle_{\mathbb{U}^*, \mathbb{U}} \geq 0, \quad \forall\, (u, y) \in U_\partial \times \mathbb{Y}_1, \tag{3.131}$$

where $\mathcal{D}_u \mathcal{L}(\cdot, \cdot, \cdot, \cdot)$ and $\mathcal{D}_y \mathcal{L}(\cdot, \cdot, \cdot, \cdot)$ are the corresponding Gâteaux derivatives of $\mathcal{L}$. Moreover, if $\operatorname{int} \Xi \cap \left\{(u^0, y^0) + \operatorname{Ker} L\right\} \neq \emptyset$, then $\lambda = 1$. Conversely, if I is a convex functional, $(u^0, y^0) \in \Xi$ and there exists a $\psi \in \mathbb{Z}^*$ such that the collection $(u^0, y^0, 1, \psi)$ satisfies relations (3.131), then (u^0, y^0) is an optimal pair to the problem (3.126)–(3.127).

Let us consider the case when the condition $\operatorname{int} U_\partial \neq \emptyset$ and, hence, $\operatorname{int} \Xi \neq \emptyset$ are too restrictive. Let $\mathbb{U}_0$ and $\mathbb{Y}_0$ be two Banach spaces such that $\mathbb{U}_0 \subset \mathbb{U}$ and $\mathbb{Y}_0 \subset \mathbb{Y}$. Instead of the condition $\operatorname{int} \Xi \neq \emptyset$, we consider the following one:

$$\operatorname{int}_{\mathbb{U}_0 \times \mathbb{Y}_0} \left((\mathbb{U}_0 \times \mathbb{Y}_0) \cap (\Xi - (u^0, y^0)) \right) \neq \emptyset, \tag{3.132}$$

where $\operatorname{int}_{\mathbb{U}_0 \times \mathbb{Y}_0} M$ is the interior of M with respect to the topology of $\mathbb{U}_0 \times \mathbb{Y}_0$ and $\Xi - (u^0, y^0) = \left\{(u - u^0, y - y^0) \ : \ \forall\, (u, y) \in \Xi\right\}$.

In what follows, by $\operatorname{lin} \left(\Xi - (u^0, y^0)\right)$ we denote the linear span of the set $\Xi - (u^0, y^0)$ and by $\overline{\operatorname{lin} \left(\Xi - (u^0, y^0)\right)}$ its closure with respect to the space $\mathbb{U} \times \mathbb{Y}_1$. Since the condition (3.132) is weaker than $\operatorname{int} \Xi \neq \emptyset$, the following lemma guarantees its fulfillment.

Lemma 3.38 ([111]). *Let (u^0, y^0) be a fixed pair of Ξ. Assume that Ξ is a convex closed separable subset of $\mathbb{U} \times \mathbb{Y}_1$. Then there exist Banach spaces $\mathbb{U}_0$ and $\mathbb{Y}_0$ such that the embeddings $\mathbb{U}_0 \hookrightarrow \mathbb{U}$ and $\mathbb{Y}_0 \hookrightarrow \mathbb{Y}$ are continuous, $\mathbb{U}_0 \times \mathbb{Y}_0$ is a dense subset of $\overline{\operatorname{lin}}\,(\Xi - (u^0, y^0))$ and condition (3.132) holds true.*

Definition 3.39. *We say that the cost functional $I : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ is Gâteaux differentiable at (u^0, y^0) along the space $\mathbb{U}_0 \times \mathbb{Y}_0$ if there exist linear functionals $I'_u : \mathbb{U}_0 \to \mathbb{R}$ and $I'_y : \mathbb{Y}_0 \to \mathbb{R}$ such that*

$$I(u^0 + \lambda v, y^0 + \lambda h) - I(u^0, y^0) \\ - \lambda \left\langle I'_u(u^0, y^0), v \right\rangle_{\mathbb{U}_0^*, \mathbb{U}_0} - \lambda \left\langle I'_y(u^0, y^0), h \right\rangle_{\mathbb{Y}_0^*, \mathbb{Y}_0} = o(\lambda),$$

where $|o(\lambda)|/|\lambda| \to 0$ as $\lambda \to 0$ for every $v \in \mathbb{U}_0$ and $h \in \mathbb{Y}_0$.

Let $\mathbb{Z}_0$ be a Banach space such that $L : \Xi \subset \mathbb{Z}_0$, $L : \Xi \to \mathbb{Z}_0$ is a continuous mapping and $L\Xi$ is a closed set with respect to the topology of $\mathbb{Z}_0$. Then the following result can be viewed as a natural generalization of Theorem 3.37 to the case of (3.132) (see Fursikov [111]).

Theorem 3.40. *Let $(u^0, y^0) \in \Xi$ be a solution of the OCP (3.126)–(3.127). Assume that there exist Banach spaces $\mathbb{U}_0$ and $\mathbb{Y}_0$ satisfying condition (3.132) and the cost functional I is Gâteaux differentiable at (u^0, y^0) along $\mathbb{U}_0 \times \mathbb{Y}_0$. Then there exists a pair $(\lambda, \psi) \in \mathbb{R}_+ \times \mathbb{Z}_0^* \setminus \{0\}$ such that*

$$\left\langle \mathcal{D}_y \mathcal{L}(u^0, y^0, \lambda, \psi), y \right\rangle_{\mathbb{Y}_0^*, \mathbb{Y}_0} + \left\langle \mathcal{D}_u \mathcal{L}(u^0, y^0, \lambda, \psi), u \right\rangle_{\mathbb{U}_0^*, \mathbb{U}_0} \geq 0, \\ \forall\, (u, y) \in (\mathbb{U}_0 \times \mathbb{Y}_0) \cap (\Xi - (u^0, y^0)), \quad (3.133)$$

where $\mathcal{D}_u \mathcal{L}(\cdot, \cdot, \cdot, \cdot)$ and $\mathcal{D}_y \mathcal{L}(\cdot, \cdot, \cdot, \cdot)$ are the corresponding Gâteaux derivatives of $\mathcal{L}$. Moreover, if $\operatorname{int}_{\mathbb{U}_0 \times \mathbb{Y}_0} \left((\mathbb{U}_0 \times \mathbb{Y}_0) \cap (\Xi - (u^0, y^0)) \right) \cap \operatorname{Ker} L \neq \emptyset$ then $\lambda = 1$. Conversely, if $(u^0, y^0) \in \Xi$, I is a convex functional, I is Gâteaux differentiable at (u^0, y^0) along $\mathbb{U}_0 \times \mathbb{Y}_0$, there exists a $\psi \in \mathbb{Z}^*$ such that the collection $(u^0, y^0, 1, \psi)$ satisfies relations (3.133) and

$$(\mathbb{U}_0 \times \mathbb{Y}_0) \cap \left(\operatorname{Ker} L \cap (\Xi - (u^0, y^0)) \right) \quad \text{is dense in} \quad \operatorname{Ker} L \cap (\Xi - (u^0, y^0))$$

with respect to the topology where I is continuous, then (u^0, y^0) is an optimal pair to the problem (3.126)–(3.127).

In fact, the principal feature of Theorems 3.37 and 3.40 is the fact that each of these results can be viewed as a key point to the study of OCPs for ill-posed processes. As for the control systems described by well-posed boundary value problems, we have the following result (see Ioffe and Tikhomirov [125])

Theorem 3.41. *Let $(u^0, y^0) \in \mathbb{U} \times \mathbb{Y}_1$ be a solution to the problem*

$$I(u,y) \to \inf, \tag{3.134}$$

$$L(u,y) + F(y) = 0,\ u \in U_\partial. \tag{3.135}$$

Assume that for every $u \in U_\partial$ the mappings $y \mapsto I(u,y)$, $y \mapsto L(u,y)$ and $y \mapsto F(y)$ are continuously differentiable at $y \in \mathcal{O}(y^0)$, where $\mathcal{O}(y^0)$ is some neighborhood of $y^0 \in \mathbb{Y}_1$. Assume that the image of the mapping

$$\mathcal{D}_y L(u^0,y^0) + \mathcal{D}_y F(y^0) : \mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z} \tag{3.136}$$

is closed in $\mathbb{Z}$ and has a finite codimension. Assume also that for every $y \in \mathcal{O}(y^0)$ the mapping $u \to I(u,y)$ is convex and Gâteaux differentiable at (u^0,y^0) and the mapping $u \to L(u,y)$ is affine and continuous. Then there exists a pair $(\lambda,\psi) \in \mathbb{R}^1_+ \times \mathbb{Z}' \setminus \{0\}$ such that

$$\left.\begin{aligned} \big\langle \mathcal{D}_y \mathcal{L}(u^0,y^0,\lambda,\psi), h\big\rangle_{\mathbb{Y}_1^*,\mathbb{Y}_1} = 0, \quad \forall\, h \in \mathbb{Z}, \\ \big\langle \mathcal{D}_u \mathcal{L}(u^0,y^0,\lambda,\psi), u-u^0\big\rangle_{\mathbb{U}^*,\mathbb{U}} \geq 0, \quad \forall\, u \in U_\partial, \end{aligned}\right\} \tag{3.137}$$

where the Lagrange function $\mathcal{L}$ is defined in (3.129). Furthermore, if the mapping (3.136) is an epimorphism, that is,

$$\operatorname{Im}\big(\mathcal{D}_y L(u^0,y^0) + \mathcal{D}_y F(y^0)\big) = \mathbb{Z}, \tag{3.138}$$

then the parameter λ can be taken as 1.

To illustrate the application of the results given above, we consider a few examples. Let us begin with the following OCP

$$I(u,y) = \frac{1}{2}\int_\Omega |\nabla y(x) - \nabla z_\partial(x)|^2\,\mathrm{d}x + \frac{N}{2}\int_\Omega |u(x)|^2\,\mathrm{d}x \longrightarrow \inf, \tag{3.139}$$

$$\triangle y(x) = u(x), \quad x \in \Omega, \quad y|_{\partial\Omega} = 0, \tag{3.140}$$

$$u \in U_\partial, \tag{3.141}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ with a Lipschitz continuous boundary $\partial\Omega$, $z_\partial \in H^1(\Omega)$, $N > 0$, and U_∂ is a convex closed subset of $L^2(\Omega)$.

Theorem 3.42. *A pair $(u^0,y^0) \in L^2(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))$ is optimal for the problem (3.139)–(3.141) if and only if for some $\psi \in L^2(\Omega)$ the triplet (u^0,y^0,ψ) satisfies the relations*

$$\triangle y^0(x) = u^0(x), \quad x \in \Omega, \quad y^0\big|_{\partial\Omega} = 0, \tag{3.142}$$

$$\triangle\psi(x) - \triangle y^0(x) = \triangle z_\partial(x), \quad x \in \Omega, \quad \psi|_{\partial\Omega} = 0, \tag{3.143}$$

$$u^0 \in U_\partial, \quad \int_\Omega (Nu^0(x) - \psi(x))(u(x) - u^0(x))\,\mathrm{d}x \geq 0 \quad \forall\, u \in U_\partial. \tag{3.144}$$

Proof. Let us set

$$\mathbb{Y} = H_0^1(\Omega), \quad \mathbb{Y}_1 = H^2(\Omega) \cap H_0^1(\Omega),$$
$$\mathbb{U} = L^2(\omega), \quad \mathbb{Z} = L^2(\Omega), \quad L(u,y) = \triangle y - u, \quad F(y) = F_0 = 0.$$

First, we note that by Theorem 3.2 the OCP (3.139)–(3.141) has a unique solution $(u^0, y^0) \in L^2(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))$. Indeed, it remains only to verify hypotheses (H1) and (H2) because the remaining conditions of Theorem 3.2 are obvious. Since the Dirichlet boundary value problem (3.140) has a unique solution $y = y(u) \in H^2(\Omega) \cap H_0^1(\Omega)$ for every $u \in L^2(\Omega)$ with the estimate (see Theorem 2.20) $\|y\|^2_{H^2(\Omega)} \le C\|u\|^2_{L^2(\Omega)}$, it follows that hypothesis (H1) is obviously true. As for the coerciveness condition (H2), this immediately follows from (3.139) and the fact that

$$\|y\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla y(x)|^2 \, \mathrm{d}x,$$

by Friedrichs inequality.

To derive the optimality system for the OCP, we make use of Theorem 3.41. To do so, we have to verify condition (3.138). However, since the boundary value problem (3.140) has a unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$ for every $u \in L^2(\Omega)$, it means that

$$\operatorname{Im}\left(\mathcal{D}_y L(u^0, y^0) + \mathcal{D}_y F(y^0)\right) = \operatorname{Im}\left(\mathcal{D}_y L(u^0, y^0)\right) = \mathbb{Z} = L^2(\Omega).$$

Thus, by Theorem 3.41, we take a Lagrange function as follows:

$$L(u, y, \psi) = \frac{1}{2}\|\nabla y - \nabla z_\partial|^2_{L^2(\Omega)} + \frac{N}{2}\|u\|^2_{L^2(\Omega)} + (\triangle y - u, \psi)_{L^2(\Omega)}, \quad (3.145)$$

where $\psi \in L^2(\Omega)$.

It is clear that, in this case, condition $(3.137)_2$ implies inequality (3.144), whereas from $(3.137)_1$, it follows that

$$\int_\Omega \left(\nabla y^0 - \nabla z_\partial\right) \cdot \nabla z \, \mathrm{d}x + \int_\Omega \triangle z(x)\psi(x) \, \mathrm{d}x = 0 \quad (3.146)$$

for all $z \in H^2(\Omega) \cap H_0^1(\Omega)$.

Let us show that the last relation leads us to (3.143). Indeed, by Friedrichs inequality, we have $\|y\|_{H_0^1(\Omega)} = \|\nabla y\|_{\mathbf{L}^2(\Omega)}$. So, from (3.146) we deduce that

$$\left|(\triangle z, \psi)_{L^2(\Omega)}\right| \le C\|\nabla z\|_{\mathbf{L}^2(\Omega)}.$$

Therefore, the mapping $z \mapsto (\triangle z, \psi)_{L^2(\Omega)}$ can be extended by continuity from $H^2(\Omega) \cap H_0^1(\Omega)$ onto $H_0^1(\Omega)$. Due to the Riesz theorem, an element $q \in H_0^1(\Omega)$ can be found such that

$$(\triangle z, \psi)_{L^2(\Omega)} = (\nabla z, \nabla q)_{\mathbf{L}^2(\Omega)} = \langle \triangle z, q \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \quad (3.147)$$

So, $q = \psi \in H_0^1(\Omega)$, and hence the boundary condition in (3.143) is valid. In addition, combining (3.146) and (3.147), we obtain the conjugate equation (3.143) which should be interpreted in the distributional sense. This concludes the proof. □

We close this subsection with the following optimal boundary control problem:

$$I(u,y) = \frac{1}{2}\,\|\partial_\nu y - z_\partial\|^2_{L^2(\partial\Omega)} + \frac{N}{2}\,\|u\|^2_{L^2(\partial\Omega)} \longrightarrow \inf, \tag{3.148}$$

$$\triangle y(x) = f(x), \quad x \in \Omega, \tag{3.149}$$

$$y|_{\partial\Omega} = u, \quad u \in U_\partial, \tag{3.150}$$

where $z_\partial \in L^2(\partial\Omega)$, $f \in L^2(\Omega)$, $N > 0$, $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative to $\partial\Omega$ and U_∂ is a convex closed subset of $L^2(\partial\Omega)$ such that $U_\partial \cap H^1(\partial\Omega) \neq \emptyset$. As follows from Theorem 3.16, there exists a unique solution $(u^0, y^0) \in L^2(\partial\Omega) \times H_\Delta^{3/2}(\Omega)$ to the optimal boundary control problem (3.148)–(3.150). We recall that $H_\Delta^{3/2}(\Omega) = \left\{y \in H^{3/2}(\Omega) \,:\, \triangle y \in L^2(\Omega)\right\}$ and the trace operator $y \mapsto (y|_{\partial\Omega}\,, \partial y/\partial\nu|_{\partial\Omega})$ is well defined and continuous as the mapping

$$\Pi : H_\Delta^{3/2}(\Omega) \to H^1(\partial\Omega) \times L^2(\partial\Omega). \tag{3.151}$$

Theorem 3.43. *A pair $(u^0, y^0) \in H^1(\partial\Omega) \times H_\Delta^{3/2}(\Omega)$ is optimal for the problem (3.148)–(3.150) if and only if for some $\psi \in H_\Delta^{1/2}(\Omega)$ the triplet $(u^0, y^0, \psi) \in H^1(\partial\Omega) \times H_\Delta^{3/2}(\Omega) \times H_\Delta^{1/2}(\Omega)$ satisfies the relations*

$$\triangle y^0(x) = f(x), \quad x \in \Omega, \quad y^0\big|_{\partial\Omega} = u^0, \tag{3.152}$$

$$\triangle \psi(x) = 0 \quad x \in \Omega, \quad \psi|_{\partial\Omega} + \partial_\nu y^0\big|_{\partial\Omega} = z_\partial, \tag{3.153}$$

$$u^0 \in U_\partial, \quad \int_\Omega (Nu^0 - \partial_\nu\psi)(u - u^0)\,\mathrm{d}\mathcal{H}^{n-1} \geq 0, \quad \forall\, u \in U_\partial. \tag{3.154}$$

Proof. In order to prove this assertion, we make use of Theorem 3.2. To this end, we set

$$\mathbb{Y} = \mathbb{Y}_1 = H_\Delta^{3/2}(\Omega), \quad \mathbb{U} = H^1(\partial\Omega),$$
$$\mathbb{Z} = L^2(\Omega) \times H_0^1(\partial\Omega), \quad L(u,y) = (\triangle y - f,\, y|_{\partial\Omega} - u),$$
$$F(y) = F_0 = (0,0).$$

As in the previous case, the existence of a unique solution $z \in H_\Delta^{3/2}(\Omega)$ to the problem $\triangle z = h_1$, $z|_{\partial\Omega} = h_2$ for every $h_1 \in L^2(\Omega)$ and $h_2 \in H^1(\partial\Omega)$ implies the fulfillment of the condition (3.138) of Theorem 3.41. Since the remaining assumptions of this theorem are obviously true, we can apply relations (3.137) to the case when the Lagrange function is

$$\mathcal{L}(u,y,\psi)=\frac{1}{2}\,\|\partial_\nu y-z_\partial\|^2_{L^2(\partial\Omega)}+\frac{N}{2}\,\|u\|^2_{L^2(\partial\Omega)}$$
$$+(\triangle y-f,\psi)_{L^2(\Omega)}+\langle q,\,y|_{\partial\Omega}-u\rangle_{H^{-1}(\partial\Omega),H^1_0(\partial\Omega)}\,, \quad (3.155)$$

where $\psi\in L^2(\Omega)$ and $q\in H^{-1}(\partial\Omega)$.

Then, using (3.155) in $(3.137)_1$, we get

$$\big(\partial_\nu y^0\big|_{\partial\Omega}-z_\partial,\,\partial_\nu z|_{\partial\Omega}\big)_{L^2(\partial\Omega)}+(\triangle z,\psi)_{L^2(\Omega)}$$
$$+\langle q,\,z|_{\partial\Omega}\rangle_{(H^1(\partial\Omega))^*,H^1(\partial\Omega)}=0,\quad \forall\, z\in H^{3/2}_\Delta(\Omega). \quad (3.156)$$

This implies that $\triangle\psi=0$ in $\mathcal{D}'(\Omega)$. Hence, $\psi\in H^0_\Delta(\Omega)$. Then if we combine $\triangle\psi=0$ with (3.156), we get

$$\big(\partial_\nu y^0\big|_{\partial\Omega}-z_\partial+\psi|_{\partial\Omega}\,,\,\partial_\nu z|_{\partial\Omega}\big)_{L^2(\partial\Omega)}$$
$$+\langle q-\partial_\nu\psi|_{\partial\Omega}\,,\,z|_{\partial\Omega}\rangle_{(H^1(\partial\Omega))^*,H^1(\partial\Omega)}=0\quad\forall\, z\in H^{3/2}_\Delta(\Omega).$$

As a result, we deduce

$$\psi|_{\partial\Omega}+\partial_\nu y^0\big|_{\partial\Omega}=z_\partial, \quad (3.157)$$
$$\partial_\nu\psi|_{\partial\Omega}=q. \quad (3.158)$$

Hence, the conditions $\triangle\psi=0$ in $\mathcal{D}'(\Omega)$ and (3.157) imply the required relations (3.153). As for the inequality (3.154), it remains only to apply $(3.137)_2$ to (3.155) and take into account the condition (3.158). This concludes the proof. □

3.6.2 Necessary optimality conditions in the form of variational inequalities

The question we will discuss in this subsection is the optimality condition to essentially nonlinear OCPs in Banach spaces with control constraints. Typically, these optimality conditions consist of some operator equations and variational inequalities. We refer to Barbu [19], Barbu and Precupanu [20], Lions [171, 172], Tiba [239] as some basic textbooks on various questions and results related to this subdomain of control theory.

Let $\mathbb{Y}$ be a reflexive Banach space, $\mathbb{Y}^*$ be its dual, and $\mathbb{U}$ be a control space which is assumed to be dual to some separable Banach space $\mathbb{V}$ ($\mathbb{U}=\mathbb{V}^*$). Let U_∂ be a subset of admissible controls in $\mathbb{U}$. The OCPs we consider can be stated as follows:

$$I(u,y)\longrightarrow\inf, \quad (3.159)$$
$$A(u,y)=f,\quad u\in U_\partial\subset\mathbb{U}, \quad (3.160)$$

where $A:\mathbb{U}\times\mathbb{Y}\to\mathbb{Y}^*$ is nonlinear mapping, $I:\mathbb{U}\times\mathbb{Y}\to\overline{\mathbb{R}}$ is a cost functional, f is a given element of $\mathbb{Y}^*$ and U_∂ is a subset of admissible controls.

We assume that there exists at least one admissible pair (u, y) for the problem (3.159)–(3.160) (i.e., satisfying (3.160) and such that $I(u, y)$ is finite). As indicated in Theorem 3.12, under assumptions (A1)–(A3) and (A6), the OCP (3.159)–(3.160) has a nonempty set of solutions. Let $(u^0, y^0) \in \Xi$ be an optimal pair and let $V = \mathcal{O}(u^0, y^0) \subset \mathbb{U} \times \mathbb{Y}$ be an open neighborhood of this pair. To derive the optimality system, we begin with the following assumptions:

(B1) The mapping $A : \mathbb{U} \times \mathbb{Y} \to \mathbb{Y}^*$ has partial Gâteaux derivatives in V, $\mathcal{D}_y A : V \to \mathcal{L}(\mathbb{Y}, \mathbb{Y}^*)$ and $\mathcal{D}_u A : V \to \mathcal{L}(\mathbb{U}, \mathbb{Y}^*)$, which are continuous with respect to the uniform operator topology.

(B2) The cost functional $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is Gâteaux differentiable in V and its partial derivatives $\mathcal{D}_y I : V \to \mathbb{Y}^*$ and $\mathcal{D}_u I : V \to \mathbb{U}^*$ are continuous.

(B3) If a pair $(u, y) \in V$ is admissible then $\operatorname{Ker} \mathcal{D}_y A(u, y) = \{0\}$ and

$$\frac{\langle \mathcal{D}_y A(u, y)\xi, \xi \rangle_{\mathbb{Y}^*, \mathbb{Y}}}{\|\xi\|_{\mathbb{Y}}} \to +\infty \quad \text{as} \quad \|\xi\|_{\mathbb{Y}} \to +\infty.$$

Before proceeding further, we recall some basic notation of nonlinear analysis.

Definition 3.44. *We say that the cost functional $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is sequentially lower semicompact if for any sequences $\{u_k\}_{k=1}^{\infty} \subset \mathbb{U}$ and $\{y_k\}_{k=1}^{\infty} \subset \mathbb{Y}$ such that $u_k \overset{*}{\rightharpoonup} u$ in $\mathbb{U}$ and $y_k \rightharpoonup y$ in $\mathbb{Y}$ as $k \to \infty$, there exist subsequences $\{u_{k_i}\}_{i=1}^{\infty} \subset \mathbb{U}$ and $\{y_{k_i}\}_{i=1}^{\infty} \subset \mathbb{Y}$ satisfying the inequality*

$$\liminf_{i \to \infty} I(u_{k_i}, y_{k_i}) \geq I(u, y).$$

An operator $A(u, \cdot) : \mathbb{Y} \to \mathbb{Y}^*$ is said to be bounded if for any $u \in \mathbb{U}$, the mapping $A(u, \cdot) : \mathbb{Y} \to \mathbb{Y}^*$ transforms a bounded subset of $\mathbb{Y}$ to a bounded subset of $\mathbb{Y}^*$. We also say that an operator $A(u, \cdot) : \mathbb{Y} \to \mathbb{Y}^*$ is monotone if for any $u \in \mathbb{U}$

$$\langle A(u, y_1) - A(u, y_2), y_1 - y_2 \rangle_{\mathbb{Y}^*, \mathbb{Y}} \geq 0, \quad \forall\, y_1, y_2 \in \mathbb{Y},\ y_1 \neq y_2.$$

Now we are in a position to state the main result of this subsection.

Theorem 3.45. *Assume that (i) U_∂ is a sequentially weakly-$*$ closed bounded convex subset of $\mathbb{U}$; (ii) $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is a sequentially lower semicompact functional; (iii) the operator $A : \mathbb{U} \times \mathbb{Y} \to \mathbb{Y}^*$ is coercive and possesses the property $(\mathfrak{M})$ (see (A2)–(A3)); (iv) for any $u \in \mathbb{U}$ the operator $A(u, \cdot) : \mathbb{Y} \to \mathbb{Y}^*$ is monotone and bounded. Then the OCP (3.159)–(3.160) is solvable under conditions (B1)–(B3) and any optimal pair $(u^0, y^0) \in U_\partial \times \mathbb{Y}$ satisfies the relations*

$$A(u^0, y^0) = f, \tag{3.161}$$

$$\left[\mathcal{D}_y A(u^0, y^0)\right]^* p = \mathcal{D}_y I(u^0, y^0), \tag{3.162}$$

$$\left\langle -\left[\mathcal{D}_u A(u^0, y^0)\right]^* p + \mathcal{D}_u I(u^0, y^0), u - u^0 \right\rangle_{\mathbb{U}^*, \mathbb{U}} \geq 0, \quad \forall\, u \in U_\partial. \tag{3.163}$$

Proof. Since $A(u,\cdot)$ is a bounded coercive operator satisfying the property $(\mathfrak{M})$, it follows that for any $u \in U_\partial \subset \mathbb{U}$ the operator equation $A(u,y) = f$ has at least one solution (see [126, 132]). Moreover, due to Theorem 3.12 and the fact that the lower semicompactness property of I is essentially weaker than property (A6) (see Sect. 3.2), the OCP (3.159)–(3.160) is solvable.

Let $(u^0, y^0) \in U_\partial \times \mathbb{Y}$ be some optimal pair and let (u,y) be any admissible pair belonging to the neighborhood $V = \mathcal{O}(u^0, y^0) \subset \mathbb{U} \times \mathbb{Y}$. First, we show that the inequality

$$\langle (\mathcal{D}_y A(u,y))\,\xi, \xi \rangle_{\mathbb{Y}^*,\mathbb{Y}} \geq 0, \quad \forall\, \xi \in \mathbb{Y}, \tag{3.164}$$

holds true for every $u \in U_\partial$. Indeed, let $s \in (0,1]$. Then, for any $u \in U_\partial$, $y \in \mathbb{Y}$ and $\xi \in \mathbb{Y}$, there exists $\tau \in [0,s]$ such that, by the monotonicity property of $A(u,\cdot)$ and the definition of the Gâteaux derivative $\mathcal{D}_y A(u,y)$, we have

$$\begin{aligned} 0 &\leq \langle A(u, y+s\xi) - A(u,y), s\xi \rangle_{\mathbb{Y}^*,\mathbb{Y}} \\ &= \int_0^s \langle (\mathcal{D}_y A(u, y+t\xi))\,\xi, s\xi \rangle_{\mathbb{Y}^*,\mathbb{Y}}\, \mathrm{d}t \\ &= s^2 \langle (\mathcal{D}_y A(u, y+\tau\xi))\,\xi, \xi \rangle_{\mathbb{Y}^*,\mathbb{Y}}\,. \end{aligned} \tag{3.165}$$

Since the mapping $\tau \mapsto \langle (\mathcal{D}_y A(u, y+\tau\xi))\,\xi, \xi \rangle_{\mathbb{Y}^*,\mathbb{Y}}$ is continuous on $[0,s]$, after dividing both sides of (3.165) by s^2 and letting $s \to 0$, we get (3.164).

As a result, inequality (3.164) together with the coerciveness property of the operator $\mathcal{D}_y A(u,y)$ (i.e., $\|\xi\|_{\mathbb{Y}}^{-1} \langle (\mathcal{D}_y A(u,y))\,\xi, \xi \rangle_{\mathbb{Y}^*,\mathbb{Y}} \to +\infty$ as $\|\xi\|_{\mathbb{Y}} \to \infty$) implies the fact that the operator $\mathcal{D}_y A(u,y)$ is surjective at the pair $(u,y) \in V$. Since $\operatorname{Ker} \mathcal{D}_y A(u,y) = \{0\}$, it means that the operator $\mathcal{D}_y A(u,y)$ is invertible. In fact, $[\mathcal{D}_y A(u,y)]^{-1}$ is a bounded operator. Indeed, let us set $\mathcal{D}_y A(u,y)\xi = \eta$, where $\|\eta\|_{\mathbb{Y}^*} \leq M$. Then

$$\|\xi\|_{\mathbb{Y}} \gamma(\|\xi\|_{\mathbb{Y}}) = \langle \mathcal{D}_y A(u,y)\xi, \xi \rangle_{\mathbb{Y}^*,\mathbb{Y}} \leq \|\eta\|_{\mathbb{Y}^*} \|\xi\|_{\mathbb{Y}},$$

where

$$\gamma(\|\xi\|_{\mathbb{Y}}) = \langle \mathcal{D}_y A(u,y)\xi, \xi \rangle_{\mathbb{Y}^*,\mathbb{Y}} / \|\xi\|_{\mathbb{Y}}.$$

Hence, $\gamma(\|\xi\|_{\mathbb{Y}}) \leq \|\eta\|_{\mathbb{Y}^*}$. However, by the initial assumptions, we have $\gamma(s) \to +\infty$ as $s \to \infty$. So, there exists a constant C such that $\|\xi\|_{\mathbb{Y}} = \|\,[\mathcal{D}_y A(u,y)]^{-1}\,\eta\|_{\mathbb{Y}} \neq C$ and C is independent of M. We may conclude that (see [126]) for every $(u,y) \in \mathcal{O}(u^0, y^0)$ the corresponding solutions $y = y(u)$ of the operator equation $A(u,y) = f$ are differentiable with respect to $u \in U_\partial$ and

$$y'_u(u)h = -\,[\mathcal{D}_y A(u,y)]^{-1}\,[\mathcal{D}_u A(u,y)]\,h, \quad \forall\, h \in U_\partial. \tag{3.166}$$

For a given optimal pair (u^0, y^0), we set $(u,y) \in V$ and $(w,z) = (1-\tau)(u^0,y^0) + \tau(u,y)$, where $\tau \in (0,1)$. Then

$$I(u^0, y^0) \leq I(w,z) = I\left((u^0,y^0) + \tau(u-u^0, y-y^0)\right)$$

for all $\tau \in (0,1)$ and $(u,y) \in V$. Passing to the limit in the last inequality as $\tau \to +0$, we obtain

$$\left\langle \mathcal{D}_u I(u^0,y^0),(u-u^0)\right\rangle_{\mathbb{U}^*,\mathbb{U}} + \left\langle \mathcal{D}_y I(u^0,y^0),(y-y^0)\right\rangle_{\mathbb{Y}^*,\mathbb{Y}} \geq 0, \quad \forall (u,y) \in V.$$

Using (3.166), we get

$$\begin{aligned}
&\left\langle \mathcal{D}_u I(u^0,y^0),u-u^0\right\rangle_{\mathbb{U}^*,\mathbb{U}} + \left\langle \mathcal{D}_y I(u^0,y^0),y-y^0\right\rangle_{\mathbb{Y}^*,\mathbb{Y}} \\
&\quad = \left\langle \mathcal{D}_u I(u^0,y^0),u-u^0\right\rangle_{\mathbb{U}^*,\mathbb{U}} \\
&\qquad - \left\langle \mathcal{D}_y I(u^0,y^0),\left[\mathcal{D}_y A(u^0,y^0)\right]^{-1}\left[\mathcal{D}_u A(u^0,y^0)\right](u-u^0)\right\rangle_{\mathbb{Y}^*,\mathbb{Y}} \\
&\quad = \Big\langle \mathcal{D}_u I(u^0,y^0) - \left[\mathcal{D}_u A(u^0,y^0)\right]^* \left(\left[\mathcal{D}_y A(u^0,y^0)\right]^*\right)^{-1} \\
&\qquad \times \mathcal{D}_y I(u^0,y^0),u-u^0\Big\rangle_{\mathbb{U}^*,\mathbb{U}} \geq 0
\end{aligned}$$

for all $u \in U_\partial$. Let us introduce the adjoint state $p \in \mathbb{Y}$ as a solution of the problem

$$\left[\mathcal{D}_y A(u^0,y^0)\right]^* p = \mathcal{D}_y I(u^0,y^0).$$

Since the operator $\left[\mathcal{D}_y A(u^0,y^0)\right]^*$ is invertible at the point (u^0,y^0), we have $p = \left(\left[\mathcal{D}_y A(u^0,y^0)\right]^*\right)^{-1} \mathcal{D}_y I(u^0,y^0)$. Taking this into account, the last inequality just implies the required relation (3.163). The proof is complete. □

3.7 Optimal control of distributed singular systems

Let Ω be an open bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Let $T > 0$, $\nu > 0$, $r > 1$, $\rho > 0$ and $q > 1$ be given values. The OCPs we consider in this section can be described as follows: Find a control function u and a corresponding state y such that a functional

$$I(u,y) = \frac{1}{r}\|y - z_\partial\|^r_{L^r(\Omega)} + \frac{\nu}{q}\|u\|^q_{L^q(\Omega)} \tag{3.167}$$

is minimized subject to the constraints

$$\dot{y} - \Delta y - |y|^\rho y - u = 0 \quad \text{in} \quad Q = (0,T)\times\Omega, \tag{3.168}$$

$$y|_{t=0} = y^0 \quad \text{on} \quad \Omega, \quad y = 0 \quad \text{on} \quad \Sigma = (0,T)\times\partial\Omega, \tag{3.169}$$

$$u \in U_\partial. \tag{3.170}$$

Here, $\dot{y}(t,x) = \partial y(t,x)/\partial t$, U_∂ is a convex closed subset of $L^q(\Omega)$, $y^0 \in H^1_0(\Omega)$ and $z_\partial \in L^r(Q)$ are given functions.

Let us note that the state equation (3.168) with the sign "–" before the nonlinear term has no a priori estimates. Hence, this equation is singular in the

sense that for a given control $u \in U_\partial$, the boundary value problem (3.168)–(3.169) has no global solution, in general (see [172]). So, we deal with the OCP for an ill-posed (singular) initial-boundary value problem. However, for some controls, such a solution exists, and we can suppose that hypothesis (H1) holds true. Let us set

$$\mathbb{U} = L^q(\Omega),\ \mathbb{Y} = L^r(Q),$$
$$\mathbb{Y}_1 = L^r(Q) \cap \left\{ y \in W^{(1,2),\,s}(Q)\ :\ y|_{t=0} = y^0 \right\},$$
$$L(u,y) = (\dot{y} - \triangle y - u,\, y|_{t=0}),\ F(y) = (-|y|^\rho y, -y^0),$$

where by $W^{(1,2),s}(Q)$ we denote the Sobolev space formed by all functions $y \in L^s(Q)$ such that $\partial y/\partial t$ and $D_x^\alpha y$ exist in the weak sense for each multi–index α with $|\alpha| \le 2$ and the norm

$$\|y\|_{W^{(1,2),s}(Q)} = \left(\int_0^T \int_\Omega \left[\left| \frac{\partial y(t,x)}{\partial t} \right|^s + \sum_{|\alpha| \le 2} |D_x^\alpha y(t,x)|^s \right] \mathrm{d}x\, \mathrm{d}t \right)^{1/s}$$

is finite. Here, $s = \min\{q, r/(\rho+1)\}$, for $r > \rho + 1$ and $s = q$ if $r < \rho + 1$.

According to the general theory of singular control systems [172], we define the set of admissible pairs $\Xi \subset \mathbb{U} \times \mathbb{Y}_1$ to the above problem in the form (3.4).

Then we have the following result.

Theorem 3.46 ([228]). *If $r > \rho + 1$, then the OCP (3.167)–(3.170) admits at least one solution.*

Proof. Since the cost functional I is bounded below, the extremal problem $\inf_{(u,y)\in\Xi} I(u,y)$ possesses a minimizing sequence, i.e., there exists a sequence $\{(u_k, y_k) \in \Xi\}_{k=1}^\infty$ of admissible pairs such that

$$I(u_k, y_k) \longrightarrow \inf_{(u,y)\in\Xi} I(u,y). \tag{3.171}$$

By coerciveness of the functional in the space $L^q(Q) \times L^r(Q)$, condition (3.171) implies that the sequence under consideration is bounded on the set Ξ.

Since the pair (u_k, y_k) is admissible, its components are connected by the equality

$$\dot{y}_k = \Delta y_k + |y_k|^\rho y_k + u_k.$$

Thereby the function y_k is a solution to the following initial-boundary value problem

$$\dot{y}_k - \Delta y_k = f_k \quad \text{in}\ \ Q = (0,T) \times \Omega, \tag{3.172}$$
$$y_k|_{t=0} = y^0 \quad \text{on}\ \ \Omega, \quad y_k = 0 \quad \text{on}\ \ \Sigma = (0,T) \times \partial\Omega, \tag{3.173}$$

where $f_k = |y_k|^\rho y_k + u_k$. Using the fact that the sequence $\{f_k\}_{k=1}^\infty$ is bounded in $L^s(Q)$ for $r > \rho + 1$, we can establish the boundedness of the solutions

$\{y_k\}_{k=1}^{\infty}$ to the problem (3.172)–(3.173) in $W^{(1,2),\,s}(Q)$, and hence in $\mathbb{Y}_1$. Dropping down to a subsequence (keeping the former notation), we obtain the convergence

$$u_k \rightharpoonup u \ \text{ in } \ L^q(Q), \quad y_k \rightharpoonup y \ \text{ in } \ \mathbb{Y}_1. \tag{3.174}$$

By compactness of the embedding of $W^{(1,2),s}(Q)$ into $L^{\beta-\alpha}(Q)$ (see [168], Theorem 5.1]), where $\alpha > 0$ and $1/\beta = 1/s + 1/(n+2)$, we establish that $y_k \to y$ in $L^{\beta-\alpha}(Q)$ and a.e. in Q. By the convergence $|y_k|^{\rho} y_k \to |y|^{\rho} y$ a.e. in Q, the boundedness of the sequence $\{|y_k|^{\rho} y_k\}_{k=1}^{\infty}$ in $L^{\alpha}(Q)$, we see that (see [168]) $|y_k|^{\rho} y_k \rightharpoonup |y|^{\rho} y$ in $L^{\alpha}(Q)$. As a result, passing to the limit in the problem (3.172)–(3.173) as k tends to ∞, we come to relations (3.168)–(3.169). Recalling the convexity and closure properties of U_{∂}, we conclude that the limit pair (u, y) is admissible to the original problem (3.167)–(3.170).

It remains to prove that this pair is optimal. Using (3.174), we obtain the inequality

$$\liminf_{k\to\infty} I(u_k, y_k) \geq I(u, y).$$

Then it follows by (3.171) that the pair (u, y) is optimal. This concludes the proof. □

The particular case of this OCP (with $\rho = 2$, $q = 2$ and $r = 6$) was studied by Lions in [172]. So, Theorem 3.46 can be viewed as a natural generalization of Theorem 1.3 in [172]. Note that in [172] this problem was later regularized by an adaptive penalty method and the necessary optimality conditions were derived for the regularized problem. However, these results can be easily extended to the general case, provided that the problem (3.167)–(3.17) is solvable (i.e., when $r > \rho + 1$). The main reason is the fact that in the proof of convergence of the penalty method (see [172]), these constraints on the parameters of the problem play a principle role and the smoothness of the functions under consideration is insufficient for passing to the limit in the optimality conditions for the regularized problem as the regularization parameter tends to 0. As we will see in the next chapter, it can be done through the application of a somewhat different form of the regularization method and by weakening the notion of an approximate solution to the problem.

4

Suboptimal and Approximate Solutions to Extremal Problems

In this chapter, we continue the discussion of the optimal control problems (OCPs) described in Chap. 3. Our main focus here is a proper regularization and approximation of these problems. In order to clarify the significance of these notions, we begin with the following abstract extremal problem.

Let $\mathbb{Y}$ be a reflexive Banach space, $\mathbb{Y}^*$ be its dual, and $\mathbb{Z}$ be a Banach space partially ordered by a closed reproducing pointed cone $\Lambda \subset \mathbb{Z}$. Let $\mathbb{U}$ be a control space which is assumed to be dual to some separable Banach space $\mathbb{V}$ ($\mathbb{U} = \mathbb{V}^*$). Let U_∂ be a subset of admissible controls in $\mathbb{U}$ and let K be a subset of admissible states in $\mathbb{Y}$. The OCPs we consider can be described in a general manner as follows (see (3.14)–(3.17)):

$$I(u, y) \longrightarrow \inf \tag{4.1}$$

subject to

$$A(u, y) = f, \tag{4.2}$$

$$F(u, y) \geq_\Lambda 0, \tag{4.3}$$

$$y \in K, \quad u \in U_\partial \subset \mathbb{U}, \tag{4.4}$$

where $A : \mathbb{U} \times \mathbb{Y} \to \mathbb{Y}^*$, $F : \mathbb{U} \times \mathbb{Y} \to \mathbb{Z}$ are nonlinear mappings, $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is a cost functional, and f is a given element of $\mathbb{Y}^*$. Let us recall that, in general, the mapping $u \to y(u)$ can be multivalued.

As usual, by Ξ we denote the set of all admissible pairs for the problem (4.1)–(4.4), that is,

$$\Xi = \{(u, y) \in U_\partial \times K \ : \ A(u, y) = f, \ F(u, y) \geq_\Lambda 0, \ I(u, y) < +\infty\}. \tag{4.5}$$

We associate with the problem (4.1)–(4.4) the constrained minimization problem

$$\left\langle \inf_{(u,y) \in \Xi} I(u, y) \right\rangle, \tag{4.6}$$

which, for the sake of simplicity, we will identify with the pair (I, Ξ).

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_4,
© Springer Science+Business Media, LLC 2011

As we said earlier, the problem (4.1)–(4.4) is regular if its set of admissible pairs is nonempty. Thus, we ask: Does there exist a pair (u, y) in $U_\partial \times K \subset \mathbb{U} \times \mathbb{Y}$ satisfying the state equation (4.2) and the inequality constraint (4.3) such that $I(u, y) < +\infty$? In fact, one needs the set of admissible pairs to be sufficiently rich in some sense; otherwise the OCP (4.1)–(4.4) becomes trivial. However, from a mathematical point of view, to deal directly with all constraints presented above is typically very difficult and, except for some special cases, this question is largely open. Nevertheless, in many applications and, for example, in "inexact" numerical methods, it is important to find at least an approximately admissible – in a sense to be made precise – pair when both state and control constraints are given. On the other hand, since U_∂ and K are closed and convex subsets, they may be very "thin" and it is possible that the original problem has no solutions. In view of this, it is reasonable to weaken the requirements on approximate solutions to the OCPs. In particular, we may accept that the cost functional is minimized with some error. However, it would also be reasonable to assume that the relations $A(u, y) = 0$ and $F(u, y) \geq_\Lambda 0$ and the inclusions $u \in U_\partial$ and $y \in K$ hold not strictly but rather with some (possibly high) accuracy. Moreover, the set of solutions of the extremal problem (4.6) can possibly be empty, that is, the greatest lower bound of the cost functional is often unattainable on the given set Ξ. Nevertheless, the absence of a minimum of the functional does not mean that the problem does not make any sense (see, e.g., [252]), since its greatest lower bound exists and hence can be approached with some accuracy. Thus, an extremal problem may have an approximate or suboptimal solution even if it is not solvable.

4.1 The notion of suboptimal and approximate solutions

Let τ be the product of the weak-$*$ topology of $\mathbb{U}$ and the weak topology of $\mathbb{Y}$. Let (I, Ξ) be a constrained minimization problem on $\mathbb{U} \times \mathbb{Y}$ which is associated with some OCP. We assume that this control problem can be described in a general manner as the problem (4.1)–(4.4). If the cost functional I is bounded below on the set Ξ and $\Xi \neq \emptyset$, then I has the greatest lower bound $\mu = \inf_{(u,y)\in\Xi} I(u, y)$. Hence, in view of the "direct method" in the calculus of variations, we can find a minimizing sequence $\{(u_k, y_k)\}_{k=1}^{\infty}$ in Ξ (see, e.g., [169]) such that $I(u_k, y_k) \to \mu$. As a result, we can introduce the following notion.

Definition 4.1. *For a given $\varepsilon > 0$, the OCP (4.1)–(4.4) is said to be ε-solvable if there exists a pair $(u_\varepsilon, y_\varepsilon) \in \Xi$ such that*

$$I(u_\varepsilon, y_\varepsilon) \leq \inf_{(u,y)\in\Xi} I(u, y) + \varepsilon.$$

In this case, the pair $(u_\varepsilon, y_\varepsilon)$ is called the ε-approximate solution to the problem.

Indeed, having a minimizing sequence $\{(u_k, y_k)\}_{k=1}^{\infty} \subset \Xi$, we can take an ε-approximate solution to the problem (4.1)–(4.4) as an element of this sequence with a sufficiently large index. Thus, any ε-approximate solution is an admissible pair such that the corresponding value of the cost functional is arbitrarily close to its greatest lower bound on the set Ξ. However, even if an optimal pair (u^{opt}, y^{opt}) is available, we cannot guarantee the proximity of an ε-approximate solution $(u_\varepsilon, y_\varepsilon)$ to (u^{opt}, y^{opt}) with respect to the τ topology of $\mathbb{U} \times \mathbb{Y}$.

Definition 4.2. *The OCP (4.1)–(4.4) is said to be well-posed in the Tikhonov sense if it is solvable and every minimizing sequence $\{(u_k, y_k)\}_{k=1}^{\infty} \subset \Xi$ converges to a solution $(u^{opt}, y^{opt}) \in \Xi$. In this case, a minimizing sequence is called a Tikhonov minimizing sequence.*

Note that having a Tikhonov minimizing sequence, we can guarantee both the proximity of the corresponding cost functional to its greatest lower bound and the proximity of the approximation itself to the solution (u^{opt}, y^{opt}) of the problem. As a result, for every $\varepsilon > 0$ and every neighborhood $\mathcal{O}(u^{opt}, y^{opt})$ of (u^{opt}, y^{opt}) in $\mathbb{U} \times \mathbb{Y}$, we can define a Tikhonov approximate solution $(u_\varepsilon^*, y_\varepsilon^*) \in \Xi$ to the problem (4.1)–(4.4), for which we have $(u_\varepsilon^*, y_\varepsilon^*) \in \mathcal{O}(u^{opt}, y^{opt})$ and

$$I(u_\varepsilon^*, y_\varepsilon^*) \leq \inf_{(u,y) \in \Xi} I(u, y) + \varepsilon.$$

Nevertheless, it should be stressed that even in simple applications, the construction of Tikhonov minimizing sequences and corresponding Tikhonov approximate solutions is very difficult. We introduce the notion of a weakly minimizing sequence.

Definition 4.3. *We say that a sequence $\{(u_k, y_k)\}_{k=1}^{\infty}$ in $\mathbb{U} \times \mathbb{Y}$ is a weakly minimizing sequence, if $(u_k, y_k) \in U_\partial \times K$ for every $k \in \mathbb{N}$, $u_k \xrightarrow{*} u^{opt}$ in $\mathbb{U}$, $y_k \rightharpoonup y^{opt}$ in $\mathbb{U}$, $A(u_k, y_k) \to f$ in $\mathbb{Y}^*$, $\lim_{k\to\infty} F(u_k, y_k) \geq_\Lambda 0$ and*

$$I(u^{opt}, y^{opt}) \leq \liminf_{k \to \infty} I(u_k, y_k) \leq \inf_{(u,y) \in \Xi} I(u, y).$$

Moreover, if for every $\varepsilon > 0$, every neighborhood $\mathcal{O}(u^{opt}, y^{opt})$ of (u^{opt}, y^{opt}) in the τ-topology of $\mathbb{U} \times \mathbb{Y}$ and every neighborhood $\mathcal{O}_\mathbb{Z}(0)$ of the origin in $\mathbb{Z}$, there is a pair $(u_\varepsilon^w, y_\varepsilon^w) \in U_\partial \times K$ such that the conditions

$$(u_\varepsilon^w, y_\varepsilon^w) \in \mathcal{O}(u^{opt}, y^{opt}), \ \|A(u_\varepsilon^w, y_\varepsilon^w) - f\|_{\mathbb{Y}^*} < \varepsilon,$$
$$F(u_\varepsilon^w, y_\varepsilon^w) \in \mathcal{O}_\mathbb{Z}(0) + \Lambda$$

hold true, then $(u_\varepsilon^w, y_\varepsilon^w)$ is said to be a weak ε-approximate solution to the problem (4.1)–(4.4).

It is clear that any Tikhonov approximate solution to the problem (4.1)–(4.4) is also a weak approximate solution. However, the original OCP may fail to have an exact solution $(u^{opt}, y^{opt}) \in \Xi$, that is, the greatest lower

bound of the cost functional is often unattainable on the set of admissible pairs Ξ. An extremal problem may have an approximate solution even in the absence of solvability. However, in this case, the notions of Tikhonov and weak approximate solutions make no sense, for they explicitly appeal to the notion of an exact solution. Removing the optimality of the pair (u^{opt}, y^{opt}) from the definitions of a weak minimizing sequence and a weak approximate solution problem, we arrive at the weakened forms of these notions.

Definition 4.4. *We say that a sequence* $\{(u_k, y_k)\}_{k=1}^{\infty}$ *in* $\mathbb{U} \times \mathbb{Y}$ *is a weakened minimizing sequence if* $(u_k, y_k) \in U_\partial \times K$ *for every* $k \in \mathbb{N}$, $u_k \overset{*}{\rightharpoonup} \widehat{u}$ *in* $\mathbb{U}$, $y_k \rightharpoonup \widehat{y}$ *in* $\mathbb{U}$, $A(u_k, y_k) \to f$ *in* $\mathbb{Y}^*$, $\lim_{k\to\infty} F(u_k, y_k) \geq_\Lambda 0$ *and*

$$I(\widehat{u}, \widehat{y}) \leq \liminf_{k \to \infty} I(u_k, y_k) \leq \inf_{(u,y)\in \Xi} I(u, y).$$

Moreover, if for some $(\widehat{u}, \widehat{y}) \in \mathbb{U} \times \mathbb{Y}$, *every* $\varepsilon > 0$, *every neighborhood* $\mathcal{O}(\widehat{u}, \widehat{y})$ *of* $(\widehat{u}, \widehat{y})$ *in the* τ*-topology of* $\mathbb{U} \times \mathbb{Y}$ *and every neighborhood* $\mathcal{O}_{\mathbb{Z}}(0)$ *of the origin in* $\mathbb{Z}$, *there is a pair* $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)$ *such that the conditions*

$$(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \mathcal{O}(\widehat{u}, \widehat{y}) \cap (U_\partial \times K), \quad \|A(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) - f\|_{\mathbb{Y}^*} < \varepsilon,$$
$$F(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \mathcal{O}_{\mathbb{Z}}(0) + \Lambda$$

hold true, then $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)$ *is said to be a weakened* ε*-approximate solution to the problem (4.1)–(4.4).*

Thus, a weakened approximate solution is a pair arbitrarily close to the set of admissible solutions, guaranteeing in a sense the proximity of the cost functional to its greatest lower bound. Since the only difference from the above notion (see Definition 4.3) is the absence of the optimality requirement on $(\widehat{u}, \widehat{y})$, a weak approximate solution to the OCP is certainly a weakened approximate solution.

Observe also that the convex closed subsets U_∂ and K may be very "thin" with rather complicated structures. Because of this, a direct numerical computation of solutions to the corresponding OCPs with such constraints can be extremely difficult. Usually a very fine discretization mesh is needed, which means a significant computation time. Therefore, it is reasonable to introduce the following notion.

Definition 4.5. *If for every* $\varepsilon > 0$ *there exists a pair* $(u_\varepsilon^{sub}, y_\varepsilon^{sub}) \in \mathbb{U} \times \mathbb{Y}$ *such that*

$$\left| I(u_\varepsilon^{sub}, y_\varepsilon^{sub}) - \inf_{(u,y)\in \Xi} I(u, y) \right| \leq \varepsilon, \tag{4.7}$$

$$(u_\varepsilon^{sub}, y_\varepsilon^{sub}) \in \Xi_\varepsilon = \left\{(u, y) \in \mathbb{U} \times \mathbb{Y} \ : \ \operatorname{dist}\left((u, y), \Xi\right)_{\mathbb{U}\times\mathbb{Y}} \leq \varepsilon\right\}, \tag{4.8}$$

then the pair $(u_\varepsilon^{sub}, y_\varepsilon^{sub})$ *is said to be an* ε*-suboptimal solution to the problem (4.1)–(4.4).*

In the next sections, we consider a large class of OCPs for nonlinear operator equations. Their solvability is established only under certain additional assumptions. Nevertheless, even for the general statement, we describe an algorithm, following in many aspects Serovaĭskiĭ [228] and Ivanenko and Mel'nik [126], that enables us to construct weak and weakened minimizing sequences and hence to find the corresponding approximate solutions to these problems.

4.2 Regularization of optimal control problems

The main object of our consideration in this section is the OCP (4.1)–(4.4). We begin with the following observation: The set of admissible pairs Ξ to the original problem can be presented in the form $\Xi = \Xi_1 \cap \Xi_2$, where

$$\Xi_1 = \{(u,y) \in U_\partial \times \mathbb{Y} \ : \ A(u,y) = f, \ y \in K\},$$
$$\Xi_2 = \{(u,y) \in U_\partial \times \mathbb{Y} \ : \ F(u,y) \geq_\Lambda 0\}.$$

Let us assume that $\Xi = \emptyset$, provided that $\Xi_1 \neq \emptyset$. In this case, we associate with the set Ξ_2 a functional $J_{\Xi_2} : \Xi_1 \to \overline{R}$ characterizing a "deviation measure" from the set Ξ_2. We suppose that $J_{\Xi_2}(u,y) = 0$ if and only if $(u,y) \in \Xi_2$. In what follows, we will call each of the sets Ξ_i the corresponding constraint system.

Definition 4.6. *Assume that $\Xi_1 \neq \emptyset$. We say that the OCP (4.1)–(4.4) is J_{Ξ_2}-regularized if there exists a pair $(\widehat{u},\widehat{y})$ in $U_\partial \times \mathbb{Y}$ such that $J_{\Xi_2}(\widehat{u},\widehat{y}) = \inf_{(u,y)\in\Xi_1} J_{\Xi_2}(u,y)$. We also say that Ξ_1 is a J_{Ξ_2}-attainable restriction system if there exists a pair $(\widehat{u},\widehat{y}) \in U_\partial \times \mathbb{Y}$ satisfying the condition $J_{\Xi_2}(\widehat{u},\widehat{y}) = 0$.*

Before proceeding further, for every $(u,y) \in \Xi_1$ we define the set

$$\mathfrak{L}^*(u,y) = \left\{\xi \in \Lambda^* \ : \ \langle \xi, F(u,y)\rangle_{\mathbb{Z}^*,\mathbb{Z}} < 0\right\} \cup \{0\},$$

where $\Lambda^* \subset \mathbb{Z}^*$ is the conjugate semigroup with respect to the cone Λ (see (3.23)). It is clear that $\mathfrak{L}^*(u,y)$ is a convex pointed cone in $\mathbb{Z}^*$. Let S_1^* be the unit sphere in $\mathbb{Z}^*$. We set

$$J_{\Xi_2}(u,y) = \sup_{\xi \in \mathfrak{L}^*(u,y)\cap S_1^*} \left|\langle \xi, F(u,y)\rangle_{\mathbb{Z}^*,\mathbb{Z}}\right|. \tag{4.9}$$

A pair $(\widehat{u},\widehat{y}) \in \Xi_1$ is said to be a regularizer of the constraint system Ξ_2 to the OCP (4.1)–(4.4) if

$$J_{\Xi_2}(\widehat{u},\widehat{y}) = \inf_{(u,y)\in\Xi_1} \sup_{\xi \in \mathfrak{L}^*(u,y)\cap S_1^*} \left|\langle \xi, F(u,y)\rangle_{\mathbb{Z}^*,\mathbb{Z}}\right|.$$

By $\mathfrak{R}$ we denote a set of all regularizers of Ξ_2.

Theorem 4.7. *Assume that $\Xi_1 \neq \emptyset$ and conditions (A1)–(A6) of Sect. 3.2 hold true. Then the OCP (4.1)–(4.4) is J_{Ξ_2}-regularized. Moreover, there exists a pair $(\overline{u}, \overline{y}) \in \mathfrak{R}$ (optimal regularizer to this problem) such that $I(\overline{u}, \overline{y}) \leq I(u, y)$ for all $(u, y) \in \mathfrak{R}$.*

Proof. We divide our proof into three steps.

Step 1. By analogy with notation of Sect. 3.2, we denote by $\mathcal{F}(\mathbb{R}, \mathbb{R}_+)$ the set of lower semicontinuous monotone decreasing functions $\mu : \mathbb{R} \to \mathbb{R}_+$ such that $\mu(0) = 0$ and μ is strictly monotone on $\mathbb{R}_-$. To be specific, we set

$$\mu(\xi) = \begin{cases} |\xi| & \text{if } \xi < 0, \\ 0 & \text{if } \xi \geq 0. \end{cases}$$

Let us consider the following approximating control problem

$$I_\mu(u, y) = \sup_{\psi \in S_1^* \cap \Lambda^*} \left[\mu \left(\langle \psi, F(u, y) \rangle_{\mathbb{Z}^*; \mathbb{Z}} \right) \right] \to \inf, \tag{4.10}$$

$$A(u, y) = f, \tag{4.11}$$

$$y \in K, \quad u \in U_\partial \subset \mathbb{U}, \tag{4.12}$$

where $\mu \in \mathcal{F}(\mathbb{R}, \mathbb{R}_+)$.

It is easy to see that (4.10)–(4.12) is a special case of the penalized problem studied in Sect. 3.2. Since $\Xi_1 \neq \emptyset$ and suppositions (A1)–(A6) are valid, the existence at least one optimal pair $(\widehat{u}_\mu, \widehat{y}_\mu) \in \Xi_1$ to the problem (4.10)–(4.12) immediately follows by Lemma 3.10. Note that this optimal pair is not necessarily unique. To begin, we show that $(\widehat{u}_\mu, \widehat{y}_\mu)$ is a regularizer for the OCP (4.1)–(4.4) (i.e., $(\widehat{u}_\mu, \widehat{y}_\mu) \in \mathfrak{R}$). Indeed, if $I_\mu(\widehat{u}_\mu, \widehat{y}_\mu) = 0$ then $F(\widehat{u}_\mu, \widehat{y}_\mu) \in \Lambda$ and hence $J_{\Xi_2}(\widehat{u}_\mu, \widehat{y}_\mu) = 0$. So, in this case, $(\widehat{u}_\mu, \widehat{y}_\mu) \in \mathfrak{R}$, and moreover, $(\widehat{u}_\mu, \widehat{y}_\mu) \in \Xi$ is an optimal regularizer to the original problem.

For now we assume that $I_\mu(\widehat{u}_\mu, \widehat{y}_\mu) > 0$. This implies $F(\widehat{u}_\mu, \widehat{y}_\mu) \notin \Lambda$. Therefore,

$$\begin{aligned} &\sup_{\psi \in S_1^* \cap \Lambda^*} \left[\mu \left(\langle \psi, F(\widehat{u}_\mu, \widehat{y}_\mu) \rangle_{\mathbb{Z}^*; \mathbb{Z}} \right) \right] \\ &\qquad = \sup_{\psi \in S_1^* \cap \mathfrak{L}^*(\widehat{u}_\mu, \widehat{y}_\mu)} \left[\mu \left(\langle \psi, F(\widehat{u}_\mu, \widehat{y}_\mu) \rangle_{\mathbb{Z}^*; \mathbb{Z}} \right) \right] \\ &\qquad = \sup_{\xi \in \mathfrak{L}^*(\widehat{u}_\mu, \widehat{y}_\mu) \cap S_1^*} \left| \langle \xi, F(\widehat{u}_\mu, \widehat{y}_\mu) \rangle_{\mathbb{Z}^*, \mathbb{Z}} \right| = J_{\Xi_2}(\widehat{u}_\mu, \widehat{y}_\mu). \end{aligned} \tag{4.13}$$

Assume that there is a pair $(u_*, y_*) \in \Xi_1$ such that

$$0 < J_{\Xi_2}(u_*, y_*) < J_{\Xi_2}(\widehat{u}_\mu, \widehat{y}_\mu).$$

Taking into account (4.13), we come to the contradiction that $I_\mu(u_*, y_*) < I_\mu(\widehat{u}_\mu, \widehat{y}_\mu)$. Thus, $(\widehat{u}_\mu, \widehat{y}_\mu) \in \Xi_1$ is an regularizer of the OCP (4.1)–(4.4).

Step 2. Existence of an optimal regularizer. Having set

$$d = \inf_{(u,y)\in\Xi_1} J_{\Xi_2}(u,y),$$

we consider the following penalized OCP:

$$\widehat{I}_\varepsilon(u,y) = I(u,y) + \varepsilon^{-1}\left[\sup_{\psi\in S_1^*\cap\Lambda^*} \mu\left(\langle\psi, F(u,y)\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right) - d\right]^2 \longrightarrow \inf \tag{4.14}$$

subject to the restrictions (4.11)–(4.12).

Following Lemma 3.10, for every $\varepsilon > 0$, the problem (4.11)–(4.12), (4.14) has a nonempty set of solutions. Let $\{(u_\varepsilon, y_\varepsilon) \in \Xi_1\}_{\varepsilon>0}$ be a sequence of optimal pairs to this problem when the small parameter $\varepsilon > 0$ varies in a strictly decreasing sequence of positive numbers which converge to 0. In the same way as in the proof of Lemma 3.10, we can conclude that the sequence $\{(u_\varepsilon, y_\varepsilon) \in \Xi_1\}_{\varepsilon>0}$ is relatively w-compact in $\mathbb{U}\times\mathbb{Y}$ and, passing to a subsequence, we get

$$u_\varepsilon \overset{*}{\rightharpoonup} u \ \text{ in } \ \mathbb{U}, \quad y_\varepsilon \rightharpoonup y \ \text{ in } \ \mathbb{Y}, \quad (u,y) \in U_\partial \times K, \quad A(u,y) = f.$$

Let us prove

$$\sup_{\xi\in\mathfrak{L}^*(u,y)\cap S_1^*} \left|\langle\xi, F(u,y)\rangle_{\mathbb{Z}^*,\mathbb{Z}}\right| = d. \tag{4.15}$$

To do so, we observe that, as follows from the previous step, the extremal problem (4.11)–(4.12), (4.14) with the additional restriction

$$\sup_{\psi\in S_1^*\cap\Lambda^*} \mu\left(\langle\psi, F(u,y)\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right) = d$$

is regular. Let (v,q) be any admissible pair to this problem. Then $\widehat{I}_\varepsilon(u_\varepsilon, y_\varepsilon) \le I(v,q)$. Hence,

$$\left[\sup_{\psi\in S_1^*\cap\Lambda^*} \mu\left(\langle\psi, F(u_\varepsilon, y_\varepsilon)\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right) - d\right]^2 \le C\varepsilon.$$

So using the lower semicontinuity property of $\widehat{I}_\varepsilon$ with respect to the w-convergence, we have

$$\left[\sup_{\psi\in S_1^*\cap\Lambda^*} \mu\left(\langle\psi, F(u,y)\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right) - d\right]^2$$
$$\le \liminf_{\varepsilon\to 0}\left[\sup_{\psi\in S_1^*\cap\Lambda^*} \mu\left(\langle\psi, F(u_\varepsilon, y_\varepsilon)\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right) - d\right]^2 \le C\lim_{\varepsilon\to 0}\varepsilon = 0.$$

Therefore, $\liminf_{\varepsilon\to 0}\widehat{I}_\varepsilon(u_\varepsilon, y_\varepsilon) \ge I(u,y)$, and in view of the (4.13), we obtain the required relation (4.15).

Step 3. It remains to prove that (u, y) is an optimal regularizer. Let us assume the converse: There exists an admissible pair (v, q) to the problem (4.11)–(4.12), (4.14), (4.15) such that $I(v, q) < I(u, y)$. Then due to the construction of the penalized cost functional (4.14), we come to the opposite inequality

$$I(u, y) \le \liminf_{\varepsilon \to 0} I(u_\varepsilon, y_\varepsilon) \le \liminf_{\varepsilon \to 0} \widehat{I}_\varepsilon(u_\varepsilon, y_\varepsilon) \le I(v, q).$$

This concludes the proof. □

We now proceed with the discussion of the J_{Ξ_1}-regularization of the OCP (4.1)–(4.4). We assume that the constraint system

$$\Xi_1 = \{(u, y) \in U_\partial \times \mathbb{Y} \ : \ A(u, y) = f, \ y \in K\}$$

is not considered as level constraints and we may satisfy the state equation $A(u, y) = f$ and the corresponding state constraint $y \in K$ with some accuracy. For this purpose, we make use the following observation: If a pair (u, y) is admissible to the original problem, then this pair satisfies the relation

$$\langle A(u, y), \zeta - y \rangle_{\mathbb{Y}^*, \mathbb{Y}} \ge \langle f, \zeta - y \rangle_{\mathbb{Y}^*, \mathbb{Y}}, \quad \forall \zeta \in K. \tag{4.16}$$

Note that the reverse statement is not true, in general. In fact, we discuss a variant of the penalization approach, called the "variational inequality method". This idea was first studied in Mel'nik [186]. Further references and a complete theoretical analysis may be found in Tiba [238, 239], Neittaanmäki and Tiba [204], Ivanenko and Mel'nik [126]. We reconsider the problem (4.1)–(4.4) with the given assumptions and we apply the variational inequality procedure. For every fixed $u \in U_\partial$, the operator $A(u, \cdot) : \mathbb{Y} \to \mathbb{Y}^*$ is pseudo-monotone bounded and coercive. So, there exists at least one solution $y = y(u)$ of the variational inequality (4.16) such that $y \in K$ (see [19, 104]). Note that it is not necessary that the corresponding pair (u, y) satisfy the operator equation $A(u, y) = f$. In view of this, we can use the penalized term $\varepsilon^{-1} \|A(u, y) - f\|_{\mathbb{Y}^*}$ as a deviation measure in an associated cost functional. As a result, we consider the following penalized OCP:

$$\begin{aligned} \widehat{I}_\varepsilon(u, y) = I(u, y) &+ \varepsilon^{-1} \|A(u, y) - f\|_{\mathbb{Y}^*} \\ &+ \varepsilon^{-1} \sup_{\psi \in S_1^* \cap \Lambda^*} \left[\mu \left(\langle \psi, F(u, y) \rangle_{\mathbb{Z}^*; \mathbb{Z}} \right) \right] \to \inf_{u \in U_\partial} \end{aligned} \tag{4.17}$$

subject to the restriction (4.16).

Our aim is to show that each cluster pair of any sequence of optimal pairs to the problem (4.16)–(4.17), when the small parameter $\varepsilon > 0$ varies in a strictly decreasing sequence of positive numbers converging to 0, is a weak ε-approximate solution to the original problem (4.1)–(4.4). We begin with the following result.

Lemma 4.8. *Assume that $A : \mathbb{U} \times \mathbb{Y} \to \mathbb{Y}^*$ is a quasi-monotone operator (see Remark 3.13). Then, under suppositions (A1), (A2), and (A4)–(A6) of*

Sect. 3.2, the OCP (4.16)–(4.17) admits at least one solution for every fixed $\varepsilon > 0$.

Proof. Let $\{(u_k, y_k)\}_{k=1}^{\infty} \subset U_\partial \times K$ be a minimizing sequence of admissible pairs to the problem (4.16)–(4.17). The coerciveness property (A2) and the estimate

$$\langle A(u_k, y_k), y_k - \zeta\rangle_{\mathbb{Y}^*,\mathbb{Y}} \le \langle f, y_k - \zeta\rangle_{\mathbb{Y}^*,\mathbb{Y}} \le \|f\|_{\mathbb{Y}^*}\|y_k - \zeta\|_{\mathbb{Y}} \tag{4.18}$$

immediately imply that the sequence $\{y_k\}_{k=1}^{\infty}$ is bounded. Since the set $U_\partial \times K$ is sequentially closed with respect to the w-convergence, we may assume that there exists a pair $(u_\varepsilon^0, y_\varepsilon^0) \in U_\partial \times K$ such that $(u_k, y_k) \overset{w}{\to} (u_\varepsilon^0, y_\varepsilon^0)$. Then passing to the limit in (4.18) as $k \to \infty$, we obtain

$$\limsup_{k\to\infty} \langle A(u_k, y_k), y_k - \zeta\rangle_{\mathbb{Y}^*,\mathbb{Y}} \le 0 \quad \text{for all} \quad \zeta \in K.$$

Hence,

$$\liminf_{k\to\infty} \langle A(u_k, y_k), y_k - \zeta\rangle_{\mathbb{Y}^*,\mathbb{Y}} \ge \left\langle A(u_\varepsilon^0, y_\varepsilon^0), y_\varepsilon^0 - \zeta\right\rangle_{\mathbb{Y}^*,\mathbb{Y}}, \quad \forall\, \zeta \in K,$$

by the quasi-monotonicity property of the operator A. Combining this inequality with

$$\limsup_{k\to\infty} \langle A(u_k, y_k), y_k - \zeta\rangle_{\mathbb{Y}^*,\mathbb{Y}} \le \left\langle f, y_\varepsilon^0 - \zeta\right\rangle_{\mathbb{Y}^*,\mathbb{Y}}, \quad \forall\, \zeta \in K,$$

we come to the relation

$$\left\langle A(u_\varepsilon^0, y_\varepsilon^0), \zeta - y_\varepsilon^0\right\rangle_{\mathbb{Y}^*,\mathbb{Y}} \ge \left\langle f, \zeta - y_\varepsilon^0\right\rangle_{\mathbb{Y}^*,\mathbb{Y}} \quad \forall\, \zeta \in K.$$

Thus, $(u_\varepsilon^0, y_\varepsilon^0) \in U_\partial \times K$ is an admissible pair to the problem (4.16)–(4.17).

Let us show that $(u_\varepsilon^0, y_\varepsilon^0)$ is an optimal pair to this problem. Let d be a weak limit in $\mathbb{Y}^*$ of the sequence $\{A(u_k, y_k)\}_{k=1}^{\infty}$ as $k \to \infty$. Then

$$\begin{aligned}\limsup_{k\to\infty} \ & \langle A(u_k, y_k), y_k\rangle_{\mathbb{Y}^*,\mathbb{Y}} \\ & \le \left\langle f, y_\varepsilon^0 - \zeta\right\rangle_{\mathbb{Y}^*,\mathbb{Y}} + \langle d, \zeta\rangle_{\mathbb{Y}^*,\mathbb{Y}} \\ & = \left\langle d, y_\varepsilon^0\right\rangle_{\mathbb{Y}^*,\mathbb{Y}} + \left\langle d - f, \zeta - y_\varepsilon^0\right\rangle_{\mathbb{Y}^*,\mathbb{Y}} \quad \forall\, \zeta \in K.\end{aligned}$$

Substituting y_ε^0 for ζ in the last inequality, we get

$$\limsup_{k\to\infty} \langle A(u_k, y_k), y_k\rangle_{\mathbb{Y}^*,\mathbb{Y}} \le \left\langle d, y_\varepsilon^0\right\rangle_{\mathbb{Y}^*,\mathbb{Y}}.$$

Since the quasi-monotone operator possesses the $(\mathfrak{M})$-property (see Remark 3.13), it follows that $d = A(u_\varepsilon^0, y_\varepsilon^0)$. Moreover, it is easy to see that for every $\psi \in \Lambda^*$, we have

$$\langle \psi, F(u_k, y_k)\rangle_{\mathbb{Z}^*;\mathbb{Z}} \to \left\langle \psi, F(u_\varepsilon^0, y_\varepsilon^0)\right\rangle_{\mathbb{Z}^*;\mathbb{Z}} \quad \text{as} \quad k \to \infty.$$

As a result, using the w-lower semicontinuity property of the cost functional (4.17), we finally obtain

$$\liminf_{k\to\infty} \widehat{I}_\varepsilon(u_k, y_k) \geq I(u^0_\varepsilon, y^0_\varepsilon) + \varepsilon^{-1}\|A(u^0_\varepsilon, y^0_\varepsilon) - f\|_{\mathbb{Y}^*} + \varepsilon^{-1} \sup_{\psi\in S_1^*\cap\Lambda^*} \left[\mu\left(\langle\psi, F(u^0_\varepsilon, y^0_\varepsilon)\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right)\right] = \widehat{I}_\varepsilon(u^0_\varepsilon, y^0_\varepsilon).$$

Thus, $(u^0_\varepsilon, y^0_\varepsilon)$ is an optimal pair to the penalized problem (4.16)–(4.17). □

The next step of our analysis is to consider a sequence of optimal pairs $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon>0} \subset U_\partial \times K$ in the limit as ε tends to 0.

Theorem 4.9. *In addition to the assumptions of Lemma 4.8, assume that K is a bounded subset of the state space $\mathbb{Y}$ and that the OCP (4.1)–(4.4) is regular. Let $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon>0}$ be a sequence of optimal pairs to the penalized problem (4.16)–(4.17). Then $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon>0}$ is a weak minimizing sequence for the original OCP (4.1)–(4.4). Moreover, each element of this sequence can be taken as the corresponding weak ε-approximate solution to the problem (4.1)–(4.4).*

Proof. In view of our assumptions, the sequence $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon>0}$ is bounded in $U_\partial \times K$. Hence, we may assume that there exists a pair $(u^0, y^0) \in U_\partial \times K$ such that $(u^0_\varepsilon, y^0_\varepsilon) \overset{w}{\to} (u^0, y^0)$ as $\varepsilon \to 0$. Since

$$\langle A(u^0_\varepsilon, y^0_\varepsilon), y^0_\varepsilon - \zeta\rangle_{\mathbb{Y}^*,\mathbb{Y}} \leq \langle f, y^0_\varepsilon - \zeta\rangle_{\mathbb{Y}^*,\mathbb{Y}}, \quad \forall\, \zeta \in K,\ \forall\, \varepsilon > 0.$$

and the operator A is quasi-monotone and bounded, it follows that we can use the arguments of the proof of Lemma 4.8. As a result, we have $A(u^0_\varepsilon, y^0_\varepsilon) \rightharpoonup d$ in $\mathbb{Y}^*$, $d = A(u^0, y^0)$ and

$$\begin{aligned}\langle f, \zeta - y^0\rangle_{\mathbb{Y}^*,\mathbb{Y}} &\leq \limsup_{\varepsilon\to 0} \langle A(u^0_\varepsilon, y^0_\varepsilon), \zeta - y^0_\varepsilon\rangle_{\mathbb{Y}^*,\mathbb{Y}} \\ &\leq \langle A(u^0, y^0), \zeta - y^0\rangle_{\mathbb{Y}^*,\mathbb{Y}}, \quad \forall \zeta \in \mathbb{Y}.\end{aligned}$$

Thus, the limit pair $(u^0, y^0) \in U_\partial \times K$ satisfies the variational inequality (4.16). Let us show that this pair is admissible to the original problem (4.1)–(4.4).

Let $(u, y) \in \Xi$ be any admissible pair. Then $\widehat{I}_\varepsilon(u, y) = I(u, y)$. On the other hand, since $(u^0_\varepsilon, y^0_\varepsilon)$ is an optimal pair to the problem (4.16)–(4.17), this yields $\widehat{I}_\varepsilon(u^0_\varepsilon, y^0_\varepsilon) \leq \widehat{I}_\varepsilon(u, y) = I(u, y)$ for every $\varepsilon > 0$. So, the sequence $\{\widehat{I}_\varepsilon(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded with respect to ε. Hence, in view of the structure of the cost functional (4.17), we deduce

$$\|A(u^0_\varepsilon, y^0_\varepsilon) - f\|_{\mathbb{Y}^*} \leq C_1\varepsilon, \qquad \sup_{\psi\in S_1^*\cap\Lambda^*} \left[\mu\left(\langle\psi, F(u^0_\varepsilon, y^0_\varepsilon)\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right)\right] \leq C_2\varepsilon.$$

From this, we immediately conclude that

$$A(u^0, y^0) = f, \quad F(u^0, y^0) \geq 0.$$

Thus, $(u^0, y^0) \in \Xi$.

It remains to prove that (u^0, y^0) is an optimal pair. If, on the contrary, we assume that the exists a pair $(u, y) \in \Xi$ such that $I(u, y) < I(u^0, y^0)$, then

$$I(u_\varepsilon^0, y_\varepsilon^0) \le I(u_\varepsilon^0, y_\varepsilon^0) + \varepsilon^{-1}\|A(u_\varepsilon^0, y_\varepsilon^0) - f\|_{\mathbb{Y}^*} + \varepsilon^{-1} \sup_{\psi \in S_1^* \cap \Lambda^*} \left[\mu\left(\left\langle \psi, F(u_\varepsilon^0, y_\varepsilon^0)\right\rangle_{\mathbb{Z}^*;\mathbb{Z}}\right)\right] \le I(u, y), \quad \forall\, \varepsilon > 0.$$

Therefore, passing to the limit in this inequality as $\varepsilon \to 0$ and using the w-lower semicontinuity property of the cost functional (4.1), we get

$$I(u^0, y^0) \le \liminf_{\varepsilon \to 0} \le I(u, y).$$

This contradiction immediately leads us to the conclusion: The sequence of optimal pairs $\left\{(u_\varepsilon^0, y_\varepsilon^0)\right\}_{\varepsilon > 0}$ satisfies all the properties of Definition 4.3. So, the proof is complete. □

4.3 ε-Suboptimal solutions to optimal control problems

As was noted in Sect. 4.1, many problems coming from applications are lacking solvability. Nevertheless, as discussed above, an inexact or relaxed formulation may seem reasonable. Thus, an extremal problem may have an approximate solution even in the absence of solvability. Taking as an example the OCP (4.1)–(4.4), we note that one of the most restrictive assumptions is the coerciveness property of the operator $A : \mathbb{U} \times \mathbb{Y} \to \mathbb{Y}^*$. Below we give a particular result concerning a noncoercive OCPs. We show that in spite of the absence of optimal solutions to such problems, for every fixed $\varepsilon > 0$, there exists an ε-suboptimal pair in the sense of Definition 4.5.

We begin with the following result.

Theorem 4.10. *Assume that the hypotheses (A1) and (A3)–(A6) of Sect. 3.2 and the regularity property $\Xi \neq \emptyset$ for the problem (4.1)–(4.4) hold true. Assume also that the sets $U_\partial \subset \mathbb{U}$ and $K \subset \mathbb{Y}$ are not necessarily bounded. Then for every $\varepsilon > 0$ an ε-suboptimal solution to the problem (4.1)–(4.4) can be taken as a solution to the following extremal problem:*

$$J_\varepsilon(u, y) = I(u, y) + \varepsilon\left(\|u\|_{\mathbb{U}} + \|y\|_{\mathbb{Y}}\right) \longrightarrow \inf, \tag{4.19}$$

$$A(u, y) = f, \tag{4.20}$$

$$F(u, y) \ge_\Lambda 0, \tag{4.21}$$

$$y \in K, \quad u \in U_\partial \subset \mathbb{U}. \tag{4.22}$$

Proof. It is clear that for every fixed $\varepsilon > 0$, the cost functional $J_\varepsilon : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ possesses the following property: $\forall\, C > 0$, the set

$$\{(u, y) \in U_\partial \times K \ : \ J_\varepsilon(u, y) \le C\}$$

is bounded in $\mathbb{U} \times \mathbb{Y}$. Hence, the solution set to the problem (4.19)–(4.22) is nonempty for every $\varepsilon > 0$.

Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a strictly decreasing sequence of positive numbers converging to 0 as $k \to \infty$. Let $\{(u_\varepsilon, y_\varepsilon) \in U_\partial \times K\}_{k=1}^{\infty}$ be the corresponding solutions of the extremal problem (4.19)–(4.22) with $\varepsilon = \varepsilon_k$. Let us show that this sequence is ε-minimizing to the problem (4.1)–(4.4), that is, for every $\varepsilon > 0$ a value $j \in \mathbb{N}$ can be found such that

$$I(u_{\varepsilon_k}, y_{\varepsilon_k}) \leq \inf_{(u,y)\in \Xi} I(u, y) + \varepsilon, \quad \forall\, k \geq j.$$

Indeed, let (u, y) be any admissible pair to the original noncoercive problem (4.1)–(4.4). Then for any ε_i and ε_k such that $\varepsilon_i < \varepsilon_k$, we have $J_{\varepsilon_i}(u, y) \leq J_{\varepsilon_k}(u, y)$. Hence,

$$d \leq \inf_{(u,y)\in \Xi} J_{\varepsilon_{i+1}}(u, y) \leq \inf_{(u,y)\in \Xi} J_{\varepsilon_i}(u, y) \leq \cdots \leq \inf_{(u,y)\in \Xi} J_{\varepsilon_{i-k}}(u, y) \leq \cdots .$$

Using the fact that the problem (4.19)–(4.22) is solvable for all $\varepsilon > 0$, we deduce

$$d \leq J_{\varepsilon_{i+1}}(u_{\varepsilon_{i+1}}, y_{\varepsilon_{i+1}}) \leq J_{\varepsilon_i}(u_{\varepsilon_i}, y_{\varepsilon_i}) \leq \cdots \leq J_{\varepsilon_{i-k}}(u_{\varepsilon_{i-k}}, y_{\varepsilon_{i-k}}) \leq \cdots .$$

Thus, the non-negative numerical sequence $\{J_{\varepsilon_i}(u_{\varepsilon_i}, y_{\varepsilon_i})\}_{i=1}^{\infty}$ is monotonically decreasing and bounded below. As a result, for any $\varepsilon > 0$, a value $j(\varepsilon)$ can be found such that

$$I(u_{\varepsilon_i}, y_{\varepsilon_i}) \leq J_{\varepsilon_i}(u_{\varepsilon_i}, y_{\varepsilon_i}) \leq d + \varepsilon \quad \text{for all} \quad i \geq j(\varepsilon).$$

This concludes the proof. □

As a special case of the OCP (4.1)–(4.4), let us consider the OCP of the distributed singular system (3.167)–(3.170). To rewrite it in the form of the OCP (4.1)–(4.4), we make use of the following notation: Let Ω be an open bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and let $T > 0$, $\nu > 0$, $r > 1$, $\rho > 0$ and $q > 1$ be given values. Let U_∂ be a convex closed subset of $L^q(\Omega)$ and let $y^0 \in H_0^1(\Omega)$ and $z_\partial \in L^r(Q)$ be given functions, where $Q = (0, T) \times \Omega$. Let us set

$$\mathbb{U} = L^q(\Omega),\ \mathbb{Y} = L^r(Q) \cap \left\{ y \in W^{(1,2),\,s}(Q)\ :\ y|_{t=0} = y^0 \right\},$$

$$A(u, y) = \dot{y} - \triangle y - |y|^\rho y - u,\ I(u, y) = \frac{1}{r}\|y - z_\partial\|^r_{L^r(\Omega)} + \frac{\nu}{q}\|u\|^q_{L^q(\Omega)},$$

where $s = \min\{q, r/(\rho + 1)\}$ for $r > \rho + 1$ and $s = q$ for $r < \rho + 1$. We define the set of admissible pairs as follows:

$$\Xi = \{(u, y) \in \mathbb{U} \times \mathbb{Y}\ :\ A(u, y) = 0,\ u \in U_\partial\}.$$

As a result, the OCP (3.167)–(3.170) can be reduced to the abstract extremal problem

$$\inf_{(u,y)\in\Xi} I(u,y). \tag{4.23}$$

Let us note that this problem admits at least one solution in the case when $r > \rho + 1$ (see Theorem 3.46). So, our prime interest in this section is to consider the general statement of the problem (4.23), that is, the case in which there are no constraints (as in [172]) on the parameters, the inequality $r > \rho+1$ may be violated, and the problem (4.23) may be unsolvable. Following Serovaĭskiĭ [227], to construct ε-suboptimal solutions for this problem, we will use the penalty method. In view of this, we introduce the following auxiliary cost functional:

$$I_\varepsilon(u,y) = I(u,y) + \frac{\varepsilon\gamma}{b}\|y\|^b_{L^b(Q)} + \frac{1}{\varepsilon q}\|A(u,y)\|^q_{L^q(Q)}, \tag{4.24}$$

where $\varepsilon > 0$ is the regularization parameter, $\gamma = 0$ for $r > \rho + 1$, $\gamma = 1$ for $r \le \rho + 1$, $b = \rho + 1 + \alpha$ and $\alpha > 0$.

Let us note that if the problem (4.23) is solvable, then we have the standard penalty method (see [172]), whereas for $r \le \rho + 1$ the cost functional involves one more summand which guarantees the availability of an additional a priori estimate and tends to 0 as $\varepsilon \to 0$.

We begin with the following result.

Lemma 4.11. *The approximated problem*

$$\textit{Minimize the functional } I_\varepsilon \textit{ on the set } U_\partial \times \mathbb{Y} \tag{4.25}$$

is solvable for every $\varepsilon > 0$.

Proof. Since the cost functional I_ε is bounded below on the set $U_\partial \times \mathbb{Y}$, it follows that for every $\varepsilon > 0$, there is a sequence $\{(u_k, y_k)\}_{k=1}^\infty$ in $U_\partial \times \mathbb{Y}$ such that the condition

$$\lim_{k\to\infty} I_\varepsilon(u_k, y_k) = \inf_{(u,y)\in U_\partial\times\mathbb{Y}} I_\varepsilon(u,y)$$

holds true. Then, as follows from (4.24), the sequence $\{y_k\}_{k=1}^\infty$ is bounded in $L^c(Q)$ and the sequences $\{u_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are bounded in $L^q(Q)$, where

$$g_k = \dot{y}_k - \triangle y_k - |y_k|^\rho y_k - u_k, \quad c = \begin{cases} r \text{ for } & r > \rho + 1, \\ b \text{ for } & r \le \rho + 1. \end{cases}$$

From this we can deduce that the sequence $\{f_k = g_k + |y_k|^\rho y_k + u_k\}_{k=1}^\infty$ is bounded in $L^d(Q)$, where $d = \min\{q, c/(\rho+1)\}$. As a result, it turns out that each of the functions $y_k(t,x)$ is a solution to the problem

$$\dot{y}_k - \Delta y_k = f_k \quad \text{in } \ Q = (0,T)\times\Omega, \tag{4.26}$$

$$y_k|_{t=0} = y^0 \ \text{ on } \ \Omega, \quad y_k = 0 \ \text{ on } \ \Sigma = (0,T)\times\partial\Omega. \tag{4.27}$$

Hence, due to Theorem 3.46, the sequence $\{y_k\}_{k=1}^{\infty}$ is uniformly bounded in $W^{(1,2),d}(Q)$. So, passing to subsequences, we may suppose that $u_k \rightharpoonup u$ in $L^q(Q)$ and $y_k \rightharpoonup y$ in $W^{(1,2),\,d}(Q) \cap L^c(Q)$. Repeating the arguments of the proof of Theorem 3.46, it is easy to derive the relation

$$\lim_{k\to\infty} I_\varepsilon(u_k, y_k) \geq I_\varepsilon(u, y).$$

Hence, the limit pair (u, y) is a solution to the problem (4.25). The proof is complete. □

Since the approximated problem (4.25) has a solution $(u_\varepsilon, y_\varepsilon) \in U_\partial \times \mathbb{Y}$ for every $\varepsilon > 0$, the next step of our consideration is to consider the asymptotic behavior of the sequence of such solutions $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ as ε tends to 0. Having supposed that the small parameter ε discretely diminishes in the sequence $\{\varepsilon > 0\}$, we can state the following result:

Theorem 4.12. *Assume that $\Xi \neq \emptyset$. Then for every $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that an optimal pair $(u_{\varepsilon_\delta}, y_{\varepsilon_\delta})$ to the corresponding problem (4.25) (with $\varepsilon = \varepsilon_\delta$) is δ-suboptimal to the original OCP (3.167)–(3.170).*

Proof. By our assumptions, the set of admissible pairs Ξ to the problem (3.167)–(3.170) is nonempty. Since the cost functional I is bounded below on Ξ, it has the greatest lower bound $\inf_{(u,y)\in\Xi} I(u, y)$ on the set Ξ. Then we can find a minimizing sequence $\{(u_k, y_k)\}_{k=1}^{\infty}$ in Ξ such that $I(u_k, y_k) \to \inf_{(u,y)\in\Xi} I(u, y)$. Having a minimizing sequence, we can take an approximate solution to the problem as an element of the sequence $\{(u_k, y_k)\}_{k=1}^{\infty}$ with a sufficiently large index. Thereby, for every $\delta > 0$, we can define a pair $(u^\delta, y^\delta) \in \Xi$, called an approximate solution to the problem, such that the inequality $I(u^\delta, y^\delta) < \inf_{(u,y)\in\Xi} I(u, y) + \delta$ is valid. Since $\Xi \subset U_\partial \times \mathbb{Y}$ and the pair $(u_\varepsilon, y_\varepsilon)$ is optimal for the problem (4.23), we come to the relation

$$\begin{aligned} I_\varepsilon(u_\varepsilon, y_\varepsilon) &= \min_{(u,y)\in U_\partial\times\mathbb{Y}} I_\varepsilon(u, y) \leq I_\varepsilon(u^\delta, y^\delta) \\ &= I(u^\delta, y^\delta) + \frac{\varepsilon\gamma}{b}\|y^\delta\|^b_{L^b(Q)} \\ &\leq \inf_{(u,y)\in\Xi} I(u, y) + \delta + \frac{\varepsilon\gamma}{b}\|y^\delta\|^b_{L^b(Q)}. \end{aligned} \tag{4.28}$$

Taking into account the structure of the functional I_ε (see (4.24)) and passing to the limit in (4.28) as $\varepsilon \to 0$, we get

$$\liminf_{\varepsilon\to 0} I(u_\varepsilon, y_\varepsilon) \leq \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon) \leq \inf_{(u,y)\in\Xi} I(u, y) + \delta. \tag{4.29}$$

From this, we immediately deduce: For a given δ, there exist $\varepsilon_\delta > 0$ and a constant $C > 0$ independent of ε and $\delta > 0$ such that

$$(u_{\varepsilon_\delta}, y_{\varepsilon_\delta}) \in U_\partial \times \mathbb{Y}, \quad \left| I(u_{\varepsilon_\delta}, y_{\varepsilon_\delta}) - \inf_{(u,y)\in\Xi} I(u, y) \right| < \delta,$$

$$\|A(u_{\varepsilon_\delta}, y_{\varepsilon_\delta})\|_{L^q(Q)} \leq \varepsilon_\delta^{1/q} C < \delta.$$

Thus, $(u_{\varepsilon_\delta}, y_{\varepsilon_\delta})$ is the δ-suboptimal pair in the sense of Definition 4.5. □

As was noted earlier, the approximated problem (4.25) has a solution $(u_\varepsilon, y_\varepsilon) \in U_\partial \times \mathbb{Y}$ for every $\varepsilon > 0$. To define these solutions, we derive the optimality system for the OCP (4.25). However, before we do so, we should emphasize that the necessary conditions for optimality in the approximated problem (4.25) will differ slightly in form from those in [172] for the case $p = 2$, $q = 2$ and $r = 6$. Using equivalent transformations, we can nevertheless rewrite them in the same form.

Theorem 4.13. *Let $(u_{\varepsilon_\delta}, y_{\varepsilon_\delta}) \in U_\partial \times \mathbb{Y}$ be an optimal pair to the problem (4.25). Then there is a function $p_\varepsilon \in L^{q'}(Q)$, where $1/q + 1/q' = 1$, satisfying the relations*

$$\int_0^T \int_\Omega \left(\nu |u_\varepsilon|^{q-2} u_\varepsilon + p_\varepsilon\right)(w - u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \geq 0, \quad \forall\, w \in U_\partial, \tag{4.30}$$

$$\begin{aligned} \dot{p}_\varepsilon + \Delta p_\varepsilon + (\rho+1)|y_\varepsilon|^\rho p_\varepsilon \\ = -|y_\varepsilon - z_\partial|^{r-2}(y_\varepsilon - z_\partial) \\ - \gamma\varepsilon |y_\varepsilon|^{b-2} y_\varepsilon \quad \text{in} \quad Q = (0,T)\times\Omega, \end{aligned} \tag{4.31}$$

$$p_\varepsilon|_{t=T} = 0 \quad \text{on} \quad \Omega, \quad p_\varepsilon = 0 \quad \text{on} \quad \Sigma = (0,T)\times\partial\Omega. \tag{4.32}$$

where y_ε is a solution to the problem

$$\dot{y}_\varepsilon - \Delta y_\varepsilon - |y_\varepsilon|^\rho y_\varepsilon - u_\varepsilon = \varepsilon^{1/(q-1)} |p_\varepsilon|^{q'-1} p_\varepsilon \quad \text{in} \quad Q, \tag{4.33}$$

$$y_\varepsilon|_{t=0} = y^0 \quad \text{on} \quad \Omega, \quad y_\varepsilon = 0 \quad \text{on} \quad \Sigma. \tag{4.34}$$

Proof. Following [169] (see also Theorems 3.37 and 3.41) and taking into account the structure of the set of admissible pairs $U_\partial \times \mathbb{Y}$ to the problem (4.25), we see that the necessary conditions for this problem take the form of the variational inequality in the first argument and the stationary state condition in the second argument, that is,

$$\mathcal{D}_u I_\varepsilon(u_\varepsilon, y_\varepsilon)[w - u_\varepsilon] \geq 0, \quad \forall\, w \in U_\partial, \tag{4.35}$$

$$\mathcal{D}_y I_\varepsilon(u_\varepsilon, y_\varepsilon) = 0, \tag{4.36}$$

where by $\mathcal{D}_u I_\varepsilon(u_\varepsilon, y_\varepsilon)$ and $\mathcal{D}_y I_\varepsilon(u_\varepsilon, y_\varepsilon)$ we denote the corresponding Gâteaux derivatives of I_ε calculated at the pair $(u_\varepsilon, y_\varepsilon)$. Naturally, the existence of these derivatives needs justification.

As Lemma 4.11 indicates, the optimal state y_ε for the problem (4.25) turns out to be an element of the set $Z = W^{(1,2),\,d}(Q) \cap L^c(Q)$. Therefore, by $\mathcal{D}_y I_\varepsilon$ we mean the corresponding derivative over the subspace Z of $\mathbb{Y}$. It means (see [11]) that, varying the functional in the second argument, we can choose the increment in Z. In view of the conditions (3.169), this increment must satisfy zero initial conditions, which agrees again with the definition of the derivative over the subspace Z_0 of Z constituted by the functions vanishing

at $t = 0$. Thus, by $\mathcal{D}_y I_\varepsilon$ in (4.36) we mean the partial Gâteaux derivative over the subspace Z_0.

To find the derivative of the cost functional I_ε in the first argument, we note that the following equality is valid for arbitrary functions u and g in $L^q(Q)$, y in Z and a number σ:

$$\begin{aligned} &I_\varepsilon(u+\sigma g, y) - I_\varepsilon(u,y) \\ &\quad = \frac{\nu}{q}\int_0^T \int_\Omega [|u+\sigma g|^q - |u|^q]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{1}{q\varepsilon}\int_0^T \int_\Omega [|A(u+\sigma g, y)|^q - |A(u,y)|^q]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \sigma\nu \int_0^T \int_\Omega |u|^{q-2}ug\,\mathrm{d}x\,\mathrm{d}t + \frac{\sigma}{\varepsilon}\int_0^T \int_\Omega |A(u,y)|^{q-2}A(u,y)g\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + o(\sigma). \end{aligned}$$

As a result, we have

$$\begin{aligned} \mathcal{D}_u I_\varepsilon(u,y)[g] &= \nu\int_0^T \int_\Omega |u|^{q-2}ug\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{1}{\varepsilon}\int_0^T \int_\Omega |A(u,y)|^{q-2}A(u,y)g\,\mathrm{d}x\,\mathrm{d}t, \quad \forall\, g \in L^q(Q). \end{aligned}$$

Inserting this value into (4.35), we just establish (4.30), where

$$p_\varepsilon = \varepsilon^{-1}|A(u_\varepsilon, y_\varepsilon)|^{q-2}A(u_\varepsilon, y_\varepsilon).$$

Since the value of the operator A at a solution to the approximated problem (4.25) belongs to $L^q(Q)$, we obtain $p_\varepsilon \in L^{q'}(Q)$, where $1/q + 1/q' = 1$. The preceding equality also implies the validity of (4.33). The boundary conditions (4.34) hold because the solution to the problem (4.25) satisfies (3.169).

Similarly, we can deduce the validity of the following equality for arbitrary functions u in $L^q(Q)$, y in Z, h in Z_0 and a number σ:

$$\begin{aligned} &I_\varepsilon(u, y+\sigma h) - I_\varepsilon(u,y) \\ &\quad = \frac{1}{r}\int_0^T \int_\Omega [|y+\sigma h - z_\partial|^r - |y-z_\partial|^r]\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \frac{\varepsilon\gamma}{b}\int_0^T \int_\Omega \left[|y+\sigma h|^b - |y|^b\right]\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \frac{1}{q\varepsilon}\int_0^T \int_\Omega [|A(u,y+\sigma h)|^q - |A(u,y)|^q]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \sigma\int_0^T \int_\Omega |y-z_\partial|^{r-2}(y-z_\partial)h\,\mathrm{d}x\,\mathrm{d}t + \sigma\varepsilon\gamma\int_0^T \int_\Omega |y|^{b-2}yh\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \frac{\sigma}{\varepsilon}\int_0^T \int_\Omega \left\{|A(u,y)|^{q-2}A(u,y)\left[\dot{h} - \Delta h - (\rho+1)|y|^\rho h\right]\right\}\,\mathrm{d}x\,\mathrm{d}t + o(\sigma). \end{aligned}$$

From this, we obtain the relation

$$\begin{aligned}
&\mathcal{D}_y I_\varepsilon(u,y)[h] \\
&\quad = \int_0^T \int_\Omega |y - z_\partial|^{r-2}(y - z_\partial)h \,\mathrm{d}x\,\mathrm{d}t + \varepsilon\gamma \int_0^T \int_\Omega |y|^{b-2} yh \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{\varepsilon}\int_0^T \int_\Omega \left\{ |A(u,y)|^{q-2} A(u,y) \left[\dot{h} - \Delta h - (\rho+1)|y|^\rho h \right] \right\} \mathrm{d}x\,\mathrm{d}t, \\
&\qquad\qquad \forall\, h \in Z.
\end{aligned}$$

Consequently, condition (4.36) takes the form

$$\begin{aligned}
\int_0^T \int_\Omega \Big\{ &\left[|y_\varepsilon - z_\partial|^{r-2}(y_\varepsilon - z_\partial) + \varepsilon\gamma |y_\varepsilon|^{b-2} y_\varepsilon \right] h \\
&+ p_\varepsilon \left[\dot{h} - \Delta h - (\rho+1)|y_\varepsilon|^\rho h \right] \Big\} \mathrm{d}x\,\mathrm{d}t = 0, \quad \forall\, h \in Z.
\end{aligned}$$

As a result, we see that the function p_ε is a solution to the boundary value problem (4.31)–(4.32) in $L^{q'}(Q)$. Thus, for every fixed $\varepsilon > 0$, a solution $(u_\varepsilon, y_\varepsilon)$ to the problem (4.25) can be found by solving the relations (4.30)–(4.34). This concludes the proof. □

4.4 Approximate solutions to distributed singular systems

This section deals with the design of weak and weakened ε-approximate solutions to one class of OCPs for distributed singular systems. In many aspects, we follow Serovaĭskiĭ (see [228, 229]). Let Ω be an open bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Let $T > 0$, $\nu > 0$, $r > 1$, $\rho > 0$ and $q > 1$ be given values. Let U_∂ be a convex closed subset of $L^q(\Omega)$ and let $y^0 \in H_0^1(\Omega)$ and $z_\partial \in L^r(Q)$ be given functions. We consider the following OCP:

$$I(u,y) = \frac{1}{r}\|y - z_\partial\|^r_{L^r(\Omega)} + \frac{\nu}{q}\|u\|^q_{L^q(\Omega)} \longrightarrow \inf, \tag{4.37}$$

$$\dot{y} - \Delta y - |y|^\rho y - u = 0 \quad \text{in} \quad Q = (0,T)\times\Omega, \tag{4.38}$$

$$y|_{t=0} = y^0 \quad \text{on} \quad \Omega, \quad y = 0 \quad \text{on} \quad \Sigma = (0,T)\times\partial\Omega, \tag{4.39}$$

$$u \in U_\partial. \tag{4.40}$$

By analogy with the previous section, we set

$$\mathbb{U} = L^q(\Omega),\ \mathbb{Y} = L^r(Q) \cap \left\{ y \in W^{(1,2),\,s}(Q) \ :\ y|_{t=0} = y^0 \right\},$$

$$A(u,y) = \dot{y} - \triangle y - |y|^\rho y - u,\ I(u,y) = \frac{1}{r}\|y - z_\partial\|^r_{L^r(\Omega)} + \frac{\nu}{q}\|u\|^q_{L^q(\Omega)},$$

where $s = \min\{q, r/(\rho+1)\}$ for $r > \rho+1$ and $s = q$ for $r < \rho+1$. We define the set of admissible pairs as follows:

$$\Xi = \{(u,y) \in \mathbb{U} \times \mathbb{Y} \ : \ A(u,y) = 0, \ u \in U_\partial\}\,.$$

Let us recall that (see Theorem 3.46) the problem (4.37)–(4.40) is solvable in $\mathbb{U} \times \mathbb{Y}$ if $\Xi \neq \emptyset$ and $r > \rho+1$. In the general case, this question remains open. However, as we will see later, this problem has weakened ε-approximate solutions, which in the case $r > \rho+1$ become weak approximate solutions.

We turn our attention to the approximate problem

$$I_\varepsilon(u,y) = I(u,y) + \frac{\varepsilon\gamma}{b}\|y\|^b_{L^b(Q)} + \frac{1}{\varepsilon q}\|A(u,y)\|^q_{L^q(Q)} \to \inf_{(u,y)\in U_\partial\times\mathbb{Y}}, \tag{4.41}$$

where the regularization parameter $\varepsilon > 0$ takes its values in a sequence which tends to 0, $\gamma = 0$ for $r > \rho+1$, $\gamma = 1$ for $r \le \rho+1$, $b = \rho+1+\alpha$ and $\alpha > 0$. Let us point out that this problem has a solution $(u_\varepsilon, y_\varepsilon) \in U_\partial \times \mathbb{Y}$ for every $\varepsilon > 0$ (see Lemma 4.11). Let $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ be a sequence of such solutions. We begin with the following result:

Theorem 4.14. *Assume that $\Xi \neq \emptyset$. Then the sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ is a weakened minimizing sequence (in the sense of Definition 4.4) for the problem (4.37)–(4.40).*

Proof. Let $\{(u_k, y_k)\}_{k=1}^\infty \subset \Xi$ be a minimizing sequence for the cost functional I on the set Ξ. Then for every $\delta > 0$, we can define an approximate solution $(u^\delta, y^\delta) \in \Xi$ such that the inequality

$$I(u^\delta, y^\delta) < \inf_{(u,y)\in\Xi} I(u,y) + \delta$$

is valid. Since $\Xi \subset U_\partial \times \mathbb{Y}$ and the pair $(u_\varepsilon, y_\varepsilon)$ is optimal to the problem (4.41), we get

$$\begin{aligned} I_\varepsilon(u_\varepsilon, y_\varepsilon) &= \min_{(u,y)\in U_\partial\times\mathbb{Y}} I_\varepsilon(u,y) \le I_\varepsilon(u^\delta, y^\delta) \\ &= I(u^\delta, y^\delta) + \frac{\varepsilon\gamma}{b}\|y^\delta\|^b_{L^b(Q)} \\ &\le \inf_{(u,y)\in\Xi} I(u,y) + \delta + \frac{\varepsilon\gamma}{b}\|y^\delta\|^b_{L^b(Q)}. \end{aligned}$$

This leads us to the relation

$$\lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon) \le \inf_{(u,y)\in\Xi} I(u,y) + \delta.$$

Passing to the limit as $\delta \to 0$, we see that

$$\lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon) \le \inf_{(u,y)\in\Xi} I(u,y). \tag{4.42}$$

In view of the structure of the functional I_ε (see (4.41)), we have the following estimates:

$$\|u_\varepsilon\|_{L^q(Q)} \le C, \quad \|y_\varepsilon)\|_{L^r(Q)} \le C, \tag{4.43}$$

$$\|A(u_\varepsilon, y_\varepsilon)\|_{L^q(Q)} \le \varepsilon_\delta^{1/q} C, \tag{4.44}$$

with constants C on the right-hand sides independent of ε. Using the above inequalities and the Banach–Alaoglu theorem, we can extract a subsequence from $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ (keeping the former notation for simplicity) such that

$$y_\varepsilon \rightharpoonup y_0 \ \text{ in } \ L^r(Q), \quad u_\varepsilon \rightharpoonup u_0 \ \text{ in } \ L^q(Q), \tag{4.45}$$

$$A(u_\varepsilon, y_\varepsilon) \rightharpoonup 0 \ \text{ in } \ L^q(Q). \tag{4.46}$$

However, conditions (4.45) are equivalent to the relation $(u_\varepsilon, y_\varepsilon) \rightharpoonup (u_0, y_0)$ in $L^q(Q) \times L^r(Q)$. Then, using convexity and continuity of the cost functional I in $L^q(Q) \times L^r(Q)$, we obtain

$$\lim_{\varepsilon\to 0} I(u_\varepsilon, y_\varepsilon) \ge I(u_0, y_0). \tag{4.47}$$

From the definition of the regularized functional (see (4.41)), we deduce the inequality $I_\varepsilon(u, y) \ge I(u, y)$ for all $(u, y) \in U_\partial \times \mathbb{Y}$. Hence, $I_\varepsilon(u_\varepsilon, y_\varepsilon) \ge I(u_\varepsilon, y_\varepsilon)$ and, therefore,

$$\lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon) \ge \lim_{\varepsilon\to 0} I(u_\varepsilon, y_\varepsilon).$$

Using (4.42) and (4.45), we then find that

$$I(u_\varepsilon, y_\varepsilon) \le \lim_{\varepsilon\to 0} I(u_\varepsilon, y_\varepsilon) \le \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon) \le \inf_{(u,y)\in\Xi} I(u, y). \tag{4.48}$$

Then, as follows from (4.45), (4.46) and (4.48), $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ is a weakened minimizing sequence for the problem (4.37)–(4.40), and the proof is complete. □

Let us show that under the condition $r > \rho + 1$ (see Theorem 3.46), the sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ defined by the relations (4.30)-(4.34) is also a weak minimizing sequence.

Theorem 4.15. *If $\Xi \neq \emptyset$, then for $r > \rho + 1$ the sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ is a weak minimizing sequence for the problem (4.37)–(4.40).*

Proof. First, we note that, as follows from (4.44), for every $\varepsilon > 0$ the function y_ε is a solution to the boundary value problem

$$\dot{y}_\varepsilon - \Delta y_\varepsilon = |y_\varepsilon|^\rho y_\varepsilon + u_\varepsilon + f_\varepsilon \quad \text{in } \ Q = (0, T) \times \Omega, \tag{4.49}$$

$$y_\varepsilon|_{t=0} = y^0 \quad \text{on } \ \Omega, \quad y_\varepsilon = 0 \quad \text{on } \ \Sigma = (0, T) \times \partial\Omega, \tag{4.50}$$

where the function f_ε is such that

$$f_\varepsilon \rightharpoonup 0 \ \text{ in } \ L^q(Q). \tag{4.51}$$

When the inequality $r > \rho + 1$ holds true, by (4.43) and (4.51), the right-hand side of (4.49) is bounded in $L^s(Q)$ (see the proof of Theorem 3.46). Then the sequence $\{y_\varepsilon\}_{\varepsilon>0}$ is bounded in $W^{(1,2),\,s}(Q)$ and hence in $\mathbb{Y}$. Thus, the following convergence holds:

$$y_\varepsilon \rightharpoonup y_0 \quad \text{in} \quad \mathbb{Y}. \tag{4.52}$$

Repeating the arguments of the proof of Theorem 3.46, we establish that

$$|y_\varepsilon|^\rho y_\varepsilon \rightharpoonup |y_0|^\rho y_0 \quad \text{in} \quad L^a(Q), \quad \text{where} \quad a = r/(\rho+1). \tag{4.53}$$

Passing to the limit in (4.49) and using (4.43) and (4.51)–(4.53), we arrive at the relation

$$\dot{y}_0 = \triangle y_0 - |y_0|^\rho y_0 - u_0 \quad \text{in} \quad Q.$$

Since the embedding of $\mathbb{Y}$ into $C(0,T;L^2(\Omega))$ is continuous (see [113, Theorem 1.17]), using (4.53) we obtain the convergence

$$y_\varepsilon|_{t=0} \rightharpoonup y_0|_{t=0} \quad \text{in} \quad L^2(\Omega).$$

Passing to the limit in (4.50), we conclude that the function y_0 satisfies the boundary conditions

$$y_0|_{t=0} = y^0 \quad \text{on} \quad \Omega, \quad y_0 = 0 \quad \text{on} \quad \Sigma.$$

By convexity and closure properties of U_∂, we have $u_0 \in U_\partial$. From the above results, we conclude that the pair (u_0, y_0) is admissible for the problem (4.37)–(4.40). Thereby the equality $A(u_0, y_0) = 0$ is valid. Together with (4.43) and (4.44), this means continuity of the operator $A : L^q(Q) \times L^r(Q) \to \mathbb{W}$ at (u_0, y_0), where $\mathbb{W}$ denotes the space $L^q(Q)$ furnished with the weak topology.

As mentioned earlier, the continuity of A at the pair (u_0, y_0) guarantees that $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ is a weak minimizing sequence for the problem (4.37)–(4.40). Indeed, in the proof of Theorem 4.14, we established relation (4.48). Hence, since $(u_0, y_0) \in \Xi$, it follows that (u_0, y_0), as the limit of the sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$, is a solution to the problem (4.37)–(4.40). □

Thus, we have shown that for sufficiently small values of the regularization parameter ε, the approximate solution $(u_\varepsilon, y_\varepsilon)$ satisfies the state equation with any accuracy and, in a sense, approximates the greatest lower bound of the cost functional I. Moreover, if the inequality $r > \rho + 1$ is valid, $(u_\varepsilon, y_\varepsilon)$ is also sufficiently close to a solution to the original problem (4.37)–(4.40) (in the topology of $L^q(Q) \times L^r(Q)$).

5

Introduction to the Asymptotic Analysis of Optimal Control Problems: A Parade of Examples

The main interest in this book is in mathematical models of optimal control problems (OCPs) that depend on some small parameter ε. In many mathematical problems, which come from natural or engineering sciences, industrial applications, or abstract mathematical questions all by themselves, some parameters appear (small or large, of geometric or constitutive origin, coming from approximation processes or discretization arguments) and make those control problems increasingly complex or degenerate. For instance, if we deal with OCPs on reticulated structures or perforated domains, this small parameter comes from the geometry of such domains. This dependence can be defined both by a regular and a singular occurrence of the parameter ε (the presence of ε-periodic coefficients, ε-periodically perforated zone of controls, quickly oscillating boundaries, etc.). In order to proceed with a mathematical description, we formulate the following parameterized OCP (OCP_ε), where ε is a small parameter:

$$(\mathrm{OCP}_\varepsilon): \qquad \min\{I_\varepsilon(u,y) \,:\, (u,y) \in \Xi_\varepsilon\}, \tag{5.1}$$

where

(B_1) $I_\varepsilon : \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \to \overline{\mathbb{R}}$ is a cost functional (CF_ε),
(B_2) $\mathbb{Y}_\varepsilon$ is a space of states,
(B_3) $\mathbb{U}_\varepsilon$ is a space of controls,
(B_4) $\Xi_\varepsilon \subset \{(u_\varepsilon, y_\varepsilon) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \,:\, u \in U_\varepsilon, I_\varepsilon(u,y) < +\infty\}$ is a set of all admissible pairs linked by some state equation (SE_ε).

Note that, as the small parameter ε varies, it is sometimes possible to foresee some "limiting" behavior and "guess" which *effective model* we may substitute for the complex model. In view of this, the main questions we are going to answer are:

(i) What is the *behavior* of the OCP under the variation of the parameter ε?
(ii) How does one pass to the limit in such problems, as ε tends to 0?
(iii) What should be expected as a limit problem?

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_5,
© Springer Science+Business Media, LLC 2011

(iv) Can the limit problems be classified?
(v) Is there a characterization of the variational properties of the limiting problem?

It is reasonable to consider optimal solutions to the "limit" problem as approximate solutions for the original problem.

We begin our consideration with the OCP that can be described as follows:

$$(\mathrm{OCP}_\varepsilon)\ :\quad \left\{\begin{array}{rcl} (\mathrm{CF}_\varepsilon) & : & I_\varepsilon(u,y)\to\inf \\ & \text{subject to} & \\ (\mathrm{CSC}_\varepsilon) & : & (u,y)\in \mathbb{U}_\varepsilon\times\mathbb{Y}_\varepsilon, u\in U_\varepsilon\subset\mathbb{U}_\varepsilon \\ (\mathrm{SE}_\varepsilon) & : & L_\varepsilon(u,y)+F_\varepsilon(y)=0 \end{array}\right\}. \tag{5.2}$$

In what follows, we always suppose that all ingredients of the mathematical model (5.2) satisfy the assumptions of the previous section, including hypotheses (H1)–(H3). Our prime interest in this chapter is to discuss different ways and procedures for asymptotic analysis of a wide class of OCPs as the parameter ε tends to 0.

We note that without going into details of the definition of "asymptotic behavior" of OCPs, independently of what kind of "limit passage" in $(\mathrm{OCP}_\varepsilon)$ we take, this procedure has to guarantee the following variational properties:

(V1) If $(u_\varepsilon^0, y_\varepsilon^0)$ is an optimal pair of $(\mathrm{OCP}_\varepsilon)$ and if $(u_\varepsilon^0, y_\varepsilon^0)$ tends (in some sense) to (u^0, y^0), then (u^0, y^0) is an optimal solution for the limit problem.
(V2) $\lim_{\varepsilon\to 0}\inf_{(u,y)\in\,\Xi_\varepsilon} I_\varepsilon(u,y)=\inf_{(u,y)\in\,\Xi_0} I_0(u,y)$.

To give a precise meaning to this statement, we begin with the discussion of the different examples and procedures for the definition of an appropriate limit problem (OCP_0) to the family (5.2).

5.1 Component-by-component limit analysis

As follows from the description (5.2), the typical OCP consists of the independent mathematical objects such as a state equation $(\mathrm{SE}_\varepsilon)$, control and state constraints $(\mathrm{CSC}_\varepsilon)$ and a cost functional $(\mathrm{CF}_\varepsilon)$. Therefore, the scheme of *component-by-component limit analysis* is based on the passage to the limit in each of the ingredients of the mathematical model (5.2) separately as $\varepsilon\to 0$. For this, the well-known concepts of the asymptotic analysis such as epi-convergence or Γ-convergence of functionals, G-convergence of differential operators and Painlevé–Kuratowski convergence of sets and spaces can be applied. In this case, the structure of a limit OCP is well given by

$$(\mathrm{OCP}_0)\ :\quad \left\{\begin{array}{rcl} (\mathrm{CF}_0) & : & I_0(u,y)\to\inf \\ & \text{subject to} & \\ (\mathrm{CSC}_0) & : & (u,y)\in \mathbb{U}\times\mathbb{Y}, u\in U_0\subset\mathbb{U} \\ (\mathrm{SE}_0) & : & L_0(u,y)+F_0(y)=0 \end{array}\right\}. \tag{5.3}$$

In spite of the apparent simplicity of this approach, it has rather limited application (see Akulenko [3], Lions [170] and Kokotovic [154]). The main reason is the fact that the limiting problem (5.3) does not preserve, in general, the variational properties (V1)–(V2). As an illustrative example, let us consider the following OCP governed by a nonlinear PDE:

Example 5.1. Let Ω be a bounded open subset of $\mathbb{R}^n$ $(n < 12)$ with a boundary $\partial\Omega \in C^\infty$ and let us consider the following OCP in Ω (see Haraux and Murat [117]):

$$I(u,y) = \int_\Omega |y(x) - z_d(x)|^6 \,\mathrm{d}x + N \int_\Omega u^2(x)\,\mathrm{d}x \longrightarrow \inf, \tag{5.4}$$

$$-\varepsilon \Delta y(x) - y^3(x) = u(x), \quad x \in \Omega, \tag{5.5}$$

$$y\,|_{\partial\Omega} = 0, \tag{5.6}$$

$$u \in U_\partial = \left\{ u \in L^2(\Omega) \ : \ \lambda \le u \le \nu \ \text{ a.e. on } \ \Omega \right\}, \tag{5.7}$$

where $z_d \in L^6(\Omega)$ is a fixed function, $\varepsilon > 0$ is a small parameter, U_∂ is a convex closed subset of $L^2(\Omega)$ and $0 < \lambda < \nu$.

In order to give a sense of the problem (5.4)–(5.7), we make the following assumptions: For every $\varepsilon > 0$, there exists a pair $(u_\varepsilon, y_\varepsilon) \in U_\partial \times \left[L^6(\Omega) \cap H^2(\Omega) \cap H_0^1(\Omega)\right]$ satisfying the conditions (5.5)–(5.7). To fulfill the conditions of Theorem 3.3, we put

$$\mathbb{Y} = L^6(\Omega), \quad \mathbb{Y}_1 = L^6(\Omega) \cap H_0^1(\Omega) \cap H^2(\Omega),$$
$$\mathbb{U} = L^2(\Omega), \quad \mathbb{Z} = L^2(\Omega), \quad L(u,y) = -\Delta y - u,$$
$$F(y) = -y^3, \quad \mathbb{Y}_{-1} = L^3(\Omega),$$

and make use the compactness property of the embedding $H^2(\Omega) \hookrightarrow L^3(\Omega)$ $(n < 12)$. Since the functional

$$y \mapsto \langle F(y), \varphi \rangle_{H_0^1(\Omega)} = -\int_\Omega y^3(x)\varphi(x)\,\mathrm{d}x$$

is continuous on $L^3(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$, and $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it follows that, by Theorem 3.3, the OCP (5.4)–(5.7) admits at least one solution.

Now, we deal with the question of the asymptotic behavior of this problem as ε tends to 0. Taking into account the procedure of component-by-component limit analysis, we pass to the limit in the state equation (5.5) as $\varepsilon \to 0$.

Let τ be the product of the weak topology for $L^6(\Omega)$ and the strong topology for $L^2(\Omega)$. Let $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon \to 0}$ be any τ-convergent sequence such that

$$-\varepsilon \Delta y_\varepsilon(x) = u_\varepsilon(x) + y_\varepsilon^3(x) \ \text{ in } \ \Omega, \quad y_\varepsilon\,|_{\partial\Omega} = 0$$

for every $\varepsilon > 0$. Multiplying this relation by y_ε and integrating by parts, we obtain

$$\varepsilon \int_\Omega |\nabla y_\varepsilon|^2 \, \mathrm{d}x = (u_\varepsilon, y_\varepsilon)_{L^2(\Omega)} + \|y_\varepsilon\|^4_{L^4(\Omega)}. \tag{5.8}$$

Since $(u_\varepsilon, y_\varepsilon)_{L^2(\Omega)} \to (u, y)_{L^2(\Omega)}$ (as the product of weakly and strongly convergent sequences), it follows that there exists a constant $C > 0$ such that $\varepsilon \int_\Omega |\nabla y_\varepsilon|^2 \, dx \le C$. Hence, the sequence $\{\sqrt{\varepsilon}\,\nabla y_\varepsilon\}$ is bounded in $(L^2(\Omega))^n$. Then, passing to the limit in the integral identity

$$\sqrt{\varepsilon} \int_\Omega (\sqrt{\varepsilon}\,\nabla y_\varepsilon, \varphi) \, \mathrm{d}x - (y_\varepsilon^3, \varphi)_{L^2(\Omega)} = (u_\varepsilon, \varphi)_{L^2(\Omega)}, \quad \forall\, \varphi \in H^1_0(\Omega),$$

we get $(y^3 + u, \varphi)_{L^2(\Omega)} = 0$. Since $H^1_0(\Omega)$ is a dense subset of $L^2(\Omega)$, we conclude that $u = -y^3$ a.e. on Ω. As a result, we obtain the following limit OCP

$$I_0(u, y) = \int_\Omega |y(x) - z_d(x)|^6 \, \mathrm{d}x + N \int_\Omega u^2(x) \, \mathrm{d}x \longrightarrow \inf, \tag{5.9}$$

$$-y^3(x) = u(x), \quad x \in \Omega, \tag{5.10}$$

$$y\,|_{\partial\Omega} = 0 \tag{5.11}$$

$$u \in U_\partial = \left\{ u \in L^2(\Omega) \;\; \lambda \le u \le \nu \;\; \text{a.e. on} \;\; \Omega \right\}. \tag{5.12}$$

However, this problem does not preserve the main variational properties (V1)–(V2). Indeed, let us substitute in (5.4)–(5.7) the following data: $\varepsilon = 1/n^2$, $\Omega = (0, T)$, $0 \le N < +\infty$, $0 < \lambda \le \nu \le +\infty$, and $z_d = -C$, where

$$T = \lambda^{-1/3} 2^{5/6} \int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - t^4}}, \quad C \ge 3 \left(\frac{\lambda}{2}\right)^{1/3},$$

and define a T-periodic function $\varphi \in H^1_0(0, T)$ as follows:

$$\varphi'' + \varphi^3 = \lambda, \quad \varphi(0) = \varphi(T) = 0.$$

Then, as was shown in [117], the pair $(u^0_\varepsilon, y^0_\varepsilon) = (\lambda, \varphi\,(x/\sqrt{\varepsilon}))$ is an optimal one for the problem (5.4)–(5.7). Hence, for every $\varepsilon = n^{-2}$, we have

$$\begin{aligned} \inf I(u, y) = I(u^0_\varepsilon, y^0_\varepsilon) &= \int_0^T \left| \varphi\left(\frac{x}{\sqrt{\varepsilon}}\right) \right|^6 \mathrm{d}x + N\lambda^2 T \\ &= \int_0^T \varphi^6(nx) \, \mathrm{d}x + N\lambda^2 T = \int_0^T \varphi^6(x) \, \mathrm{d}x + N\lambda^2 T. \end{aligned}$$

As a result,

$$\liminf_{\varepsilon \to 0} I(u^0_\varepsilon, y^0_\varepsilon) = \int_0^T \varphi^6(x) \, \mathrm{d}x + N\lambda^2 T.$$

On the other hand, the limit problem (5.9)–(5.12) is solvable as well. Moreover, $(u^0, y^0) = (\lambda, \lambda^{1/3})$ is an optimal pair for (5.9)–(5.12). Finally, it is easy to see that

$$\inf I_0(u,y) = I_0(\lambda, \lambda^{1/3}) = T|\lambda^{1/3} + C|^6 + N\lambda^2 T.$$

However,

$$\int_0^T \varphi^6(x)\,\mathrm{d}x < T\left|\lambda^{1/3} + C\right|^6,$$

and we obtain the required result.

Let us represent the OCP (5.2) in the form of a constrained minimization problem, namely

$$(\mathrm{OCP}_\varepsilon) \;:\; \left\langle \inf_{(u,y)\in\,\Xi_\varepsilon} I_\varepsilon(u,y) \right\rangle, \tag{5.13}$$

where the set of admissible solutions Ξ_ε is defined as

$$(\mathrm{SAS}_\varepsilon) \;:\; \Xi_\varepsilon = \{(u,y) \in U_\varepsilon \times \mathbb{Y}_\varepsilon, U_\varepsilon \subset \mathbb{U}_\varepsilon,\; L_\varepsilon(u,y) + F_\varepsilon(y) = 0\}. \tag{5.14}$$

It is clear that the set of OCPs $(\mathrm{OCP}_\varepsilon)$ correlates well with the set of pairs $\langle I_\varepsilon, \Xi_\varepsilon\rangle$. However, we cannot assert that there is a one-to-one correspondence between these sets. Actually, this would be true only when each of the pairs $\langle I_\varepsilon, \Xi_\varepsilon\rangle$ could be interpreted as some optimal control problem (i.e., when the sets Ξ_ε have the representation $\Xi_\varepsilon = \{(u,y) \;:\; (\mathrm{SE}_\varepsilon) \cap (\mathrm{CSC}_\varepsilon)\}$). However, this is by no means always the case.

Thus, we come to the following scheme of a limit analysis that can be viewed as one more variant of the procedure of component-by-component passage to the limit (see Muthukumar [197]):

1. Taking into account the concepts of Γ-convergence of functionals and Painlevé–Kuratowski convergence of sets, find a limit of the sequence $\{\langle I_\varepsilon, \Xi_\varepsilon\rangle\,;\; \varepsilon \to 0\}$ – namely $\operatorname*{Lim}_{\varepsilon\to 0} \langle I_\varepsilon, \Xi_\varepsilon\rangle = \langle I_0, \Xi_0\rangle$, where

$$\Xi_0 = \mathrm{K}\text{–}\operatorname*{Lim}_{\varepsilon\to 0} \Xi_\varepsilon \quad \text{and} \quad I_0(u,y) = \Gamma\text{–}\operatorname*{Lim}_{\varepsilon\to 0} \inf I_\varepsilon(u,y).$$

2. If the set Ξ_0 can be represented in the form

$$\Xi_0 = \{(u,y) \;:\; (u,y) \in (\mathrm{SE}_0) \cap (\mathrm{CSC}_0)\},$$

then the OCP (5.2) should be taken as a limit to (5.2).

As in the previous case, let us show that this scheme does not guarantee the fulfillment of the main variational properties for the limit problems. For this, we consider a particular model OCP in the 1D periodic media.

Example 5.2. Let $\beta > \alpha > 0$, $\gamma > 0$, and c and d $(c < d)$ be given values and $f \in L^2(c,d)$. Let $a \in L^\infty(c,d)$ be a positive, 1-periodic function such that $0 < \alpha \le a(x) \le \beta < +\infty$ almost everywhere in $\Omega = (c,d)$. In Ω, we consider the following OCP

$$I(u,y)=\int_c^d [y'(x)]^2\ \mathrm{d}x+\int_c^d u^2(x)\,\mathrm{d}x\ \longrightarrow \inf, \tag{5.15}$$

$$-\left[a\left(\frac{x}{\varepsilon}\right)y'\right]'=u(x)+f(x),\quad x\in(c,d),\quad y(c)=y(d)=0, \tag{5.16}$$

$$u\in U_\partial=\left\{u\in L^2(c,d)\ :\ \|u\|_{L^2(c,d)}\le\gamma\right\}, \tag{5.17}$$

where $\varepsilon=1/n$ is a small parameter and $u\in L^2(c,d)$ is a control.

For each ε, this problem has a unique solution. We denote it by $(u_\varepsilon^0,y_\varepsilon^0)\in L^2(c,d)\times H_0^1(c,d)$. So,

$$I(u_\varepsilon^0,y_\varepsilon^0)=\inf_{(u,y)\in\Xi_\varepsilon} I(u,y),$$

where Ξ_ε is the set of admissible solutions for the original problem, that is,

$$\Xi_\varepsilon=\left\{(u,y)\ \middle|\ \begin{array}{c}-\left[a\left(\frac{x}{\varepsilon}\right)y'\right]'=u(x)+f(x),\quad x\in(c,d),\\ y\in H_0^1(c,d),\quad u\in L^2(c,d),\\ \|u\|_{L^2(c,d)}\le\gamma\end{array}\right\}. \tag{5.18}$$

Let τ be the product of the weak topologies for $L^2(c,d)$ and $H_0^1(c,d)$, respectively. In accordance with the procedure of limit analysis given above, we begin with the identification of the Painlevé–Kuratowski limit of the sequence $\{\Xi_\varepsilon;\varepsilon\to0\}$ with respect to the τ-topology of $L^2(c,d)\times H_0^1(c,d)$.

As immediately follows from (5.16), the estimate

$$\|y_\varepsilon\|_{H_0^1(c,d)}\le\alpha^{-1}\left(\|u_\varepsilon\|_{L^2(c,d)}+\|f\|_{L^2(c,d)}\right) \tag{5.19}$$

is valid for every $\varepsilon>0$. Hence, the sets $\{\Xi_\varepsilon;\varepsilon\to0\}$ are uniformly bounded. Indeed, in this case, we have

$$\sup_{\varepsilon>0}\sup_{(u,y)\in\Xi_\varepsilon}\left[\|y\|_{H_0^1(c,d)}+\|u\|_{L^2(c,d)}\right]\le\gamma(\alpha^{-1}+1)+\alpha^{-1}\|f\|_{L^2(c,d)}. \tag{5.20}$$

Since the space $H_0^1(c,d)$ is separable and reflexive, there exists a metric ρ on $H_0^1(c,d)$ such that the τ-topology for $\cup_{\varepsilon>0}\Xi_\varepsilon\subset L^2(c,d)\times H_0^1(c,d)$ is associated to the metric ρ on $\cup_{\varepsilon>0}\Xi_\varepsilon$. Hence, by the property of Painlevé–Kuratowski convergence (see Kuratowski [161] and Dal Maso [78]), we can use its sequential version for the sets $\{\Xi_\varepsilon;\varepsilon\to0\}$ (see also Definition 7.21).

Let $\{(u_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon\}$ be any sequence of admissible pairs. Let us define

$$\left\{\nu_\varepsilon=a\left(\frac{x}{\varepsilon}\right)y_\varepsilon'\right\}_{\varepsilon>0}.$$

From the initial supposition and the estimates (5.19)–(5.20), it follows that there exists a subsequence, still denoted by ε, such that

$$\begin{array}{c}y_\varepsilon\rightharpoonup y_*\ \text{ in }\ H_0^1(c,d),\quad u_\varepsilon\rightharpoonup u_*\ \text{ in }\ L^2(c,d),\\ \nu_\varepsilon\rightharpoonup\nu\ \text{ in }\ H^1(c,d),\quad \nu_\varepsilon\to\nu\ \text{ in }\ L^2(c,d),\\ a(\cdot/\varepsilon)\overset{*}{\rightharpoonup}a_*=\displaystyle\int_0^1 a(\xi)\,\mathrm{d}\xi\ \text{ in }\ L^\infty(c,d).\end{array} \tag{5.21}$$

Using the continuity of the trace operator, we have $y_*(c) = y_*(d) = 0$.

Now, passing to the limit as $\varepsilon \to 0$ in the integral identity

$$\int_c^d a\left(\frac{x}{\varepsilon}\right) y_\varepsilon' \varphi' \,\mathrm{d}x = \int_c^d (u_\varepsilon + f)\varphi \,\mathrm{d}x, \quad \varphi \in H_0^1(c,d),$$

we get

$$\int_c^d \nu \varphi' \,\mathrm{d}x = \int_c^d (u_* + f)\varphi \,\mathrm{d}x. \tag{5.22}$$

To establish the relation between y_* and ν, we note that ν_ε is the product of two weakly converging quantities. In general, this does not imply that the limit ν is the product of the limits a_* and $\mathrm{d}y_*/\mathrm{d}x$. To identify it, we recall that

$$y_\varepsilon' = a^{-1}\left(\frac{x}{\varepsilon}\right)\nu_\varepsilon, \tag{5.23}$$

where $a^{-1}(\cdot/\varepsilon) \in L^\infty(0,1)$, $a^{-1}(\cdot/\varepsilon)$ is the 1-periodic function, $1 < 1/\alpha \le 1/a(\cdot/\varepsilon) \le 1/\beta < +\infty$ and, hence,

$$a^{-1}(\cdot/\varepsilon) \overset{*}{\rightharpoonup} \left\langle a^{-1}\right\rangle \equiv \int_0^1 a^{-1}(\xi)\, d\xi \quad \text{in } L^\infty(c,d).$$

Since the term $a^{-1}(\cdot/\varepsilon)\nu_\varepsilon$ is the product of the weakly convergent quantity $a^{-1}(\cdot/\varepsilon)$ and the strongly convergent one ν_ε, its limit is the product of the limits of $a^{-1}(\cdot/\varepsilon)$ and ν_ε. Thus,

$$a^{-1}(\cdot/\varepsilon)\nu_\varepsilon \rightharpoonup \left\langle a^{-1}\right\rangle \nu \ \text{ in } \ L^2(c,d).$$

Making use of (5.22), we come to the following "homogenized" state equation

$$\left\langle a^{-1}\right\rangle^{-1} \int_c^d y_*' \varphi' \,\mathrm{d}x = \int_c^d (u_* + f)\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in H_0^1(c,d).$$

Note that for each fixed $u_* \in L^2(c,d)$, this problem has a unique solution $y_* \in H_0^1(c,d)$. Thus, taking into account the fact that $\|u_\varepsilon\|_{L^2(c,d)} \le \gamma$ for every $\varepsilon > 0$ and the norm $\|\cdot\|_{L^2(c,d)}$ is lower semicontinuous with respect to the weak convergence in $L^2(c,d)$, we have

$$\gamma \ge \varliminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^2(c,d)} \ge \left\| \varliminf_{\varepsilon\to 0} u_\varepsilon \right\|_{L^2(c,d)} = \|u_*\|_{L^2(c,d)}.$$

Hence, $(u_\varepsilon, y_\varepsilon) \overset{\tau}{\to} (u_*, y_*)$, where the τ-limit pair (u_*, y_*) belongs to the set

$$\Xi_0 = \left\{ (u,y) \;\middle|\; \begin{array}{c} -\left\langle a^{-1}\right\rangle^{-1} y'' \;=\; u(x) + f(x), \quad x \in (c,d), \\ y \in H_0^1(c,d), \quad u \in L^2(c,d), \\ \|u\|_{L^2(c,d)} \le \gamma \end{array} \right\}. \tag{5.24}$$

Let us consider the opposite situation. Let (v, y) be any pair from the set Ξ_0 and let $\{(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)\}_{\varepsilon\to 0}$ be a sequence such that $\widehat{u}_\varepsilon = v$ for every $\varepsilon > 0$ and the functions $\widehat{y}_\varepsilon$ are the corresponding solutions of the problem (5.16) with $u = v$. Then, in a similar manner as earlier, it is easy to prove that $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \xrightarrow{\tau} (v, y)$ and $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi_\varepsilon$ for every $\varepsilon > 0$. Hence, the set Ξ_0 is the sequential τ-limit of $\{\Xi_\varepsilon\}_{\varepsilon>0}$ in the Painlevé–Kuratowski sense (shortly, $K(\tau)$-limit) [161] (see also Sect. 7.4).

Thus, in accordance with the above scheme, the limit OCP can be written as

$$I(u, y) = \int_c^d [y'(x)]^2 \, \mathrm{d}x + \int_c^d u^2(x)\, \mathrm{d}x \longrightarrow \inf, \tag{5.25}$$

$$-\left\langle a^{-1}\right\rangle^{-1} y'' = u(x) + f(x), \quad x \in (c, d), \quad y(c) = y(d) = 0, \tag{5.26}$$

$$u \in U_\partial = \left\{u \in L^2(c, d) \;\; \|u\|_{L^2(c,d)} \le \gamma\right\}. \tag{5.27}$$

However, for this kind of "homogenized" problem, the variational properties (V1)–(V2) do not hold. Indeed, let $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}$ be any τ-convergent sequence of admissible pairs and let (u_*, y_*) be its τ-limit. Then, by the property of K-limits (see Proposition 7.23), the pair (u_*, y_*) belongs to the set Ξ_0. Taking into account (5.23) and the properties (5.21), one has

$$\begin{gathered} a\,(x/\varepsilon)\, y'_\varepsilon = \nu_\varepsilon \rightharpoonup \nu = \left\langle a^{-1}\right\rangle^{-1} y_* \quad \text{in} \;\; H^1(c, d), \\ \nu_\varepsilon^2 \to \nu^2 \quad \text{in} \;\; L^1(c, d), \\ a^{-2}(\cdot/\varepsilon) \xrightarrow{*} \left\langle a^{-2}\right\rangle = \int_0^1 a^{-2}(\xi)\, \mathrm{d}\xi \quad \text{in} \;\; L^\infty(c, d). \end{gathered}$$

Then, passing to the limit in the sequence $\{I(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ as ε tends to 0, we obtain

$$\begin{aligned} \liminf_{\varepsilon\to 0} I(u_\varepsilon, y_\varepsilon) &= \liminf_{\varepsilon\to 0} \int_c^d \frac{1}{a^2\left(\frac{x}{\varepsilon}\right)} \left[a\left(\frac{x}{\varepsilon}\right) y'_\varepsilon\right]^2 \mathrm{d}x + \liminf_{\varepsilon\to 0} \int_c^d u_\varepsilon^2 \, \mathrm{d}x \\ &= \left[\int_0^1 a^{-2}(\xi)\, \mathrm{d}\xi\right] \int_c^d \nu^2 \, \mathrm{d}x + \liminf_{\varepsilon\to 0} \int_c^d u_\varepsilon^2 \, dx \\ &\ge \left[\int_0^1 a^{-2}(\xi)\, \mathrm{d}\xi\right] \int_c^d \left(\left\langle a^{-1}\right\rangle^{-1} y_*\right)^2 \mathrm{d}x + \int_c^d u_*^2 \, \mathrm{d}x \\ &= \left[\int_0^1 a^{-2}(\xi)\, \mathrm{d}\xi\right] \left[\int_0^1 a^{-1}(\xi)\, \mathrm{d}\xi\right]^{-2} \int_c^d (y'_*)^2 \, \mathrm{d}x \\ &\quad + \int_c^d u_*^2 \, \mathrm{d}x. \end{aligned} \tag{5.28}$$

On the other hand, for any pair $(\widehat{u}, \widehat{y}) \in \Xi_0$, we can consider a sequence $\{(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)\}_{\varepsilon\to 0}$ as follows: $\widehat{u}_\varepsilon = \widehat{u}$ for every $\varepsilon > 0$ and $\widehat{y}_\varepsilon$ are the solutions

of (5.16) provided $u = \widehat{u}$. Then, by analogy with the previous case, it can be easily proved that $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi_\varepsilon$ for all $\varepsilon > 0$ and $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \xrightarrow{\tau} (\widehat{u}, \widehat{y})$. Consequently,

$$\begin{aligned}
\limsup_{\varepsilon\to 0} I(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) &= \limsup_{\varepsilon\to 0} \int_c^d \frac{1}{a^2\left(\frac{x}{\varepsilon}\right)} \left[a\left(\frac{x}{\varepsilon}\right) \widehat{y}_\varepsilon{}'\right]^2 \,\mathrm{d}x + \int_c^d \widehat{u}^{\,2}\,\mathrm{d}x \\
&= \left[\int_0^1 a^{-2}(\xi)\,\mathrm{d}\xi\right] \int_c^d \widehat{\nu}^{\,2}\,\mathrm{d}x + \int_c^d \widehat{u}^{\,2}\,\mathrm{d}x \\
&= \left[\int_0^1 a^{-2}(\xi)\,\mathrm{d}\xi\right] \int_c^d \left(\left\langle a^{-1}\right\rangle^{-1} \widehat{y}\,'\right)^2 \,\mathrm{d}x + \int_c^d \widehat{u}^{\,2}\,\mathrm{d}x \\
&= \left[\int_0^1 a^{-2}(\xi)\,\mathrm{d}\xi\right] \left[\int_0^1 a^{-1}(\xi)\,\mathrm{d}\xi\right]^{-2} \int_c^d (\widehat{y}\,')^2 \,\mathrm{d}x \\
&\quad + \int_c^d \widehat{u}^{\,2}\,\mathrm{d}x. \qquad (5.29)
\end{aligned}$$

Introducing the functional

$$\begin{aligned}
I_{hom}(u, y) &= \int_c^d u^{\,2}\,\mathrm{d}x \\
&\quad + \left[\int_0^1 a^{-2}(\xi)\,\mathrm{d}\xi\right] \left[\int_0^1 a^{-1}(\xi)\,\mathrm{d}\xi\right]^{-2} \int_c^d (y')^2 \,\mathrm{d}x, \qquad (5.30)
\end{aligned}$$

we can rewrite the relations (5.28) and (5.29) as

$$\liminf_{\varepsilon\to 0} I(u_\varepsilon, y_\varepsilon) \geq I_{hom}(u_*, y_*), \qquad (5.31)$$

$$\limsup_{\varepsilon\to 0} I(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \leq I_{hom}(\widehat{u}, \widehat{y}). \qquad (5.32)$$

The functional $I_{hom} : \Xi_0 \to \mathbb{R}$ with the properties (5.31) and (5.32) is called in [143] (see also [140], [141, 142, 144, 147]) the S-limit of the sequence $\{I : \Xi_\varepsilon \to \mathbb{R}\}_{\varepsilon>0}$. It does not coincide in general with the Γ-closure of the original cost functional $I : L^2(c,d) \times H_0^1(c,d) \to \mathbb{R}$. However, it can be easily shown that the functional $I_{hom} : \Xi_0 \to \mathbb{R}$ coincides with the restriction on Ξ_0 of the Γ-limit of the sequence $\left\{\mathcal{I}_\varepsilon : L^2(c,d) \times H_0^1(c,d) \to \mathbb{R}\right\}_{\varepsilon>0}$ (see [147]), where

$$\mathcal{I}_\varepsilon(u, y) = \begin{cases} \displaystyle\int_c^d [y'(x)]^2 \,\mathrm{d}x + \int_c^d u^2(x)\,\mathrm{d}x & \text{if} \quad (u, y) \in \Xi_\varepsilon, \\ +\infty & \text{if} \quad (u, y) \notin \Xi_\varepsilon. \end{cases}$$

Let us prove that the limit functional (5.30) possesses the fine variational properties (V1) and (V2). To begin, we note that the OCP (5.26)–(5.27), (5.30) admits a unique solution (u^0, y^0).

Theorem 5.3. *Let $(u_\varepsilon^0, y_\varepsilon^0)$ be an optimal pair to the original problem (5.15)–(5.17). Then*

$$(u_\varepsilon^0, y_\varepsilon^0) \xrightarrow{\tau} (u^0, y^0), \tag{5.33}$$

$$\inf_{(u,\,y)\in \Xi_0} I_{hom}(u,y) = I_{hom}\left(u^0, y^0\right) = \lim_{\varepsilon\to 0} I(u_\varepsilon^0, y_\varepsilon^0). \tag{5.34}$$

Proof. In view of the estimate (5.20), the sequence of optimal pairs $\left\{(u_\varepsilon^0, y_\varepsilon^0)\right\}$ is uniformly bounded. Hence, we may extract a subsequence $\left\{(u_{\varepsilon_k}^0, y_{\varepsilon_k}^0)\right\}_{k\in\mathbb{N}}$ τ-converging in $L^2(c,d)\times H_0^1(c,d)$ to some pair (u^*, y^*). By the arguments given above, the pair belongs to the set Ξ_0. Moreover, taking (5.28) into account, we have

$$\begin{aligned}\liminf_{k\to\infty} \min_{(u,y)\in \Xi_{\varepsilon_k}} I(u,y) &= \liminf_{k\to\infty} I(u_{\varepsilon_k}^0, y_{\varepsilon_k}^0) \geq I_{hom}(u^*, y^*)\\ &\geq \min_{(u,\,y)\in \Xi_0} I_0(u,y) = I_{hom}(u^0, y^0).\end{aligned} \tag{5.35}$$

On the other hand, since $(u^*, y^*)\in \Xi_0$, it follows that there exists a sequence $\{(u_\varepsilon, y_\varepsilon)\}$ τ-converging to (u^0, y^0) such that $(u_\varepsilon, y_\varepsilon)\in \Xi_\varepsilon$ for every $\varepsilon > 0$ and $I_{hom}(u^0, y^0) \geq \limsup_{\varepsilon\to 0} I(u_\varepsilon, y_\varepsilon)$ (see (5.29)). Consequently,

$$\begin{aligned}\min_{(u,\,y\,)\in \Xi_0} I_{hom}(u,\,y) &= I_{hom}(u^0, y^0) \geq \limsup_{\varepsilon\to 0} I(u_\varepsilon, y_\varepsilon)\\ &\geq \limsup_{\varepsilon\to 0} \min_{(u,y)\,\in \Xi_\varepsilon} I(u,y) \geq \limsup_{k\to\infty} \min_{(u,y)\,\in\, \Xi_{\varepsilon_k}} I(u,y)\\ &= \limsup_{k\to\infty} I_{\varepsilon_k}(u_{\varepsilon_k}^0, y_{\varepsilon_k}^0).\end{aligned} \tag{5.36}$$

However, by (5.35), we get

$$\liminf_{k\to\infty} I(u_{\varepsilon_k}^0, y_{\varepsilon_k}^0) \geq \limsup_{k\to\infty} I(u_{\varepsilon_k}^0, y_{\varepsilon_k}^0).$$

As a result, combining (5.35) and (5.36), we come to the following conclusion:

$$I_{hom}(u^*, y^*) = I_{hom}(u^0, y^0) = \min_{(u,\,y)\in\, \Xi_0} I_{hom}(u,y),$$

$$I_{hom}(u^0, y^0) = \lim_{k\to\infty} \min_{(u,y)\,\in\, \Xi_{\varepsilon_k}} I_{\varepsilon_k}(u,y).$$

Using these relations and the fact that the OCP (5.26)–(5.27) and (5.30) admits a unique solution (u^0, y^0), we obtain $(u^*, y^*) = (u^0, y^0)$. Since this equality holds for τ-limits of all subsequences $\left\{(u_{\varepsilon_k}^0, y_{\varepsilon_k}^0)\right\}_{k\in\mathbb{N}}$ of the sequence $\left\{(u_\varepsilon^0, y_\varepsilon^0)\right\}_{\varepsilon>0}$, it follows that these limits coincide and therefore (u^0, y^0) is the τ-limit of the original sequence $\left\{(u_\varepsilon^0, y_\varepsilon^0)\right\}_{\varepsilon>0}$. Therefore, using the same argument for the sequence of minimizers as for the subsequence, we have

$$\begin{aligned}\liminf_{\varepsilon\to 0}\min_{(u,y)\in\Xi_\varepsilon} I(u,y) &= \liminf_{\varepsilon\to 0} I(u^0_\varepsilon, y^0_\varepsilon)\\ &\geq I_{hom}(u^0, y^0) = \min_{(u,\,y)\in\,\Xi_0} I_{hom}(u,y)\\ &\geq \limsup_{\varepsilon\to 0} I(u_\varepsilon, y_\varepsilon) \geq \limsup_{\varepsilon\to 0}\min_{(u,y)\,\in\Xi_\varepsilon} I(u,y)\\ &= \limsup_{\varepsilon\to 0} I(u^0_\varepsilon, y^0_\varepsilon).\end{aligned}$$

Thus, we obtain the required result. The proof is complete. □

The question is now how to compare the original cost functional I with the limit functional I_{hom}. First, we note that each of them has a sense only on the set of admissible pairs Ξ_0. Let $(\widetilde{u}_0, \widetilde{y}_0) \in \Xi_0$ be an optimal pair to the problem (5.25)–(5.27). The existence of this pair is an obvious fact. Moreover, in general, we have $(\widetilde{u}_0, \widetilde{y}_0) \neq (u^0, y^0)$. Then

$$I(\widetilde{u}_0, \widetilde{y}_0) \leq I_{hom}(\widetilde{u}_0, \widetilde{y}_0), \quad I(u^0, y^0) \leq I_{hom}(u^0, y^0).$$

The correctness of this conclusion immediately follows from the following inequality

$$\begin{aligned}1 &= \left[\int_0^1 a^{-1}(\xi)\,\mathrm{d}\xi\right]^2 \left[\int_0^1 a^{-1}(\xi)\,\mathrm{d}\xi\right]^{-2}\\ &\leq \left(\int_0^1 1\,\mathrm{d}x\right)\ \left[\int_0^1 a^{-2}(\xi)\,\mathrm{d}\xi\right]\left[\int_0^1 a^{-1}(\xi)\,\mathrm{d}\xi\right]^{-2}.\end{aligned}$$

Consequently, taking into account the structure of I, it leads us to the inequality $I(u,y) \leq I_{hom}(u,y)$ for every $(u,y) \in \Xi_0$. Moreover, if the function $a \in L^\infty(0,1)$ is such that $a(\xi) \neq a^2(\xi)$ and $a(\xi) \neq$ const almost everywhere on the interval $(0,1)$, then

$$I(\widetilde{u}_0, \widetilde{y}_0) \leq I(u^0, y^0) < I_{hom}(u^0, y^0) = \min_{(u,\,y)\in\,\Xi_0} I_{hom}(u,y).$$

Thus, the variational properties (V1)–(V2) are lost if we take the limit OCP for the family (5.15)–(5.17) in the form (5.25)–(5.27). □

5.2 Limit analysis of optimality conditions

An asymptotic analysis of OCPs for the systems with distributed parameters has been considered in [10, 48, 61, 64, 117, 118, 220]. As follows from pioneering works in this field and more recent results, the most typical approach to asymptotic analysis of OCPs consists in the application of the homogenization theory of boundary value problems for investigation of the limiting behavior of necessary optimality conditions and the corresponding optimal solutions. Thus, the algorithm of asymptotic analysis consists of the

following steps: First, write down the necessary optimality conditions for the original problem; further, look for the corresponding limiting relations and interpret them as necessary optimality conditions for some optimal control problem; finally, using the limiting necessary optimality conditions, recover an OCP which is called the limiting control problem to the initial one (see, e.g., [21, 22, 71, 88, 129, 130, 134, 135, 136, 137, 197, 222, 268]). Thus, if we denote by OCP_ε, NOC_ε, HOCP, and HNOC the original OCP on the ε-level, the corresponding necessary optimality conditions on the ε-level, the limit OCP and the limit necessary optimality system, respectively, then the above-mentioned algorithm can be represented in the form of the following diagram:

$$\begin{array}{ccc} \mathrm{OCP}_\varepsilon & \overset{?}{\Longrightarrow} & \mathrm{HOCP} \\ \downarrow & & \uparrow \\ \mathrm{NOC}_\varepsilon & \overset{\varepsilon\to 0}{\longrightarrow} & \mathrm{HNOC}. \end{array} \tag{5.37}$$

However, it is possible that this diagram is not commutative. Moreover, it should be stressed that such approach is suitable only for simple enough (from the control theory point of view) OCPs in which there are no control and state constraints. Attempts to transfer this approach to a wider class of OCPs was realized in [71, 72], where it was shown that the construction of the limit OCP is possible only under restrictive assumptions. Moreover, the diagram (5.37) does not imply, in general, the main variational properties (V1) and (V2) for the limit problems. So, the verification of these properties is an independent problem in each concrete case (see [71, 135, 197]).

Along with this, necessary optimality conditions for OCPs with constraints, as a rule, contain variational inequalities or equations in measure terms that complicate the procedure of limit analysis [98, 126, 169, 171, 250]. On the other hand, the optimal pair (i.e., the optimal control and the corresponding state of system) cannot completely characterize the entire problem. As a rule, the full identification of an OCP (including a cost functional, a state equation and existing restrictions on the control functions and state functions) is impossible by means of the optimal solution only. Moreover, the diagram (5.37) is unacceptable when the existence of solutions to the problem (5.2) may fail [38, 48] or when these solutions have nonclassical character (solutions satisfying only the requirements of Tonelli's partial regularity theorem and having no "variational character") [17, 235].

Indeed, as is well known, the necessary conditions of optimality can be derived by the abstract Lagrange multiplier principle [111, 169] or the Pontryagin maximum principle [4]. In particular, the so-called direct analogue of Pontryagin maximum principle [219] leads us (under some natural assumptions for the problem (5.2)) to the following relation (see [4]):

$$\begin{aligned} I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) - \left\langle \Psi_\varepsilon, L_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) + F_\varepsilon(y_\varepsilon^0) \right\rangle_{\mathbb{Z}^*, \mathbb{Z}} \\ \leq I_\varepsilon(u_\varepsilon, y_\varepsilon^0) - \left\langle \Psi_\varepsilon, L_\varepsilon(u_\varepsilon, y_\varepsilon^0) + F_\varepsilon(y_\varepsilon^0) \right\rangle_{\mathbb{Z}^*, \mathbb{Z}}, \quad \forall\, u_\varepsilon \in U_\varepsilon, \end{aligned} \tag{5.38}$$

where $(u_\varepsilon^0, y_\varepsilon^0)$ is an optimal pair to the problem (5.2) and the Lagrange multiplier Ψ_ε satisfies some adjoint equation. Let us show that even in the 1D case, this principle may fail.

Example 5.4. Let $\Omega = \{x \in \mathbb{R} \ : \ -1 < x < 1\}$, let $a \in \mathbb{R}$ be a given positive value and let U_∂ be a set of 1-periodic function on $\mathbb{R}$ such that

$$U_\partial = \{u \in L^\infty(0,1) \ : \ u(x) = a \ \text{ or } \ u(x) = 1/a \ \text{ a.e. on } \ (0,1)\}. \tag{5.39}$$

In Ω, we consider the following OCP

$$I(u,y) = \int_\Omega [y'_x - z_\partial]^2 \ \mathrm{d}x \longrightarrow \inf, \tag{5.40}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u\left(\frac{x}{\varepsilon}\right) y'_x - z_\partial\right] = 0, \quad x \in \Omega, \quad y(-1) = y(1) = 0, \tag{5.41}$$

$$u \in U_\partial. \tag{5.42}$$

Here, $\varepsilon = 1/n$ is a small parameter, $u(\cdot/\varepsilon) \in L^2(\Omega)$ is an ε-periodic control influence and the function z_∂ is given by

$$z_\partial = z_\partial(x) = \begin{cases} x - \gamma & \text{if } x < 0, \\ x + \gamma & \text{if } x \geq 0. \end{cases}$$

We always suppose that the constants a and γ satisfy the conditions: $0 < a < 1$, $\gamma > 0$.

To begin, we prove that the OCP (5.40)–(5.42) has a solution. For this, we observe that the state equation (5.41) has a unique solution $y_\varepsilon \in H_0^1(\Omega)$ for every admissible control $u \in U_\partial$, and y_ε can be defined as follows

$$(y_\varepsilon)'_x(x) = (z_\partial(x) - c_0)\, u^{-1}\left(\frac{x}{\varepsilon}\right), \tag{5.43}$$

where

$$c_0 = \left(\int_{-1}^1 u^{-1}\left(\frac{x}{\varepsilon}\right) \mathrm{d}x\right)^{-1} \int_{-1}^1 z_\partial(x) u^{-1}\left(\frac{x}{\varepsilon}\right) \mathrm{d}x.$$

Hence, substituting these values into the cost functional I, it takes the form

$$I(u(\cdot/\varepsilon), y_\varepsilon) = \int_\Omega z_\partial^2 \, \mathrm{d}x + I_2^2 \frac{I_3}{I_1^2} + I_6 - 2\frac{I_2}{I_1}(I_4 - I_2) - 2I_5,$$

where we used the notations

$$\begin{aligned} I_1 &= \int_{-1}^1 \frac{1}{u(x/\varepsilon)} \, \mathrm{d}x, \quad I_2 = \int_{-1}^1 \frac{z_\partial(x)}{u(x/\varepsilon)} \, \mathrm{d}x, \quad I_3 = \int_{-1}^1 \frac{1}{u^2(x/\varepsilon)} \, \mathrm{d}x, \\ I_4 &= \int_{-1}^1 \frac{z_\partial(x)}{u^2(x/\varepsilon)} \, \mathrm{d}x, \quad I_5 = \int_{-1}^1 \frac{z_\partial^2(x)}{u(x/\varepsilon)} \, \mathrm{d}x, \quad I_6 = \int_{-1}^1 \frac{z_\partial^2(x)}{u^2(x/\varepsilon)} \, \mathrm{d}x. \end{aligned} \tag{5.44}$$

Let us introduce the auxiliary 1-periodic function $v \in L^\infty(0,1)$ satisfying

$$v(x) = a - 1 \quad \text{or} \quad v(x) = \frac{1}{a} - 1 \ \text{ a.e. on } \ (0,1).$$

Then

$$u^{-1}\left(\frac{x}{\varepsilon}\right) = 1 + v\left(\frac{x}{\varepsilon}\right), \tag{5.45}$$

$$\left|u\left(\frac{x}{\varepsilon}\right)\right|^{-2} = -1 + \left(a + \frac{1}{a}\right)\left[v\left(\frac{x}{\varepsilon}\right) + 1\right]. \tag{5.46}$$

Interpreting the function v as a new control, we see that each of the functionals (5.44) is affine with respect to v. Now, let us consider the mapping $\mathcal{I} : V_\partial \to \mathbb{R}^6$, where

$$V_\partial = \{v \in L^\infty(0,1) \ : \ v = u - 1 \ \forall\, u \in U_\partial\},$$
$$\mathcal{I}(v) = \left[I_1\left(v\left(\frac{x}{\varepsilon}\right)\right), \ldots, I_6\left(v\left(\frac{x}{\varepsilon}\right)\right)\right].$$

By the Lyapunov theorem (see [57]),

$$\mathfrak{A}_\varepsilon = \{\mathcal{I}(v) \ : \ \forall\, v \in V_\partial\}$$

is a convex closed subset of $\mathbb{R}^6$. Since the functional I continuously depends on $I_1, \ldots, I_6$, it follows that I reaches its infimum on the set of admissible pairs. Thus, the OCP (5.40)–(5.42) is solvable.

Let $(u_\varepsilon^0, y_\varepsilon^0) = (u^0\left(\frac{\cdot}{\varepsilon}\right), y_\varepsilon^0) \in U_\partial \times H_0^1(\Omega)$ be its solution. Then, following the direct analogue of Pontryagin Maximum Principle [219], the necessary optimality conditions take the form

$$\left(y_\varepsilon^0\right)'_x(x)\left(\psi_\varepsilon\right)'_x(x) u\left(\frac{x}{\varepsilon}\right) \geq \left(y_\varepsilon^0\right)'_x(x)\left(\psi_\varepsilon\right)'_x(x) u^0\left(\frac{x}{\varepsilon}\right), \quad x \in \Omega \tag{5.47}$$

for all $u \in U_\partial$, where $\psi_\varepsilon \in H_0^1(\Omega)$ is the solution of the problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u^0\left(\frac{x}{\varepsilon}\right)\psi'_x - 2(y_\varepsilon^0 - z_\partial)\right] = 0, \quad x \in \Omega, \ \psi(-1) = \psi(1) = 0. \tag{5.48}$$

By analogy with the original system, we get

$$(\psi_\varepsilon)'_x(x) = \left(2(y_\varepsilon^0(x) - z_\partial(x)) - c_1\right)\left[u^0\left(\frac{x}{\varepsilon}\right)\right]^{-1}, \tag{5.49}$$

where

$$c_1 = \left(\int_{-1}^{1}\left[u^0\left(\frac{x}{\varepsilon}\right)\right]^{-1}\mathrm{d}x\right)^{-1}\int_{-1}^{1} 2(y_\varepsilon^0(x) - z_\partial(x))\left[u^0\left(\frac{x}{\varepsilon}\right)\right]^{-1}\mathrm{d}x.$$

Then, substituting (5.43) and (5.49) in (5.47), we come to the inequality

$$\delta_\varepsilon(x)\left\{\left(\left[u^0\left(\frac{x}{\varepsilon}\right)\right]^{-1} - 1\right)\delta_\varepsilon(x) - c_2\right\}\left(u\left(\frac{x}{\varepsilon}\right) - u^0\left(\frac{x}{\varepsilon}\right)\right) \leq 0, \tag{5.50}$$

which has to be true for all $u \in U_\partial$ and almost every $x \in \Omega$. Here,

$$\delta_\varepsilon(x) = z_\partial(x) - \left(\int_{-1}^{1} \left[u^0 \left(\frac{x}{\varepsilon} \right) \right]^{-1} \mathrm{d}x \right)^{-1} \int_{-1}^{1} z_\partial(x) \left[u^0 \left(\frac{x}{\varepsilon} \right) \right]^{-1} \mathrm{d}x,$$

$$c_2 = \left(\int_{-1}^{1} \left[u^0 \left(\frac{x}{\varepsilon} \right) \right]^{-1} \mathrm{d}x \right)^{-1} \int_{-1}^{1} \delta_\varepsilon(x) \left[u^0 \left(\frac{x}{\varepsilon} \right) \right]^{-2} \mathrm{d}x.$$

However, the direct calculations show that (5.50) does not hold for any $u^0 \in U_\partial$ if $\gamma a^2 > 1$. Thus, we have an example of a solvable OCP for which its solution does not satisfy the necessary optimality condition in the form of direct analogue of the Pontryagin maximum principle. So, we cannot derive relations (5.41), (5.47) and (5.48) on the basis of the asymptotic analysis for the OCP (5.40)–(5.42). □

Another example concerns the case when the existence of solutions to the problem (5.2) may fail. Following Luc Tartar, we consider an elementary model OCP which has no solution. However, this fact by itself does not tell much about the existence of solutions for the system of necessary conditions of optimality. Indeed, let us consider the problem

$$\text{Minimize} \quad \left\{ I(u,y) = \int_0^T \left(|y|^2 - |u|^2 \right) \mathrm{d}t \right\}, \tag{5.51}$$

where the control u belongs to the set

$$U_\partial = \left\{ u \in L^2(0,T) \mid -1 \le u(t) \le 1, \text{ a.e. } t \in (0,T) \right\}, \tag{5.52}$$

and the state y is defined by the equation of state

$$\frac{\mathrm{d}y}{\mathrm{d}t} = u \quad \text{a.e. on } (0,T), \quad y(0) = 0. \tag{5.53}$$

Before we consider the necessary conditions of optimality, we show directly that the OCP (5.51)–(5.53) has no solution. To do so, we first notice that

$$I(u,y) > -T \text{ for all } u \in U_\partial, \tag{5.54}$$

because $|y|^2 - |u|^2 \ge -1$ a.e. on $(0,T)$ for every $u \in U_\partial$ implies $I(u,y) \ge -T$ and because one cannot have $I(u,y) = -T$, which would require both $y = 0$ and $|u| = 1$ a.e. on $(0,T)$, in contradiction to the fact that $y = 0$ a.e. on $(0,T)$ implies $u = 0$ a.e. on $(0,T)$.

Let us consider the following sequence $\{u_k\}_{k\in\mathbb{N}}$, where $u_k = \operatorname{sign}(\cos kt)$ on $(0,T)$. Then $u_k \in U_\partial$ for all $k \in \mathbb{N}$ and, moreover,

$$|u_k| = 1 \quad \forall\, k \in \mathbb{N}, \tag{5.55}$$

$$u_k \overset{*}{\rightharpoonup} \int_0^{2\pi} \operatorname{sign}(\cos t)\, \mathrm{d}t = 0 \quad \text{in } L^\infty(0,T). \tag{5.56}$$

Since the weak-$*$ convergence $u_k \rightharpoonup 0$ in $L^\infty(0,T)$ implies that $y_k = y(u_k)$ converges uniformly to 0, it follows that $I(u_k, y_k) \to -T$. However, as was shown earlier, the cost functional I does not achieve the value $-T$ on the set of admissible pairs

$$\Xi = \left\{ (u,y) \ : \ u \in U_\partial, \ y(t) = \int_0^t u(\tau)\,\mathrm{d}\tau, \ t \in (0,T) \right\}.$$

Having formally used the classical scheme, we can derive the necessary optimality conditions to the problem (5.51)–(5.53) – namely, we note that U_∂ is a convex set, the map $u \mapsto y$ is affine continuous, and therefore, the map $(u,y) \mapsto I$ is quadratic continuous, and so Fréchet differentiable. Let $u^0 \in U_\partial$, let y^0 be the corresponding state and let $\delta u \in L^\infty(0,T)$ be an admissible direction at u^0 (i.e., $u = u^0 + \varepsilon\delta u \in U_\partial$ for $\varepsilon > 0$ small). Then $y = y^0 + \varepsilon\delta y$ and $I(u,y) = I(u^0, y^0) + \varepsilon\delta I + o(\varepsilon)$, where

$$\frac{\mathrm{d}(\delta y)}{\mathrm{d}t} = \delta u, \quad \delta y(0) = 0, \tag{5.57}$$

$$\delta I = 2\int_0^T (y^0\delta y - u^0\delta u)\,\mathrm{d}t, \tag{5.58}$$

and classical necessary conditions of optimality consist in writing $\delta I \geq 0$ for all admissible δu and the corresponding δy. In order to eliminate δy so that δI is expressed only in terms of δu, we introduce the adjoint state p^0 by the relation

$$-\frac{\mathrm{d}p^0}{\mathrm{d}t} = y^0, \quad p^0(T) = 0. \tag{5.59}$$

Then a simple integration by parts gives

$$\int_0^T y^0\delta y\,\mathrm{d}t = -\int_0^T \frac{\mathrm{d}p^0}{\mathrm{d}t}\,\delta y\,\mathrm{d}t = \int_0^T p^0\frac{\mathrm{d}\delta y}{\mathrm{d}t}\,\mathrm{d}t = \int_0^T p^0\delta u\,\mathrm{d}t, \tag{5.60}$$

and therefore,

$$\delta I = 2\int_0^T (p^0 - u^0)\delta u\,\mathrm{d}t. \tag{5.61}$$

The admissibility of δu means that $\delta u \geq 0$ provided $u^0 = -1$, $\delta u \leq 0$ provided $u^0 = +1$, and δu is arbitrary if $-1 < u^0 < +1$. From this, we immediately deduce the classical necessary conditions of optimality:

$$\begin{cases} p^0 < -1 & \text{implies } u^0 = +1, \\ -1 \leq p^0 \leq +1 & \text{implies } u^0 \in \{-1, p^0, +1\}, \\ p^0 > +1 & \text{implies } u^0 = -1. \end{cases} \tag{5.62}$$

Note that the system of these classical necessary conditions of optimality (i.e., (5.53), (5.59) and (5.62)) has at least one solution

$$u^0 = 0, \ y^0 = 0, \ p^0 = 0 \quad \text{on} \quad (0,T). \tag{5.63}$$

On the other hand, using the Pontryagin maximum principle, we can obtain the necessary conditions of optimality if we note that any control which jumps from u^0 to another control $w \in U_\partial$ is admissible as well. It means that

$$u = (1-\chi)u^0 + \chi w \in U_\partial$$

for every characteristic function χ of a measurable subset of $(0,T)$. Therefore, it is natural to consider a sequence of characteristic functions $\{\chi_k\}_{k\in\mathbb{N}}$ such that

$$\chi_k \overset{*}{\rightharpoonup} \theta \quad \text{in} \quad L^\infty(0,T), \text{ where } \ 0 \le \theta \le 1 \text{ a.e. in } \ (0,T).$$

Then the corresponding functions y_k, which satisfy a uniform Lipschitz condition, converge uniformly to y^∞, the solution of

$$-\frac{\mathrm{d}y^\infty}{\mathrm{d}t} = (1-\theta)u^* + \theta w \quad \text{ in } \ (0,T), \quad y^\infty(0) = 0,$$

and, using the fact that

$$F\left((1-\chi)u^* + \chi w\right) = (1-\chi)F(u^*) + \chi F(w)$$

for every function F and every characteristic function χ, we deduce that $I(u_k, y_k)$ converges to $\widehat{I}(\theta)$ given by

$$\widehat{I}(\theta) = \int_0^T \left(|y^\infty|^2 - (1-\theta)|u^0|^2 - \theta|w|^2\right)\,\mathrm{d}t.$$

Assume that I attains its minimum on Ξ at (u^0, y^0). Then

$$\widehat{I}(0) = I(u^0, y^0) \le \lim_{k\to\infty} I(u_k, y_k) = \widehat{I}(\theta),$$

and therefore, $\widehat{I}$ attains its minimum at 0. Then, noticing that the admissibility for $\delta\theta$ means $\delta\theta \ge 0$, we can write the classical necessary conditions of optimality for $\widehat{I}$ as

$$\frac{\mathrm{d}\delta y}{\mathrm{d}t} = (w - u^0)\delta\theta, \quad \delta y(0) = 0 \tag{5.64}$$

and, as $\theta = 0$ corresponds to $y^\infty = y^0$, that

$$\delta\widehat{I} = \int_0^T \left(2y^0\delta y + \left(|u^0|^2 - |w|^2\right)\delta\theta\right)\,\mathrm{d}t.$$

Using the same p^0 as defined in (5.59), integration by parts gives a different result because the equation for δy is different (see (5.60)):

$$\begin{aligned}\int_0^T 2y^0\delta y\,\mathrm{d}t = -2\int_0^T \frac{\mathrm{d}p^0}{\mathrm{d}t}\delta y\,\mathrm{d}t &= \int_0^T 2p^0\frac{\mathrm{d}\delta y}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_0^T 2p^0(w-u^0)\delta\theta\,\mathrm{d}t,\end{aligned}$$

and therefore,

$$\delta\widehat{I} = \int_0^T \left(2p^0(w-u^0) + \left(|u^0|^2 - |w|^2\right)\right)\delta\theta\,\mathrm{d}t.$$

Hence, the necessary conditions of optimality for $\widehat{I}$ become

$$2p^0(w-u^0) + \left(|u^0|^2 - |w|^2\right) \geq 0 \quad \text{a.e. on} \quad (0,T). \tag{5.65}$$

Using the fact that p^0 is independent of the choice of w, (5.65) means

$$\begin{aligned} 2p^0u^0 - |u^0|^2 &= \inf_{-1\leq w\leq +1}\left(2p^0w - |w|^2\right) \text{ a.e. on } (0,T) \\ &= -2p^0 - 1 \quad \text{a.e. on } (0,T). \end{aligned}$$

As a result, we come to the following necessary conditions of optimality:

$$\begin{cases} p^0 < 0 \text{ implies } u^0 = +1, \\ p^0 = 0 \text{ implies } u^0 = \pm 1, \\ p^0 > 0 \text{ implies } u^0 = -1, \end{cases} \tag{5.66}$$

which are obviously more restrictive than (5.62). Moreover, in this case, the triplet $(u^0, y^0, p^0) = (0,0,0)$ does not satisfy the necessary conditions of optimality (5.53), (5.59), and (5.66).

5.3 Limit analysis of optimal control problems by Γ-convergence

Among the different ways and approaches for studying the asymptotic behavior of different classes of OCPs depending on a small parameter, the primary one relies on the notion of Γ-convergence [29, 33, 41, 42, 43, 44, 92, 93, 94]. Γ-convergence has become, over more than 30 years after its introduction by Ennio De Giorgi, the commonly recognized notion of convergence for variational problems in a general setting. A complete introduction to the general theory of Γ-convergence is the by now classical book by Dal Maso [78] and the book by Braides [29]. To illustrate this approach, we rewrite the OCP (5.2) as follows

$$\min\left\{J_\varepsilon(u,y) = I_\varepsilon(u,y) + \mathbf{1}_{\Xi_\varepsilon}(u,y) \;:\; (u,y) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon\right\}. \tag{5.67}$$

Here, we denote by $\mathbf{1}_{\Xi_\varepsilon}$ the indicator function of the set of admissible pairs Ξ_ε defined by

$$\mathbf{1}_{\Xi_\varepsilon}(u,y) = \begin{cases} 0 & \text{if } (u,y) \in \Xi_\varepsilon, \\ +\infty & \text{if } (u,y) \notin \Xi_\varepsilon. \end{cases} \tag{5.68}$$

Following this representation, we say that a good definition of a "pass to the limit" in (5.67) is a convergence of functionals, which guarantees the

convergence of optimal pairs and minimum values of problems (5.67) to those of a limit problem. Note that the space $\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon$ and the cost functional J_ε at level ε may vary with ε, so that a preliminary problem is to define the convergence of a sequence of functionals which belong to different spaces. The way this convergence is defined must be quite flexible. This is usually done by choosing $\mathbb{U} \times \mathbb{Y}$ large enough so that it contains the domain of the limiting cost functional and all $\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon$ [10, 29]. As a rule, one considers all functionals J_ε as defined on the space $\mathbb{U} \times \mathbb{Y}$ by identifying them with the functionals

$$\overline{J}_\varepsilon(u,y) = \begin{cases} J_\varepsilon(u,y) & \text{if } (u,y) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon, \\ +\infty & \text{if } (u,y) \in \mathbb{U} \times \mathbb{Y} \setminus (\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon). \end{cases}$$

This type of identification is common in dealing with minimum problems and sometimes it is very useful to include constraints directly into the functional. Hence, one may therefore suppose that all $\mathbb{U}_\varepsilon = \mathbb{U}$ and $\mathbb{Y}_\varepsilon = \mathbb{Y}$. Under this supposition, the next theorem gives a definition of a limit problem such that the variational properties (V1)–(V2) hold (for the proof, see [43, Proposition 2.1]).

Theorem 5.5. *Let $\mathbb{Y}$ and $\mathbb{U}$ be two topological spaces and let $J_\varepsilon : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ be a sequence of functionals. Let $(u_\varepsilon^0, y_\varepsilon^0)$ be a minimum point for J_ε, or simply a pair such that*

$$\lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) = \lim_{\varepsilon \to 0} \left[\inf_{(u,y) \in \mathbb{U} \times \mathbb{Y}} J_\varepsilon(u,y) \right].$$

Assume that $(u_\varepsilon^0, y_\varepsilon^0)$ converges to (u^0, y^0) in $\mathbb{U} \times \mathbb{Y}$ and there exists

$$J(u,y) = \Gamma(\mathbb{E}, \mathbb{U}^-, \mathbb{Y}^-) \lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon^0, y_\varepsilon^0). \tag{5.69}$$

Then we have the following:

(i) (u^0, y^0) is a minimum point for J on $\mathbb{U} \times \mathbb{Y}$.
(ii) $\lim_{\varepsilon \to 0} \left[\inf_{(u,y) \in \mathbb{U} \times \mathbb{Y}} J_\varepsilon(u,y) \right] = \min_{(u,y) \in \mathbb{U} \times \mathbb{Y}} J(u,y)$.

Here, $\mathbb{E} = \{\varepsilon = 1/n \ : \ n \in \mathbb{N}\}$, and in this case,

$$\min \{J(u,y) \ : \ (u,y) \in \mathbb{U} \times \mathbb{Y}\} \tag{5.70}$$

is called *a limit problem*, where J is defined by the multiple Γ-limit (5.69).

A multiple Γ-limit is in general defined as follows:

Definition 5.6. *Let $\mathbb{Y}$ and $\mathbb{U}$ be two topological spaces and let $J_\varepsilon : \mathbb{U} \times \mathbb{Y} \to \overline{R}$ be a sequence of functionals. By $Z(+)$ we will denote the "sup" operator and by $Z(-)$ the "inf" one. For every $u \in \mathbb{U}$ and $y \in \mathbb{Y}$, we define:*

$$\Gamma\left(\mathbb{E}^\alpha, \mathbb{U}^\beta, \mathbb{Y}^\gamma\right) \lim_{\varepsilon \to 0} J_\varepsilon(u,y) = \underset{u_\varepsilon \in S(u)}{Z(\beta)} \underset{y_\varepsilon \in S(y)}{Z(\gamma)} \underset{\varepsilon \in \mathbb{E}}{Z(-\alpha)} \underset{\varepsilon' \le \varepsilon}{Z(\alpha)} J_{\varepsilon'}(u_{\varepsilon'}, y_{\varepsilon'}),$$

where α, β, and γ are signs $+$ or $-$ and $S(u)$ and $S(y)$ respectively denote the set of all sequences $u_\varepsilon \to u$ in $\mathbb{U}$ and that of $y_\varepsilon \to y$ in $\mathbb{Y}$. Note that when the Γ-limit does not depend on the sings $+$ or $-$, this sing will be omitted.

For the complete theory of Γ-limits, we refer to Dal Maso [78] and Braides [29].

The limit problem (5.70) can be calculated explicitly in some particular cases (see [33, 38, 41, 42, 43, 44, 47, 48]). Indeed, let us suppose that the set of admissible pairs for the problem (5.2) has the form

$$\Xi_\varepsilon = \{(u,y) \in \mathbb{U} \times \mathbb{Y} \ : \ L_\varepsilon(u) + F_\varepsilon(y) = 0\},$$

where for every fixed $\varepsilon \in \mathbb{E}$, the operators L_ε and F_ε map respectively $\mathbb{U}$ and $\mathbb{Y}$ to a third topological space $\mathbb{Z}$. The collection of OCPs is then

$$\min\{I_\varepsilon(u,y) + \mathbf{1}_{\Xi_\varepsilon}(u,y) \ : \ (u,y) \in \mathbb{U} \times \mathbb{Y}\}, \quad \varepsilon \in \mathbb{E}. \tag{5.71}$$

The main trick of this calculation is the introduction of an auxiliary variable $v = L_\varepsilon(u)$. Then the functional $J_\varepsilon = I_\varepsilon + \mathbf{1}_{\Xi_\varepsilon}$ can be split as

$$\begin{aligned} J_\varepsilon(u,y) &= I_\varepsilon(u,y) + \mathbf{1}_{\{v=L_\varepsilon(u)\}}(u,v) + \mathbf{1}_{\{v=-F_\varepsilon(y)\}}(v,y) \\ &:= G_\varepsilon(u,v,y) + \mathbf{1}_{\{v=-F_\varepsilon(y)\}}(v,y), \end{aligned} \tag{5.72}$$

and the identification of (5.70) is given by the following theorem (for the proof, see [47]).

Theorem 5.7. *Assume the following:*

HF_ε : $-F_\varepsilon$ *G-converges to* $-F$.

HL_ε : *The condition* $u_\varepsilon \to u$ *in* $\mathbb{U}$ *implies that the sequence* $\{L_\varepsilon(u_\varepsilon)\}$ *is relatively compact in* $\mathbb{Z}$.

HG_ε : *For the sequence of functionals* $\{G_\varepsilon(u,y,v) : \mathbb{U} \times \mathbb{Y} \times \mathbb{Z} \to \overline{\mathbb{R}}\}$ *there exists a* Γ*-limit*

$$G(u,v,y) := \Gamma(\mathbb{E}, \mathbb{U} \times \mathbb{Z}^-, \mathbb{Y}^-) \lim_{\varepsilon \to 0} G_\varepsilon(u,v,y). \tag{5.73}$$

Then $J(u,y) = G(u,-F(y),y)$.

In view of this result, the calculation of the limit (5.69) can be reduced to the calculation of $(\mathrm{HG}_\varepsilon)$ and to the identification of the G-limit of the operators $\{F_\varepsilon\}$. We recall the definition of G-convergence of operators (see [210, 261, 262]).

Definition 5.8. *We say that the sequence* $\{F_\varepsilon\}$ *G-converges to* F *if*

$$\Gamma(\mathbb{E}, \mathbb{Z}, \mathbb{Y}^-) \lim_{\varepsilon \to 0} \mathbf{1}_{\{v=F_\varepsilon(y)\}}(v,y) = \mathbf{1}_{\{v=F(y)\}}(v,y)$$

for every $v \in \mathbb{Z}$ *and* $y \in \mathbb{Y}$, *that is, if the following conditions hold true*

(i) If $y_\varepsilon \to y$ *in* $\mathbb{Y}$, $v_\varepsilon \to v$ *in* $\mathbb{Z}$ *and* $F_\varepsilon(y_\varepsilon) = v_\varepsilon$ *for infinitely many* $\varepsilon \in \mathbb{E}$, *then* $F(y) = v$.

(ii) If $y \in \mathbb{Y}$ *and* $v \in \mathbb{Z}$ *are such that* $F(y) = v$ *and* $v_\varepsilon \to v$ *in* $\mathbb{Z}$, *then there exists* $y_\varepsilon \to y$ *in* $\mathbb{Y}$ *satisfying* $F_\varepsilon(y_\varepsilon) = v_\varepsilon$ *for every* $\varepsilon \in \mathbb{E}$ *large enough.*

A lot of results have been established about the characterization of those conditions (see, for instance, [29, 33, 38, 41, 42, 47, 48]). The identification of the limit usually can be resumed in three steps:

Step 1. Introduce the auxiliary variable $v \in \mathbb{Z}$.
Step 2. Calculate the operator $-F$ given by $(\mathrm{HF}_\varepsilon)$ and the Γ-limit in $(\mathrm{HG}_\varepsilon)$.
Step 3. Eliminate the variable v in order to write the final expression of the limit problem setting $J(u, y) = G(u, -F(y), y)$.

Having assumed that $-F_\varepsilon$ G-converge to an operator $-F$, one focuses on the behavior of the input operators L_ε. It is clear that hypothesis $(\mathrm{HL}_\varepsilon)$ is valid when the operators L_ε converge continuously to an operator L, that is,

$$u_\varepsilon \to u \ \text{ in } \ \mathbb{U} \ \Rightarrow L_\varepsilon(u_\varepsilon) \to L(u) \ \text{ in } \ \mathbb{Z}. \tag{5.74}$$

In order to characterize condition $(\mathrm{HG}_\varepsilon)$, one requires some kind of uniform continuity on the cost functional. Usually, it is assumed that there exist a function $\Psi : \mathbb{U} \to \mathbb{R}$ bounded on the $\mathbb{U}$-bounded sets and a function $\omega : \mathbb{Y} \times \mathbb{Y} \to \mathbb{R}$ with $\lim_{z \to y} \omega(y, z) = 0$ for every $y \in \mathbb{Y}$ such that for every $u \in \mathbb{U}$ and $z, y \in \mathbb{Y}$

$$I_\varepsilon(u, y) \le I_\varepsilon(z, u) + \Psi(u)\omega(y, z). \tag{5.75}$$

In this case, the limit problem has the same structure as the one of the original OCP and can be represented as

$$\min \left\{ I(u, y) \ : \ L(u) + F(y) = 0, \ (u, y) \in \mathbb{U} \times \mathbb{Y} \right\}.$$

In order to describe the limit problems under more general assumptions on the operator L_ε, we consider a case in which the verification of the hypotheses and the corresponding calculations can be made explicitly.

Example 5.9. We deal with the following OCP governed by an ordinary differential equations (see Briani [33] for details):

$$I_\varepsilon(u, y) = \int_0^T \left[\frac{1}{2}|u|^2 + |y - \phi(t)|^2 \right] \mathrm{d}t \longrightarrow \inf, \tag{5.76}$$

$$\Xi_\varepsilon = \left\{ (u, y) \ : \ y'(t) = a_\varepsilon(t, y) + \sin\left(\frac{t}{\varepsilon}\right) u(t), \ y(0) = y_0 \right\}, \tag{5.77}$$

$$\Xi_\varepsilon \subset L^2(0, T) \times W^{1,1}(0, T),$$

where $\phi \in L^2(0, T)$ and the functions a_ε are Lipschitz continuous with respect to y and measurable with respect to t, such that

$$a_\varepsilon(\cdot, y) \longrightarrow a(\cdot, y) \quad \text{weakly in } \ L^1(0, T), \ \forall\, y \in \mathbb{R}. \tag{5.78}$$

Then assumption $(\mathrm{HF}_\varepsilon)$ is fulfilled with $F_\varepsilon(y) = -y' + a_\varepsilon(t, y)$ and $F(y) = -y' + a(t, y)$. Moreover, having set

$$G_\varepsilon(u,v,y)=\int_0^T\left[\frac{1}{2}|u|^2+|y-\phi(t)|^2+\mathbf{1}_{\{\sin(t/\varepsilon)u=v\}}(t,u,v)\right]\,\mathrm{d}t$$

and closely following along the line of the calculation scheme of the Γ-limit $G(u,v,y):=\Gamma(\mathbb{E},\mathbb{U}\times\mathbb{Z}^-,\mathbb{Y}^-)\lim_{\varepsilon\to 0}G_\varepsilon(u,v,y)$ in [33, 42], one shows that the limit problem takes the form

$$\begin{aligned}\min\Big\{&I_0(u,y)\\&:=\int_0^T\left[\frac{1}{2}|u|^2+|y-\phi(t)|^2+|y'-a(t,y)|^2\right]\,\mathrm{d}t,\ y(0)=y_0\Big\}.\end{aligned}\tag{5.79}$$

Note that in this case, the state equation has disappeared in the limit problem and it is replaced by an integral contribution in the cost functional as an additional extra term. Thus, we do not have an OCP in the limit as ε tend to 0, but rather a minimization problem.

To characterize the main distinctive features of this approach, we give a few comments relating to the asymptotic analysis of the OCP (5.2). First, since the basic spaces $\mathbb{U}_\varepsilon\times\mathbb{Y}_\varepsilon$ may differ substantially from level to level, we have no guarantee in saying that in the general case, a space $\mathbb{U}\times\mathbb{Y}$ can be found, containing the domains $\mathbb{U}_\varepsilon\times\mathbb{Y}_\varepsilon$ for every $\varepsilon\in\mathbb{E}$ and guaranteeing the following compactness property: The existence of a limit of optimal pairs to (5.2) – assuming that they exist – is ensured beforehand. As follows from Example 5.9, this property may not hold.

Second, as Theorem 5.7 indicates (see also (5.75) and Example 5.9), one always assumes that the cost functional $I_\varepsilon:\Xi_\varepsilon\to\mathbb{R}$ is well defined on the whole space $\mathbb{U}_\varepsilon\times\mathbb{Y}_\varepsilon$. However, taking into account specific OCPs coming from applications, cost functionals have a sense (economical, engineering, physical, etc.) only on the set of admissible solutions and they may be undefined outside of this set.

As was mentioned earlier, one of the primary goals of the asymptotic analysis of the OCPs (5.2) is to obtain the full identification of the structure of a limiting problem, including the precise form of the state equations, cost functional and possible restrictions on managing parameters. The next observation deals with the fact that for both physical and mathematical reasons, we always have some control (and state) constraints in the original problem. In particular, if $u\in U_\varepsilon\subset\mathbb{U}_\varepsilon$, then instead of the functional (5.72), we have to consider

$$\begin{aligned}J_\varepsilon(u,y)&=I_\varepsilon(u,y)+\mathbf{1}_{\{v=L_\varepsilon(u)\}}(u,v)+\mathbf{1}_{\{u\in U_\varepsilon\}}(u)+\mathbf{1}_{\{v=-F_\varepsilon(y)\}}(v,y)\\&:=G_\varepsilon(u,v,y)+\mathbf{1}_{\{u\in U_\varepsilon\}}(u)+\mathbf{1}_{\{v=-F_\varepsilon(y)\}}(v,y).\end{aligned}$$

However, it is a nontrivial matter to find its $\Gamma(\mathbb{E},\mathbb{U}\times\mathbb{Z}^-,\mathbb{Y}^-)$-limit as ε tends to 0 because, in general, the equality

$$\Gamma(\mathbb{E}, \mathbb{U} \times \mathbb{Z}^-, \mathbb{Y}^-) \lim_{\varepsilon \to 0} \left[G_\varepsilon(u, v, y) + \mathbf{1}_{\{u \in U_\varepsilon\}}(u) \right]$$
$$= \Gamma(\mathbb{E}, \mathbb{U} \times \mathbb{Z}^-, \mathbb{Y}^-) \lim_{\varepsilon \to 0} G_\varepsilon(u, v, y) + \Gamma(\mathbb{E}, \mathbb{U}) \lim_{\varepsilon \to 0} \mathbf{1}_{\{u \in U_\varepsilon\}}(u)$$

is not valid [10, 78].

Finally, we note that – returning to Example 5.9 – in spite of the good variational properties of the limit problem (5.79), it cannot be recovered in the form of some optimal control problem. However, if we consider the case, where the control space $L^2(0,T)$ has a topology associated with the weak two-scale convergence, then this limit problem may exhibit another mathematical description. Indeed, the weak two-scale convergence $u_\varepsilon(t) \overset{2}{\rightharpoonup} u(t,\xi)$ means, by definition, that the following relation holds:

$$\lim_{\varepsilon \to 0} \int_0^T u_\varepsilon(t) b(t\varepsilon^{-1}) \varphi(t)\, \mathrm{d}t = \int_0^T \int_0^{2\pi} u(t,\xi) b(\xi) \varphi(t)\, \mathrm{d}\xi\, \mathrm{d}t,$$
$$\forall\, \varphi \in C_0^\infty(0,T), \quad \forall\, b \in C^\infty_{\mathrm{per}}(0,2\pi),$$

where $C^\infty_{\mathrm{per}}(0,2\pi)$ is the space of infinitely differentiable 2π-periodic functions.

Many properties of weak two-scale convergence have been established which make it possible to pass to the limit in the integral identity

$$\int_0^T \left[y_\varepsilon'(t) - a_\varepsilon(t, y_\varepsilon) + \sin\left(\frac{t}{\varepsilon}\right) u_\varepsilon(t) \right] \varphi(t)\, \mathrm{d}t = 0, \tag{5.80}$$

where $y_\varepsilon \in W^{1,1}(0,T)$ is a solution of (5.77) with $u = u_\varepsilon$. One of these properties is as follows [6, 256, 259]: If u_ε is bounded in $L^2(0,T)$, then (after a possible transition to a subsequence)

$$u_\varepsilon(t) \overset{2}{\rightharpoonup} u(t,\xi) \in L^2(0,T; L^2_{\mathrm{per}}(0,2\pi)).$$

As we will see later, using the Γ-convergence technique and the scheme of the direct variational convergence, the limit OCP to the family (5.76) and (5.77) with respect to the product of the weak two-scale convergence in $L^2(0,T)$ and the weak convergence in $W^{1,1}(0,T)$ as $\varepsilon \to 0$ can be recovered in the following form:

$$I_0^{\mathrm{hom}}(u,y) = \frac{1}{2} \int_0^T \int_0^{2\pi} |u(t,\xi)|^2\, \mathrm{d}\xi\, \mathrm{d}t + \int_0^T |y - \phi(t)|^2\, \mathrm{d}t \longrightarrow \inf, \tag{5.81}$$

$$\Xi_0^{\mathrm{hom}} = \Big\{ (u,y) \in L^2(0,T; L^2_{\mathrm{per}}(0,2\pi)) \times W^{1,1}(0,T) \ : \ \int_0^T [y'(t) - a(t,y)]\, \varphi(t)\, \mathrm{d}t + \int_0^T \int_0^{2\pi} \sin(\xi) u(t,\xi) \varphi(t)\, \mathrm{d}\xi\, \mathrm{d}t = 0,$$
$$y(0) = y_0, \ \forall\, \varphi \in C_0^\infty(0,T) \Big\}. \tag{5.82}$$

It should be emphasized that in this case, the limit problem (5.81)–(5.82) is some OCP for an object the state equation of which takes the form of an

integral identity (5.82). As it is easy to see, the problem (5.81)–(5.82) admits a unique solution $(u^0, y^0) \in L^2(0,T; L^2_{\rm per}(0,2\pi)) \times W^{1,1}(0,T)$. Moreover, following the variational properties of Γ-convergence, it can be shown that

$$m_\varepsilon := \inf_{(u,y)\in\,\Xi_\varepsilon} I_\varepsilon(u,y) = I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon) \to I_0(u^0,y^0) = \inf_{(u,y)\in\,\Xi_0} I_0(u,y) =: m_0,$$
$$y^0_\varepsilon \rightharpoonup y^0 \ \text{ in } \ W^{1,1}(0,T), \quad u^0_\varepsilon(t) \overset{2}{\rightharpoonup} u^0(t,\xi) \ \text{ in } \ L^2(0,T).$$

5.4 Direct variational convergence of optimal control problems

The main idea of our approach in this work is to study the asymptotic behavior of a family of OCPs

$$(\mathrm{OCP}_\varepsilon) \ : \ \left\{ \begin{array}{ccc} (\mathrm{CF}_\varepsilon) & : & I_\varepsilon(u,y) \to \inf, \\ & \text{subject to} & \\ (\mathrm{CSC}_\varepsilon) & : & (u,y) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon, u \in U_\varepsilon \subset \mathbb{U}_\varepsilon, \\ (\mathrm{SE}_\varepsilon) & : & L_\varepsilon(u,y) + F_\varepsilon(y) = 0 \end{array} \right\} \tag{5.83}$$

not through the study of the limit properties of the solutions $(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon$, but rather by defining a limit OCP that would be a "good approximation" of the previous one as $\varepsilon \to 0$ and has a clearly defined structure, including the limit form of a state equation, control constraints and a limit cost functional. For this, we apply the so-called scheme of the direct variational convergence, which was developed in [140, 141, 144, 147]. The idea is still to be in the framework of Γ-convergence. The principal feature is the representation of the original OCP as a constrained minimization problem and subsequent study of its asymptotic behavior. As we will see later, such an approach allows reducing the procedure of asymptotic analysis to consecutive identification of the set of admissible pairs for the limit OCP and then its cost functional.

We will always associate to every OCP (5.83) the corresponding constrained minimization problem:

$$(\mathrm{CMP}_\varepsilon) \ : \qquad \left\langle \inf_{(u,y)\in\,\Xi_\varepsilon} I_\varepsilon(u,y) \right\rangle, \tag{5.84}$$

where the set of admissible solutions

$$\Xi_\varepsilon = \{(u,y) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \ : \ L_\varepsilon(u,y) + F_\varepsilon(y) = 0, \quad u \in U_\varepsilon, \ I_\varepsilon(u,y) < +\infty\} \tag{5.85}$$

features a specific structure, namely

$$\Xi_\varepsilon = \{(u,y) \ : \ (\mathrm{SE}_\varepsilon) \cap (\mathrm{CSC}_\varepsilon) \cap \{I_\varepsilon(u,y) < +\infty\}\}.$$

Having observed that the cost functional is well defined and has a sense only on the set of admissible solutions leads us to the object

$$\mathrm{epi}(I_\varepsilon; \Xi_\varepsilon) = \{(u, y, \lambda) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \times \mathbb{R} \mid (u, y) \in \Xi_\varepsilon, \quad I_\varepsilon(u, y) \le \lambda\}, \quad (5.86)$$

and we call it the epigraph of the cost functional.

Note that the set $\mathrm{epi}(I_\varepsilon; \Xi_\varepsilon)$ coincides with the epigraph of the extended cost functional $\widetilde{I}_\varepsilon : \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \to \overline{\mathbb{R}}$, namely

$$\mathrm{epi}(I_\varepsilon; \Xi_\varepsilon) \equiv \mathrm{epi}(\widetilde{I}_\varepsilon) := \left\{(u, y, \lambda) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \times \mathbb{R} \ : \ \widetilde{I}_\varepsilon(x) \le \lambda\right\}, \quad (5.87)$$

where

$$\widetilde{I}_\varepsilon(u, y) = \begin{cases} I_\varepsilon(u, y) & \text{if} \quad (u, y) \in \Xi_\varepsilon, \\ +\infty & \text{if} \quad (u, y) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \setminus \Xi_\varepsilon, \quad \forall\, \varepsilon > 0. \end{cases} \quad (5.88)$$

Obviously, $\mathrm{epi}(\widetilde{I}_\varepsilon) = \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \times \mathbb{R}$ if $\widetilde{I}_\varepsilon \equiv -\infty$ and $\mathrm{epi}(\widetilde{I}_\varepsilon) = \emptyset$ if $\widetilde{I}_\varepsilon \equiv +\infty$.

However, the set $\mathrm{epi}(I_\varepsilon; \Xi_\varepsilon)$ differs considerably from the classical definition of an epigraph

$$\mathrm{epi}(I_\varepsilon) = \{(u, y, \lambda) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \times \mathbb{R} \mid I_\varepsilon(u, y) \le \lambda\}$$

– namely, $\mathrm{epi}(I_\varepsilon; \Xi_\varepsilon)$ is the restriction of $\mathrm{epi}(I_\varepsilon)$ on the set $\Xi_\varepsilon \times \mathbb{R}$. Thus, in view of this construction, the original OCP can be associated with the subset $\mathrm{epi}(I_\varepsilon, \Xi_\varepsilon)$ of $\mathbb{U} \times \mathbb{Y} \times \mathbb{R}$ and this correspondence is one-to-one. It means that the examination of the asymptotic behavior of $(\mathrm{OCP}_\varepsilon)$ as $\varepsilon \to 0$ can be done through the study of the limit properties for the corresponding sequence of sets $\{\mathrm{epi}(I_\varepsilon; \Xi_\varepsilon)\}_{\varepsilon \to 0}$. In fact, this suggestion is motivated by the following result that will be proved in Chap. 8.

Theorem 5.10. *Assume that there is a set $C \neq \emptyset$ such that*

$$\mathrm{epi}(I_\varepsilon; \Xi_\varepsilon) \xrightarrow[\varepsilon \to 0]{K} C$$

in the Painlevé–Kuratowski sense with respect to some appropriate convergence in a variable space $\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \times \mathbb{R}$. Then $C \subset \mathbb{U} \times \mathbb{Y}$ has the structure $C = \mathrm{epi}(I_{hom}; \Xi_{hom})$.

As a result, the set $C = \mathrm{epi}(I_{hom}; \Xi_{hom})$ can be associated with the corresponding constrained minimization problem

$$(\mathrm{CMP}_{\mathrm{hom}}) \ : \qquad \left\langle \inf_{(u,y) \in \Xi_{hom}} I_{hom}(u, y) \right\rangle, \quad (5.89)$$

and we call it the variational limit of the sequence

$$\left\{ \left\langle \inf_{(u,y) \in \Xi_\varepsilon} I_\varepsilon(u, y) \right\rangle \ ; \ \varepsilon \in \mathbb{E} \right\}. \quad (5.90)$$

It leads us to the following definition.

Definition 5.11. *We say that the family of OCPs (5.83) has a limit problem as ε tends to 0 if for the corresponding sequence of constrained minimization problems (5.90), there exists a variational limit which can be recovered in the form of some OCP.*

Resuming this approach, we come to the following scheme of the limit analysis (SLA) of OCPs:

$$\begin{array}{ccccc} (\mathrm{OCP}_\varepsilon) \to \left\langle \inf\limits_{(u,y)\in \Xi_\varepsilon} I_\varepsilon(u,y) \right\rangle & & & & \left\langle \inf\limits_{(u,y)\in \Xi_{hom}} I_{hom}(u,y) \right\rangle \to (\mathrm{OCP}_{\mathrm{hom}}) \\ \downarrow & & & & \uparrow \\ \mathrm{epi}(I_\varepsilon;\Xi_\varepsilon) & & \xrightarrow{\varepsilon\to 0} & & \mathrm{epi}(I_{hom};\Xi_{hom}) \end{array} .$$

It is worth noticing that in spite of apparent differences in the formulation of the limit problem compared with the analogous one by the Γ-convergence approach, it can be proved that these notions are closely connected.

Theorem 5.12 ([140, 143]). *Assume that there exist spaces $\mathbb{U}$ and $\mathbb{Y}$ large enough such that $\Xi_\varepsilon \subset \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \subset \mathbb{U} \times \mathbb{Y}$ for all $\varepsilon \in \mathbb{E}$. Let*

$$I' = \Gamma - \liminf_{\varepsilon\to 0} P_{\Xi_\varepsilon} I_\varepsilon \quad \text{and} \quad I'' = \Gamma - \limsup_{\varepsilon\to 0} P_{\Xi_\varepsilon} I_\varepsilon$$

be the lower and upper Γ-limits of the sequence

$$\left\{ P_{\Xi_\varepsilon} I_\varepsilon : \mathbb{U}\times\mathbb{Y} \to \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\} \right\}_{\varepsilon\to 0},$$

respectively, where

$$P_{\Xi_\varepsilon} I_\varepsilon(x) = \begin{cases} I_\varepsilon(u,y) & \text{if } (u,y)\in \Xi_\varepsilon, \\ +\infty & \text{if } (u,y)\notin \Xi_\varepsilon. \end{cases}$$

Then

$$\begin{aligned} \mathrm{epi}(I'; K - \limsup_{\varepsilon\to 0} \Xi_\varepsilon) &= K - \limsup_{\varepsilon\to 0} [\mathrm{epi}(I_\varepsilon;\Xi_\varepsilon)], \\ \mathrm{epi}(I''; K - \liminf_{\varepsilon\to 0} \Xi_\varepsilon) &= K - \liminf_{\varepsilon\to 0} [\mathrm{epi}(I_\varepsilon;\Xi_\varepsilon)]. \end{aligned}$$

In particular, the sequence $\{\mathrm{epi}(I_\varepsilon;\Xi_\varepsilon)\}_{\varepsilon\to 0}$ has a limit in the sense of Painlevé–Kuratowski if and only if the following conditions hold:

$$K - \limsup_{\varepsilon\to 0} \Xi_\varepsilon = K - \liminf_{\varepsilon\to 0} \Xi_\varepsilon \text{ and } I'' = I' \text{ on the set } K - \liminf_{\varepsilon\to 0} \Xi_\varepsilon.$$

In view of this, there is a good reason to believe that the limit OCP in the sense of Definition 5.11 enjoys the following important properties:

— (SLA) implies the convergence of minimum values and the convergence of (sub)sequence of (almost-)minimizers of $(\mathrm{CMP}_\varepsilon)$ to minimizers of the variational limit $(\mathrm{CMP}_{\mathrm{hom}})$.

— It is stable under continuous perturbations. This means that this analysis valid if we add to the cost functional I_ε any fixed continuous term.
— The cost functional I_{hom} is lower semicontinuous and its domain Ξ_{hom} is closed, which usually implies the existence of optimal pairs to the limit problem.

As we will see later, a crucial role for the realization of the scheme of direct variational convergence will be played by the type of convergence in variable spaces $\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon$. To identify the clearly defined structure of a limit optimal problem including the limit form of a state equation, control constraints and a limit cost functional, it will be a matter of balance between the convenience of a stronger notion of such convergence and a weaker one.

Because our interest is in the asymptotic analysis of OCPs, we note that sometimes it is convenient to rewrite (3.2)–(3.3) in the form of the minimization problem

$$\inf_{(u,y)\in\,\Xi} I(u,y), \tag{5.91}$$

where the set of admissible pairs is chosen to be

$$\Xi = \{(u,y) \in \mathbb{U}\times\mathbb{Y} \ : \ u \in U_\partial,\ y \in \operatorname{argmin} G(u,\cdot)\}\,. \tag{5.92}$$

Here, $G : \mathbb{U}\times\mathbb{Y} \to \overline{\mathbb{R}}$ is a functional of the calculus of variations whose integrand depends on the control and its Euler–Lagrange equation takes the form $L(u,y) + F(y) = 0$. Following Bucur and Buttazzo [38], we call G the *state functional*. It should be stressed that the set Ξ can be always written in the form (5.92) by choosing

$$G(u,y) = \mathbf{1}_\Xi = \begin{cases} 0 & \text{if } \ (u,y)\in\Xi, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.93}$$

For instance, when a state equation is

$$-\Delta y + y = f + u \ \text{ in } \ \Omega, \quad y \in H_0^1(\Omega), \quad u \in U_\partial \subset L^2(\Omega),$$

the corresponding state functional $G : L^2(\Omega)\times H_0^1(\Omega) \to \overline{\mathbb{R}}$ can be taken as

$$G(u,y) = \frac{1}{2}\int_\Omega \left(|\nabla y|^2 + y^2 - 2(f+u)y\right)\,\mathrm{d}x + \mathbf{1}_{U_\partial}(u).$$

In this case, the natural topology on $\mathbb{U}$ that takes into account the convergence minimizers of G is the one related to Γ-convergence of the mappings $G(u,\cdot)$ [48]. Indeed, as soon as the convergence of controls implies the convergence of associated states, it would be enough to have the compactness of minimizing sequences in $\mathbb{U}$ and the lower semicontinuity of the cost functional I in $\mathbb{U}\times\mathbb{Y}$ to obtain the existence of an optimal pair for (5.91)–(5.92).

6

Convergence Concepts in Variable Banach Spaces

The aim of this chapter is to provide a systematic exposition of the main properties of weak and strong convergence in variable L^p- and $W^{1,p}$-spaces for $p > 1$. It will always be assumed that p and q are conjugate indices (i.e., $1 = 1/p + 1/q$) and that $p > 1$. The main objects of our consideration are sequences of the types $\{y_\varepsilon^h \in L^p(\Omega, \mathrm{d}\mu_\varepsilon^h)\}_{\varepsilon>0}$ and $\{y_\varepsilon^h \in W^{1,p}(\Omega, \mathrm{d}\mu_\varepsilon^h)\}_{\varepsilon>0}$, where μ_ε^h is a two-parametric Borel measure related to the geometry of thin periodic structures. Typically, the parameter ε defines the periodicity cell and εh is the thickness of constituting elements of such structures. However, there is a principal difference between perforated domains and thin structures. For perforated domains, the typical case is when the parameter h is either independent of ε or such that $\liminf_{\varepsilon\to 0} h(\varepsilon) = h^* > 0$, whereas the principle feature of thin structures is the fact that the parameters ε and $h = h(\varepsilon)$ are related by the supposition $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore, our main intension in this chapter is to shed some light on convergence properties in described spaces.

6.1 General setting

We begin with the following assumptions and notations. For any small positive value $\varepsilon > 0$, let $\mathbb{X}_\varepsilon$ be a real reflexive separable Banach space equipped with the norm $\|\cdot\|_{\mathbb{X}_\varepsilon}$. Let θ_ε be 0 in $\mathbb{X}_\varepsilon$ and let $\|\cdot\|_{0,\mathbb{X}_\varepsilon}$ be an auxiliary norm in $\mathbb{X}_\varepsilon$. As usual, by $\mathbb{X}_\varepsilon^*$ we denote the dual space of $\mathbb{X}_\varepsilon$. We also assume that the small parameter $\varepsilon > 0$ varies in a strictly decreasing sequence of positive numbers which converge to 0. Further, let $\mathbb{X}$ be a real reflexive separable Banach space equipped with the norm $\|\cdot\|_{\mathbb{X}}$, let θ be 0 in $\mathbb{X}$, and let $\|\cdot\|_{0,\mathbb{X}}$ be an auxiliary norm in $\mathbb{X}$ relative to which the ball $B_{\mathbb{X}}(0,1) = \{u \in \mathbb{X} \ : \ \|u\|_{\mathbb{X}} \le 1\}$ is precompact.

For any $\varepsilon > 0$, we suppose that there exists a linear mapping $Q_\varepsilon : \mathbb{X} \to \mathbb{X}_\varepsilon$ such that

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_6,
© Springer Science+Business Media, LLC 2011

$$\text{if } u \in \mathbb{X}, \text{ then } \|Q_\varepsilon u\|_{\mathbb{X}_\varepsilon} \le \|u\|_{\mathbb{X}}, \ \|Q_\varepsilon u\|_{0,\mathbb{X}_\varepsilon} \le \|u\|_{0,\mathbb{X}}; \tag{6.1}$$

$$\text{if } u \in \mathbb{X} \text{ and } \liminf_{\varepsilon\to 0} \|Q_\varepsilon u\|_{0,\mathbb{X}_\varepsilon} = 0, \text{ then } u = \theta. \tag{6.2}$$

By $\mathcal{L}_\varepsilon$ we denote the set of all continuous linear mappings of $\mathbb{X}_\varepsilon$ into $\mathbb{X}$. Further, let $\mathcal{P}$ be the set of all sequences $\{P_\varepsilon\}_{\varepsilon>0}$ satisfying the following conditions:

$$P_\varepsilon \in \mathcal{L}_\varepsilon \text{ for every } \varepsilon > 0; \quad \sup_{\varepsilon>0} \|P_\varepsilon\|_{\mathcal{L}_\varepsilon} < +\infty; \tag{6.3}$$

$$\text{if } u \in \mathbb{X}_\varepsilon, \text{ then } Q_\varepsilon(P_\varepsilon u) = u. \tag{6.4}$$

Hereinafter it is assumed that $\mathcal{P} \neq \emptyset$ and

$$\kappa = \inf\left\{\sup_{\varepsilon>0} \|P_\varepsilon\|_{\mathcal{L}_\varepsilon} \ : \ \{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}\right\}. \tag{6.5}$$

We note that $\kappa \ge 1$. Indeed, taking $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$, $u \in \mathbb{X}_\varepsilon$, and $u \neq \theta_\varepsilon$, by (6.1) we have

$$\|u\|_{\mathbb{X}_\varepsilon} = \|Q_\varepsilon(P_\varepsilon u)\|_{\mathbb{X}_\varepsilon} \le \|P_\varepsilon u\|_{\mathbb{X}} \le \|P_\varepsilon\|_{\mathcal{L}_\varepsilon} \|u\|_{\mathbb{X}_\varepsilon}.$$

Hence, $\sup_{\varepsilon>0} \|P_\varepsilon\|_{\mathcal{L}_\varepsilon} \ge 1$. Since here $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$ is arbitrary, it follows that $\kappa \ge 1$.

Definition 6.1. *Assume that $u_\varepsilon \in \mathbb{X}_\varepsilon$ for any $\varepsilon > 0$. We say that the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded if the sequence of norms $\{\|u_\varepsilon\|_{\mathbb{X}_\varepsilon}\}_{\varepsilon>0}$ is bounded. Similarly, if $f_\varepsilon \in \mathbb{X}_\varepsilon^*$ for every $\varepsilon > 0$, then boundedness of the sequence $\{f_\varepsilon\}_{\varepsilon>0}$ means boundedness of the sequence of norms $\left\{\|f_\varepsilon\|_{\mathbb{X}_\varepsilon^*}\right\}_{\varepsilon>0}$.*

Definition 6.2. *Assume that $u_\varepsilon \in \mathbb{X}_\varepsilon$ for any $\varepsilon > 0$ and that $u \in \mathbb{X}$. We say that the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ converges weakly to u (in the scale of variable spaces $\{\mathbb{X}_\varepsilon\}$) if the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded and $\lim_{\varepsilon\to 0} \|u_\varepsilon - Q_\varepsilon u\|_{0,\mathbb{X}_\varepsilon} = 0$.*

Let us note that if $u \in \mathbb{X}$, then the sequence $\{u_\varepsilon = Q_\varepsilon u\}_{\varepsilon>0}$ converges weakly to u in the above sense.

Definition 6.3. *Assume that $u_\varepsilon \in \mathbb{X}_\varepsilon$ for any $\varepsilon > 0$ and that $u \in \mathbb{X}$. We say that the sequence $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ converges strongly to u (in the scale of variable spaces $\{\mathbb{X}_\varepsilon\}$) if $\lim_{\varepsilon\to 0} \|u_\varepsilon - Q_\varepsilon u\|_{\mathbb{X}_\varepsilon} = 0$.*

Definition 6.4. *Assume that $f_\varepsilon \in \mathbb{X}_\varepsilon^*$ for any $\varepsilon > 0$ and that $f \in \mathbb{X}^*$. We say that the sequence $\{f_\varepsilon \in \mathbb{X}_\varepsilon^*\}_{\varepsilon>0}$ converges strongly to f if the fact that $u_\varepsilon \in \mathbb{X}_\varepsilon$ for any $\varepsilon > 0$, $u \in \mathbb{X}$ and the sequence $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ converges weakly to u implies*

$$\lim_{\varepsilon\to 0} \langle f_\varepsilon, u_\varepsilon\rangle_{\mathbb{X}_\varepsilon^*,\mathbb{X}_\varepsilon} = \langle f, u\rangle_{\mathbb{X}^*,\mathbb{X}}.$$

As follows from condition (6.1), the strong convergence of the sequence $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ to $u \in \mathbb{X}$ implies the weak convergence of this sequence to the same element u. Next, following Kovalevskiĭ [155], we proceed in this section with the main properties of the convergence in variable spaces (see also [156, 157, 158]).

Proposition 6.5. *Let $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence. Then from this sequence it is possible to extract a weakly convergent subsequence.*

Proof. It is easy to see that for any $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$, the sequence $\{P_\varepsilon u_\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathbb{X}$. Hence, by the reflexivity of $\mathbb{X}$, there exist an decreasing sequence of positive numbers $\{\varepsilon_k\}_{k=1}^\infty$ converging to 0 and $u \in \mathbb{X}$ such that $P_{\varepsilon_k} u_{\varepsilon_k} \rightharpoonup u$ in $\mathbb{X}$. Using the assumption regarding the norm $\|\cdot\|_{0,\mathbb{X}}$, we find that $\|P_{\varepsilon_k} u_{\varepsilon_k} - u\|_{0,\mathbb{X}} \to 0$. Hence, from (6.1) we conclude that the sequence $\{u_{\varepsilon_k} \in \mathbb{X}_\varepsilon\}_{k=1}^\infty$ converges weakly to u. The proposition is proved. □

Proposition 6.6. *Assume that $u_\varepsilon \in \mathbb{X}_\varepsilon$ for any $\varepsilon > 0$, $u \in \mathbb{X}$ and the sequence $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ converges weakly to u. Then $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $\mathbb{X}$ for any sequence $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$.*

Proof. Let $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$ be a fixed sequence. We assume that the sequence $\{P_\varepsilon u_\varepsilon\}_{\varepsilon>0}$ does not converge weakly to u in $\mathbb{X}$. Then there exist a sequence $\{\varepsilon_k\}_{k=1}^\infty$ converging to 0 and $v \in \mathbb{X}$, $v \neq u$, such that $P_{\varepsilon_k} u_{\varepsilon_k} \rightharpoonup v$ in $\mathbb{X}$. From this and the weak convergence of $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ to u it follows that $\liminf_{k\to\infty} \|Q_\varepsilon(u - v)\|_{0,\mathbb{X}_\varepsilon} = 0$. Then, by condition (6.2), we obtain $u = v$. However, this contradicts the inequality $v \neq u$. Thus, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $\mathbb{X}$. The proposition is proved. □

Proposition 6.7. *Assume that $u_\varepsilon \in \mathbb{X}_\varepsilon$ for any $\varepsilon > 0$, $u \in \mathbb{X}$ and the sequence $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ converges weakly to u. Then*

$$\liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{\mathbb{X}_\varepsilon} \geq \kappa^{-1} \|u\|_{\mathbb{X}}. \tag{6.6}$$

Proof. Having chosen an arbitrary $\delta > 0$ and a sequence $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$ such that

$$\sup_{\varepsilon\to 0} \|P_\varepsilon\|_{\mathcal{L}_\varepsilon} \leq \kappa + \delta, \tag{6.7}$$

by Proposition 6.6, we have $\liminf_{\varepsilon\to 0} \|P_\varepsilon u_\varepsilon\|_{\mathbb{X}} \geq \|u\|_{\mathbb{X}}$. From this and (6.7), it follows that

$$\liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{\mathbb{X}_\varepsilon} \geq (\kappa + \delta)^{-1} \|u\|_{\mathbb{X}}.$$

Passing to the limit in this inequality as $\delta \to 0$, we obtain (6.6). This concludes the proof. □

It is useful to note that, as follows from Proposition 6.6, if $f \in \mathbb{X}^*$ and $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$, then the sequence $\{f \circ P_\varepsilon\}_{\varepsilon>0}$ converges strongly to f.

Proposition 6.8. *Assume that $f_\varepsilon \in \mathbb{X}_\varepsilon^*$ for any $\varepsilon > 0$, $f \in \mathbb{X}^*$ and the sequence $\{f_\varepsilon \in \mathbb{X}_\varepsilon^*\}_{\varepsilon>0}$ converges strongly to f. Then, for any sequence $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$, the following relation holds true:*

$$\lim_{\varepsilon\to 0} \|f_\varepsilon - f \circ P_\varepsilon\|_{\mathbb{X}_\varepsilon^*} = 0. \tag{6.8}$$

Proof. Let $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$ be a fixed sequence. We assume that the sequence of norms $\{\|f_\varepsilon - f \circ P_\varepsilon\|_{\mathbb{X}_\varepsilon^*}\}_{\varepsilon>0}$ does not converge to 0. Then there exist $\delta > 0$ and a sequence of indices $\{\varepsilon_k\}_{k=1}^\infty$ converging to 0 such that

$$\|f_{\varepsilon_k} - f \circ P_{\varepsilon_k}\|_{\mathbb{X}_{\varepsilon_k}^*} \geq \delta, \quad k \in \mathbb{N}. \tag{6.9}$$

For any $k \in \mathbb{N}$, we set $g_k = f_{\varepsilon_k} - f \circ P_{\varepsilon_k}$. From (6.9), it follows that there exists a bounded sequence $\{w_k \in \mathbb{X}_{\varepsilon_k}\}_{k=1}^\infty$ such that

$$\langle g_k, w_k\rangle_{\mathbb{X}_{\varepsilon_k}^*,\mathbb{X}_{\varepsilon_k}} \geq \delta/2, \quad k \in \mathbb{N}. \tag{6.10}$$

Since the sequence $\{v_k = P_{\varepsilon_k} w_k\}_{k=1}^\infty$ is bounded in $\mathbb{X}$, there exist an increasing sequence $\{k_i\}_{i=1}^\infty$ and $v \in \mathbb{X}$ such that $v_{k_i} \rightharpoonup v$ in $\mathbb{X}$. From this and the strong convergence of $f_\varepsilon \in \mathbb{X}_\varepsilon^*$ to f follows that $\langle g_{k_i}, w_{k_i}\rangle_{\mathbb{X}_{\varepsilon_{k_i}}^*,\mathbb{X}_{\varepsilon_{k_i}}} \to 0$. However, this contradicts (6.10). Thus, (6.8) holds. □

Proposition 6.9. *Under the conditions of Proposition 6.8, the following inequalities hold true:*

$$\liminf_{\varepsilon\to 0} \|f_\varepsilon\|_{\mathbb{X}_\varepsilon^*} \geq \|f\|_{\mathbb{X}^*}, \tag{6.11}$$

$$\limsup_{\varepsilon\to 0} \|f_\varepsilon\|_{\mathbb{X}_\varepsilon^*} \leq \kappa\|f\|_{\mathbb{X}^*}. \tag{6.12}$$

Proof. We fix an arbitrary $\delta > 0$. Suppose that $u \in \mathbb{X}$, $\|u\|_{\mathbb{X}} \leq 1$, and

$$\|f\|_{\mathbb{X}^*} \leq \langle f, u\rangle_{\mathbb{X}^*,\mathbb{X}} + \delta. \tag{6.13}$$

Using the strong convergence of f_ε to f, the weak convergence of $Q_\varepsilon u$ to u and (6.1), we find

$$\langle f, u\rangle_{\mathbb{X}^*,\mathbb{X}} = \lim_{\varepsilon\to 0} \langle f_\varepsilon, Q_\varepsilon u\rangle_{\mathbb{X}_\varepsilon^*,\mathbb{X}_\varepsilon} \leq \liminf_{\varepsilon\to 0} \|f_\varepsilon\|_{\mathbb{X}_\varepsilon^*}.$$

From this and (6.13), we come to the inequality

$$\liminf_{\varepsilon\to 0} \|f_\varepsilon\|_{\mathbb{X}_\varepsilon^*} \geq \|f\|_{\mathbb{X}^*} - \delta.$$

Passing to the limit in this inequality as $\delta \to 0$, we obtain (6.11).

We now prove (6.12). We choose an arbitrary $\delta > 0$ and a sequence $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$ such that

$$\sup_{\varepsilon>0} \|P_\varepsilon\|_{\mathcal{L}_\varepsilon} \leq \kappa + \delta. \tag{6.14}$$

For every $\varepsilon > 0$, we have

$$\|f_\varepsilon\|_{\mathbb{X}_\varepsilon^*} \le \|f \circ P_\varepsilon\|_{\mathbb{X}_\varepsilon^*} + \|f_\varepsilon - f \circ P_\varepsilon\|_{\mathbb{X}_\varepsilon^*}.$$

From this, using (6.14), the strong convergence of f_ε to f and Proposition 6.8, we deduce that

$$\limsup_{\varepsilon \to 0} \|f_\varepsilon\|_{\mathbb{X}_\varepsilon^*} \le (\kappa + \delta)\|f\|_{\mathbb{X}^*}.$$

Passing to the limit in this quality as $\delta \to 0$, we obtain the desired relation (6.12). This concludes the proof. □

The following example illustrates the application of the results given above to the convergence formalism in the variable Sobolev spaces connected with nonperiodically perforated domains.

Example 6.10. Let Ω be a bounded open domain in $\mathbb{R}^n$ $(n \ge 2)$ with Lipschitz boundary, and let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a sequence of open domains in $\mathbb{R}^n$ contained in Ω. It is assumed that there exist a constant $\nu > 1$, a sequence of finite sets $\{J_\varepsilon\}_{\varepsilon>0}$, points $x_\varepsilon^j \in \Omega$ and numbers $r_\varepsilon^j > 0$ $(j \in J_\varepsilon, \varepsilon > 0)$ such that

$$\Omega \setminus \Omega_\varepsilon = \bigcup_{j \in J_\varepsilon} B_\varepsilon^j, \tag{6.15}$$

$$\lim_{\varepsilon \to 0} \max_{j \in J_\varepsilon} r_\varepsilon^j = 0, \tag{6.16}$$

$$2(\nu - 1) r_\varepsilon^j \le \rho_\varepsilon^j, \quad \forall\, j \in J_\varepsilon,\ \forall\, \varepsilon > 0, \tag{6.17}$$

where B_ε^j is the closed ball centered at the point x_ε^j with radius r_ε^j and ρ_ε^j is the distance from B_ε^j to the set $\bigcup_{J_\varepsilon \ni l \ne j} B_\varepsilon^l \cup \partial\Omega$.

Let us show that the spaces $\mathbb{X}_\varepsilon = W^{1,p}(\Omega_\varepsilon)$ and $\mathbb{X} = W^{1,p}(\Omega)$ satisfy all conditions which are essential to the application of the convergence concept in the variable spaces in the sense of Definitions 6.2 and 6.3. Note that conditions (6.1)–(6.4) are the same as the so-called condition of "strong connectedness" of the domains Ω_ε (see [182]).

Let $\|\cdot\|_{0,\mathbb{X}_\varepsilon}$ be an auxiliary norm in $W^{1,p}(\Omega_\varepsilon)$ coinciding with the restriction to $W^{1,p}(\Omega_\varepsilon)$ of the norm in $L^p(\Omega_\varepsilon)$ and let $\|\cdot\|_{\mathbb{X}_\varepsilon^*}$ be the norm in $(W^{1,p}(\Omega_\varepsilon))^*$. By analogy we set

$$\|\cdot\|_{0,W^{1,p}(\Omega)} = \|\cdot\|_{L^p(\Omega)}, \quad \|\cdot\|_{\mathbb{X}^*} = \|\cdot\|_{(W^{1,p}(\Omega))^*}.$$

For every $\varepsilon > 0$, we define an operator $Q_\varepsilon : W^{1,p}(\Omega) \to W^{1,p}(\Omega_\varepsilon)$ as a linear mapping of $W^{1,p}(\Omega)$ to $W^{1,p}(\Omega_\varepsilon)$ such that $Q_\varepsilon u = u|_{\Omega_\varepsilon}$ for any $u \in W^{1,p}(\Omega)$. It is clear that in this case condition (6.1) is valid. We are now in a position to establish the fulfillment of condition (6.2).

Lemma 6.11. *Suppose that $u \in W^{1,p}(\Omega)$ and*

$$\liminf_{\varepsilon \to 0} \|Q_\varepsilon u\|_{L^p(\Omega_\varepsilon)} = 0. \tag{6.18}$$

Then $u = 0$ almost everywhere on Ω.

Proof. In view of conditions (6.15)–(6.17), it follows that for any cube $\square \subset \Omega$, we have

$$\liminf_{\varepsilon \to 0} |\square \cap \Omega_\varepsilon| \geq (1 - \nu^{-n})|\square|. \tag{6.19}$$

From this we conclude that for any non-negative function $\varphi \in L^1(\Omega)$ the inequality

$$\liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \varphi \, \mathrm{d}x \geq (1 - \nu^{-n}) \int_{\Omega} \varphi \, \mathrm{d}x$$

is valid. Applying this inequality to $\varphi = |u|^p$ and using (6.18), we come to the conclusion that $u = 0$ almost everywhere in Ω. The lemma is proved. □

In what follows, we make use the following notation: If $\varepsilon > 0$ is a fixed value and $j \in J_\varepsilon$, then by $B^j_{\varepsilon,\nu}$ we denote the open ball with center at the point x^j_ε and radius νr^j_ε. Let $E^j_\varepsilon = B^j_{\varepsilon,\nu} \setminus B^j_\varepsilon$. We note that, by condition (6.17), the balls $B^j_{\varepsilon,\nu}$ are contained in Ω and $B^j_{\varepsilon,\nu} \cap B^l_{\varepsilon,\nu} = \emptyset$ for $j \neq l$. Therefore, $\bigcup_{j \in J_\varepsilon} E^j_\varepsilon \subset \Omega_\varepsilon$ for all $\varepsilon > 0$.

Lemma 6.12. *There exist a sequence of linear mappings*

$$\left\{P_\varepsilon \in \mathcal{L}(W^{1,p}(\Omega_\varepsilon), W^{1,p}(\Omega))\right\}_{\varepsilon > 0} \in \mathcal{P}$$

and a constant $C > 0$ independent of $\varepsilon > 0$ such that

$$\|\nabla(P_\varepsilon u)\|_{\mathbf{L}^p(B^j_\varepsilon)} \leq C \|\nabla u\|_{\mathbf{L}^p(E^j_\varepsilon)} \tag{6.20}$$

for any $u \in W^{1.m}(\Omega_\varepsilon)$, $j \in J_\varepsilon$ and $\varepsilon > 0$.

Proof. Let B_1 be the closed ball with center at 0 and radius 1, let B_ν be the open ball with center at 0 and radius ν, let $E = B_\nu \setminus B_1$ and for any function $u \in W^{1,p}(E)$, let $\langle u \rangle$ be the mean value of u on E. Because of the smoothness of the boundary of the domain E, there exists a continuous linear mapping $P_0 : W^{1,p}(E) \to W^{1,p}(B_\nu)$ such that $(P_0 u)|_E = u$ for all $u \in W^{1,p}(E)$. We denote the norm of the mapping P_0 by ν_0. Suppose now that P is a mapping of $W^{1,p}(E)$ into $W^{1,p}(B_\nu)$ such that

$$Pu = P_0(u - \langle u \rangle) + \langle u \rangle, \quad \forall\, u \in W^{1,p}(E).$$

It is easy to see that the mapping P is linear and continuous and that $(Pu)|_E = u$ for all $u \in W^{1,p}(E)$. Since, according to [162], there exists $\nu_1 > 0$ such that for any function $u \in W^{1,p}(E)$ with 0 mean value on E,

$$\|u\|_{W^{1,p}(E)} \leq \nu_1 \|\nabla u\|_{\mathbf{L}^p(E)},$$

it follows that for any $u \in W^{1,p}(E)$ and $i \in \{1, \ldots, n\}$, we have

$$\left\| \frac{\partial (Pu)}{\partial x_i} \right\|_{L^p(B_\nu)} \leq \nu_0 \|u - \langle u \rangle\|_{W^{1,p}(E)} \leq \nu_0 \nu_1 \|\nabla u\|_{\mathbf{L}^p(E)}. \tag{6.21}$$

We denote by ν_2 the norm of the mapping P. We define the mappings f_ε^j and g_ε^j as follows: If $j \in J_\varepsilon$, then f_ε^j is a mapping of $W^{1,p}(E_\varepsilon^j)$ into $W^{1,p}(E)$ such that for any $u \in W^{1,p}(E_\varepsilon^j)$ and $x \in E$,

$$(f_\varepsilon^j u)(x) = u(x_\varepsilon^j + r_\varepsilon^j x);$$

g_ε^j is a mapping of $W^{1,p}(B_\nu)$ into $W^{1,p}(B_{\varepsilon,\nu}^j)$ such that for any $u \in W^{1,p}(B_\nu)$ and $x \in B_{\varepsilon,\nu}^j$,

$$(g_\varepsilon^j u)(x) = u\left((r_\varepsilon^j)^{-1}(x - x_\varepsilon^j)\right).$$

For any $j \in J_\varepsilon$, we set $P_\varepsilon^j = g_\varepsilon^j \circ P \circ f_\varepsilon^j$. It is not hard to see that P_ε^j is a continuous linear mapping of $W^{1,p}(E_\varepsilon^j)$ into $W^{1,p}(B_{\varepsilon,\nu}^j)$ and $(P_e^j u)\big|_{E_\varepsilon^j} = u$ for all $u \in W^{1,p}(E_\varepsilon^j)$. Moreover, for $j \in J_\varepsilon$ and $u \in W^{1,p}(E_\varepsilon^j)$, we have

$$\|P_\varepsilon^j u\|_{L^p(B_{\varepsilon,\nu}^j)} \le \nu_2(1 + r_\varepsilon^j)\|u\|_{W^{1,p}(E_\varepsilon^j)}, \tag{6.22}$$

and by (6.21),

$$\left\|\frac{\partial(P_\varepsilon^j u)}{\partial x_i}\right\|_{L^p(B_{\varepsilon,\nu}^j)} \le \nu_0\nu_1 \sum_{k=1}^n \left\|\frac{\partial u}{\partial x_k}\right\|_{L^p(E_\varepsilon^j)}, \quad i = 1, \dots, n. \tag{6.23}$$

Suppose now that P_ε is a mapping of $W^{1,p}(\Omega_\varepsilon)$ to $W^{1,p}(\Omega)$ such that, for $u \in W^{1,p}(\Omega_\varepsilon)$, $(P_\varepsilon u)(x) = u(x)$ if $x \in \Omega_\varepsilon$ and $(P_\varepsilon u)(x) = \left(P_\varepsilon(u|_{E_\varepsilon^j})\right)(x)$ if $x \in B_\varepsilon^j$.

Using (6.22) and (6.23) and condition (6.16), we establish that $\{P_\varepsilon\}_{\varepsilon>0} \in \mathcal{P}$. We set $C = n^2\nu_0\nu_n$. Then, using (6.23), we find that for any $u \in W^{1,p}(\Omega_\varepsilon)$, and $j \in J_\varepsilon$, (6.20) holds true. The lemma is proved. □

As a result, by Lemma 6.12, we obtain

$$\mathcal{P} \neq \emptyset \quad \text{and} \quad \sup_{\varepsilon>0} \|P_\varepsilon\|_{\mathcal{L}_\varepsilon(W^{1,p}(\Omega_\varepsilon), W^{1,p}(\Omega))} < +\infty.$$

Moreover, from this, Lemma 6.11, and the obvious inequalities

$$\|Q_\varepsilon u\|_{W^{1,p}(\Omega_\varepsilon)} \le \|u\|_{W^{1,p}(\Omega)}, \quad \|Q_\varepsilon u\|_{L^p(\Omega_\varepsilon)} \le \|u\|_{L^p(\Omega)}$$

valid for any $u \in W^{1,p}(\Omega)$ and $\varepsilon > 0$, it follows that the spaces $W^{1,p}(\Omega_\varepsilon)$ and $W^{1,p}(\Omega)$ satisfy all conditions that we started with in this section.

6.2 Weak convergence in variable L^p-spaces

Let μ_ε and μ be Radon measures. Let us recall that a non-negative Borel measure μ on Ω ($\mu \in \mathcal{M}_+(\Omega)$) is said to be a Radon measure if $\mu(K) < +\infty$ for any compact subset K of Ω. As usual, we will indicate by $\mathcal{M}_b(\Omega)$ the space of Radon measures on Ω with finite total variation. Note that $\mathcal{M}_b(\Omega)$

is a Banach space when equipped with the norm $\|\mu\|_{\mathcal{M}_b(\Omega)} := |\mu|(\Omega)$. For any $\mu \in \mathcal{M}_b(\Omega)$, we adopt the usual notation $L^p(\Omega, \mathrm{d}\mu)$ to indicate the space of p-summable functions with respect to μ on Ω. We omit μ if it is the Lebesgue measure. Hereinafter in this section, we suppose that $\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0$ in the sense of Definition 2.29, that is,

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi \, \mathrm{d}\mu_\varepsilon = \int_\Omega \varphi \, \mathrm{d}\mu, \quad \forall \, \varphi \in C_0(\Omega). \tag{6.24}$$

We say that a sequence $\{v_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon > 0}$ is bounded if

$$\limsup_{\varepsilon \to 0} \int_\Omega |v_\varepsilon|^p \, \mathrm{d}\mu_\varepsilon < +\infty.$$

Remark 6.13. Note that, in contrast to the situation indicated in the previous section, the spaces $\mathbb{X}_\varepsilon = L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ do not possess properties (6.1)–(6.5). However, as will be shown in Chaps. 11, 13, 14 and 15, the spaces like $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ are natural for a wide class of OPCs in reticulated and perforated domains. Therefore, in order to provide an asymptotic analysis of such OCPs as $\varepsilon \to 0$, we give in this section a convergence formalism in the scale of variable spaces $\mathbb{X}_\varepsilon = L^p(\Omega, \mathrm{d}\mu_\varepsilon)$.

Definition 6.14. *A bounded sequence $\{v_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon > 0}$ is weakly convergent in the variable space $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ to a function $v \in L^p(\Omega, \mathrm{d}\mu)$ if*

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon \varphi \, \mathrm{d}\mu_\varepsilon = \int_\Omega v \varphi \, \mathrm{d}\mu, \quad \forall \, \varphi \in C_0^\infty(\Omega). \tag{6.25}$$

The main property concerning the weak convergence in $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ can be expressed as follows.

Proposition 6.15. *If a sequence $\{v_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon > 0}$ is bounded, then it is compact in the sense of weak convergence in $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$.*

Proof. Having set $L_\varepsilon(\varphi) = \displaystyle\int_\Omega v_\varepsilon \varphi \, \mathrm{d}\mu_\varepsilon$, $\forall \, \varphi \in C_0^\infty(\Omega)$, and making use of Hölder inequality, we get

$$\begin{aligned} |L_\varepsilon(\varphi)| &\le \left(\int_\Omega |v_\varepsilon|^p \, \mathrm{d}\mu_\varepsilon \right)^{1/p} \left(\int_\Omega |\varphi|^q \, \mathrm{d}\mu_\varepsilon \right)^{1/q} \\ &\le C \left(\int_\Omega |\varphi|^q \, \mathrm{d}\mu_\varepsilon \right)^{1/q} \le C \max |\varphi| \mu_\varepsilon(K), \end{aligned} \tag{6.26}$$

where $K = \operatorname{supp} \varphi$. From this and Lemma 2.32, we conclude that

$$|L_\varepsilon(\varphi)| \le 2C\mu(K) \max |\varphi|$$

for $\varepsilon > 0$ small enough, (i.e., $\varepsilon \le \varepsilon_0 = \varepsilon_0(K)$). Since the set

$$T(K) = \{\varphi \in C_0^\infty(\Omega) \ : \ \varphi \subseteq K\}$$

is separable with respect to the norm $\max|\varphi|$ and $\{L_\varepsilon(\varphi)\}_{\varepsilon>0}$ is a uniformly bounded sequence of linear functionals, it follows that (see [258]) there exists a subsequence of positive numbers $\{\varepsilon_j\}_{j=1}^\infty$ for which the limit (in the sense of point-by-point convergence)

$$\lim_{j\to\infty} L_{\varepsilon_j}(\varphi) = L(\varphi) \tag{6.27}$$

is well defined for every $\varphi \in T(K)$. Using the fact that the domain Ω can be represented as the combination of a sequence of compact sets $\Omega = \bigcup K_i$, one gets $C_0^\infty(\Omega) = \bigcup T(K_i)$. Therefore, following the diagonal method, we may always suppose that the limit in (6.27) is well defined for every $\varphi \in C_0^\infty(\Omega)$. As a result, using (6.26), we have

$$|L(\varphi)| \le C \lim_{\varepsilon\to 0} \left(\int_\Omega |\varphi|^q \,\mathrm{d}\mu_\varepsilon\right)^{1/q} = C\left(\int_\Omega |\varphi|^q \,\mathrm{d}\mu\right)^{1/q}.$$

Hence, $L(\varphi)$ is a continuous functional on $L^q(\Omega, \mathrm{d}\mu)$ and it admits the following representation:

$$L(\varphi) = \int_\Omega v\varphi \,\mathrm{d}\mu,$$

where v is some element of $L^p(\Omega, \mathrm{d}\mu)$. Thus, taking into account Definition 6.14, the element v can be taken as the weak limit of the sequence $\left\{v_{\varepsilon_j} \in L^p(\Omega, \mathrm{d}\mu_{\varepsilon_j})\right\}_{j=1}^\infty$. □

The next property of weak convergence in $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ shows that the variable L^p-norm is lower semicontinuous with respect to the above convergence.

Proposition 6.16. *If the sequence* $\{v_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ *converges weakly to* $v \in L^p(\Omega, \mathrm{d}\mu)$, *then*

$$\liminf_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^p \,\mathrm{d}\mu_\varepsilon \ge \int_\Omega |v|^p \,\mathrm{d}\mu. \tag{6.28}$$

Proof. Indeed, using the Young inequality $ab \le |a|^p/p + |b|^q/q$ and the initial assumptions, we have

$$\frac{1}{p}\int_\Omega |v_\varepsilon|^p \,\mathrm{d}\mu_\varepsilon \ge \int_\Omega v_\varepsilon\varphi \,\mathrm{d}\mu_\varepsilon - \frac{1}{q}\int_\Omega |\varphi|^p \,\mathrm{d}\mu_\varepsilon, \quad \forall\varphi \in C_0^\infty(\Omega),$$

$$\frac{1}{p}\liminf_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^p \,\mathrm{d}\mu_\varepsilon \ge \int_\Omega v\varphi \,\mathrm{d}\mu - \frac{1}{q}\int_\Omega |\varphi|^p \,\mathrm{d}\mu.$$

Since the last inequality is valid for all $\varphi \in C_0^\infty(\Omega)$ and $C_0^\infty(\Omega)$ is a dense subset of $L^q(\Omega, \mathrm{d}\mu)$, it follows that this inequality also holds true for $\varphi \in L^q(\Omega, \mathrm{d}\mu)$. So, taking $\varphi = |v|^{p-2}v$, we come to the required relation (6.28). □

Definition 6.17. *A sequence* $\{v_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ *is said to be strongly convergent to a function* $v \in L^p(\Omega, \mathrm{d}\mu)$ *if*

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon b_\varepsilon \, \mathrm{d}\mu_\varepsilon = \int_\Omega vb \, \mathrm{d}\mu \tag{6.29}$$

whenever $b_\varepsilon \rightharpoonup b$ *in* $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ *as* $\varepsilon \to 0$.

Remark 6.18. Note that in (6.29) the boundedness of the sequence $\{v_\varepsilon\}_{\varepsilon>0} \subset L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ is not required because it is ensured by relation (6.29). Indeed, let us suppose the converse: $\xi_\varepsilon = \|v_\varepsilon\|_{L^p(\Omega,\mathrm{d}\mu_\varepsilon)} \to \infty$ as $\varepsilon \to 0$. Let $b_\varepsilon = \xi_\varepsilon^{-(p-1)} v_\varepsilon^{p-1}$. Then

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon b_\varepsilon \, \mathrm{d}\mu_\varepsilon = \xi_\varepsilon \to \infty \text{ as } \varepsilon \to 0. \tag{6.30}$$

On the other hand, the sequence $\{b_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^q(\Omega, \mathrm{d}\mu)$ with $q = p/(p-1)$ since

$$\int_\Omega \frac{v_\varepsilon^{(p-1)q}}{\|v_\varepsilon\|^{(p-1)q}_{L^p(\Omega,\mathrm{d}\mu_\varepsilon)}} \, \mathrm{d}\mu_\varepsilon = \int_\Omega \frac{v_\varepsilon^{p}}{\|v_\varepsilon\|^{p}_{L^p(\Omega,\mathrm{d}\mu_\varepsilon)}} \, \mathrm{d}\mu_\varepsilon = 1, \ \forall \varepsilon > 0.$$

Therefore, taking into account the compactness criterium, we may assume that $\{b_\varepsilon\}_{\varepsilon>0}$ is weakly convergent in $L^q(\Omega, \mathrm{d}\mu_\varepsilon)$. Then, due to (6.29), the left-hand side of (6.30) must be finite, and we come into conflict with the initial supposition.

We have the following property of the strong convergence in the variable L^p-spaces.

Lemma 6.19. *The weak convergence of the sequence* $\{v_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ *to* $u \in L^p(\Omega, \mathrm{d}\mu)$ *and the equality*

$$\lim_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^p \, \mathrm{d}\mu_\varepsilon = \int_\Omega |v|^p \, \mathrm{d}\mu \tag{6.31}$$

are equivalent to the strong convergence of $\{v_\varepsilon\}_{\varepsilon>0}$ *in* $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ *to* $u \in L^p(\Omega, \mathrm{d}\mu)$.

Proof. We begin with the case $p = 2$. It is easy to verify that the strong convergence implies the weak one and (6.31). Indeed, to do so, we should suppose that $b_\varepsilon = \varphi \in C_0^\infty(\Omega)$ in (6.29) and then substitute $b_\varepsilon = v_\varepsilon$ in this relation. Thus, the problem is to prove the converse statement.

In view of Proposition 6.15, we may suppose that there exist two values α and β such that (up to the subsequences)

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon b_\varepsilon \, \mathrm{d}\mu_\varepsilon = \alpha, \quad \lim_{\varepsilon\to 0} \int_\Omega |b_\varepsilon|^2 \, \mathrm{d}\mu_\varepsilon = \beta.$$

Using the lower semicontinuity property (6.28) and (6.31), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega [v_\varepsilon + tb_\varepsilon]^2 \,\mathrm{d}\mu_\varepsilon &= \lim_{\varepsilon\to 0}\int_\Omega |v_\varepsilon|^2\,\mathrm{d}\mu_\varepsilon + 2t\alpha + t^2\beta\\ &\geq \int_\Omega (v+tb)^2\,\mathrm{d}\mu\\ &= \int_\Omega |v|^2\,\mathrm{d}\mu + 2t\int_\Omega vb\,\mathrm{d}\mu + t^2\int_\Omega |b|^2\,\mathrm{d}\mu.\end{aligned}$$

From this, we conclude that

$$2t\alpha + t^2\beta \geq 2t\int_\Omega vb\,\mathrm{d}\mu + t^2\int_\Omega |b|^2\,\mathrm{d}\mu \quad \forall\, t\in\mathbb{R}^1.$$

Hence, $\alpha = \int_\Omega vb\,\mathrm{d}\mu$. Thereby the strong convergence of $\{v_\varepsilon \in L^2(\Omega,\mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ is established. As for the case $p\neq 2$, it is more technical. For the details, we refer to Zhikov [258]. □

In general, it is false to assert that the strong convergence of v_ε in the variable spaces $L^p(\Omega,\mathrm{d}\mu_\varepsilon)$ to $v\in L^p(\Omega,\mathrm{d}\mu)$ implies

$$\lim_{\varepsilon\to 0}\int_\Omega |v-v_\varepsilon|^p\,\mathrm{d}\mu_\varepsilon = 0.$$

Nevertheless, the following result is valid.

Proposition 6.20. *Assume that*

$$\lim_{\varepsilon\to 0}\mu_\varepsilon(\Omega) = \mu(\Omega) > 0; \tag{6.32}$$

$$v \quad \textit{is a bounded continuous function on } \Omega (v\in C(\overline{\Omega})). \tag{6.33}$$

Then

$$v_\varepsilon \to v \quad \textit{in} \quad L^p(\Omega,\mathrm{d}\mu_\varepsilon) \implies \lim_{\varepsilon\to 0}\int_\Omega |v-v_\varepsilon|^p\,\mathrm{d}\mu_\varepsilon = 0. \tag{6.34}$$

Proof. Note that in view of Definition 6.17 and our initial assumptions (6.32) and (6.33), the sequence $\{g_\varepsilon = v\}_{\varepsilon>0}$ is strongly convergent in $L^p(\Omega,\mathrm{d}\mu_\varepsilon)$ to v. Therefore, $v_\varepsilon - v\to 0$ in $L^p(\Omega,\mathrm{d}\mu_\varepsilon)$. Hence, to obtain the required conclusion (6.34), it remains only to apply Lemma 6.19. □

6.3 Two-scale convergence in variable L^p-spaces

Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $\square = [0;1)^n$ be the semiopen cube in $\mathbb{R}^n$ and $\{\varepsilon\}$ be a sequence of positive numbers converging to 0. In 1989, Nguetseng (see [206]) proved that for each bounded sequence $\{v_\varepsilon\}_{\varepsilon>0}$

in $L^2(\Omega)$, there exists a subsequence, still indexed by ε, and a $v \in L^2(\Omega \times \square)$ such that

$$\int_\Omega v_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x \to \int_\Omega \int_\square v(x,y)\phi(x,y)\,\mathrm{d}y\mathrm{d}x \tag{6.35}$$

for every sufficiently smooth function $\phi(x,y)$ which is $\square$-periodic in y.

In fact, this idea opened a new way for the substantiation of the limit properties of quickly oscillating functions. Following Allaire (see [6]), the convergence defined by (6.35) is called two-scale convergence. Allaire also developed the theory further by studying some general properties of two-scale convergence. Two-scale convergence is now a well-known concept among people who work with homogenization. The concept of two-scale convergence associated with a fixed periodic Borel measure $\mu \in \mathcal{M}(\Omega)$ was introduced by Zhikov in [256]. In the case when $\mathrm{d}\mu = \mathrm{d}x$ is the Lebesgue measure on the cell of periodicity, this convergence coincides with the two-scale convergence in the sense of Allaire. In this section, we describe the main properties of weak and strong two-scale convergence with respect to measures in variable L^p-spaces following the results of Zhikov [258]. These convergence concepts will be used in Chap. 11, where we study the asymptotic behavior of an OCP on singular graphs.

Let μ be a periodic Borel measure on $\mathbb{R}^n$ such that $\int_\square \mathrm{d}\mu = 1$, where $\square = [0,1)^n$ is the cell of periodicity for μ. For any Borel $B \subset \mathbb{R}^n$, we define the scaling measure μ_ε as

$$\mu_\varepsilon(B) = \varepsilon^n \mu(\varepsilon^{-1}B), \quad \varepsilon^{-1}B = \{x/\varepsilon \ : \ \forall\, x \in B\}\,. \tag{6.36}$$

It is clear that μ_ε is a ε-periodic measure on $\mathbb{R}^n$ and that

$$\int_{\varepsilon\square} \mathrm{d}\mu_\varepsilon = \varepsilon^n \int_\square \mathrm{d}\mu = \varepsilon^n \mu(\square) = \varepsilon^n.$$

Hereinafter we suppose that the open bounded domain $\Omega \subset \mathbb{R}^n$ is measurable in the Jordan's sense. It means that its boundary $\partial\Omega$ has a zero n-dimensional Lebesgue measure. For the beginning, we state the following useful mean value property.

Theorem 6.21. *(Mean value property) Let $\phi(x,y) = \psi(x)b(y)$, where ψ is a bounded continuous function on Ω and b is a periodic μ-measurable function on $\mathbb{R}^n$ such that $b \in L^1(\square) = L^1_{per}(\square, \mathrm{d}\mu)$. Then*

$$\lim_{\varepsilon\to 0} \int_\Omega \phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = \int_\Omega \int_\square \phi(x,y)\,\mathrm{d}\mu(y)\mathrm{d}x = \langle b\rangle \int_\Omega \psi\,\mathrm{d}x. \tag{6.37}$$

In particular,

$$\lim_{\varepsilon\to 0} \int_\Omega b\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = |\Omega| \int_\square b(y)\,\mathrm{d}\mu. \tag{6.38}$$

Proof. Assume that $b \geq 0$. Let $\square_{\mathbf{k}} = \square + \mathbf{k}$, where $\mathbf{k}$ is a vector in $\mathbb{R}^n$ with integer components. Then $\{\varepsilon\square_{\mathbf{k}}\}$ is a partition of $\mathbb{R}^n$ and

$$\int_\Omega \psi(x) b\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = \sum \int_{\varepsilon\square_{\mathbf{k}}} \psi(x) b\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon + \sum \int_{\varepsilon\square_{\mathbf{k}}\cap\Omega} \psi(x) b\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon, \tag{6.39}$$

where the first sum is taken over all $\mathbf{k}$ such that $\varepsilon\square_{\mathbf{k}}$ is inside and the second sum is over all $\mathbf{k}$ such that $\varepsilon\square_{\mathbf{k}}$ and $\partial\Omega$ have common points. Let us first consider the first sum in (6.39). Since ψ is continuous and bounded, there exist points $x_{\mathbf{k}} \in \varepsilon\square_{\mathbf{k}}$ such that

$$\int_{\varepsilon\square_{\mathbf{k}}} \psi(x) b\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = \psi(x_{\mathbf{k}}) \int_{\varepsilon\square_{\mathbf{k}}} b\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = \psi(x_{\mathbf{k}})\varepsilon^n \int_\square b(y)\,\mathrm{d}\mu.$$

This together with the fact $\sum \psi(x_{\mathbf{k}})\varepsilon^n \to \int_\Omega \psi(x)\,\mathrm{d}x$ as $\varepsilon \to 0$ implies that

$$\lim_{\varepsilon\to 0} \sum \int_{\varepsilon\square_{\mathbf{k}}} \psi(x) b\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = \int_\Omega \int_\square \psi(x) b(y)\,\mathrm{d}\mu(y)\mathrm{d}x = \langle b\rangle \int_\Omega \psi(x)\,\mathrm{d}x. \tag{6.40}$$

Let us now consider the second sum in (6.39):

$$\left|\sum \int_{\varepsilon\square_{\mathbf{k}}\cap\Omega} \psi(x) b\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon\right| \leq \max_{x\in\Omega} |\psi(x)|\varepsilon^n \int_\square |b(y)|\,\mathrm{d}\mu M(\varepsilon),$$

where $M(\varepsilon)$ is the number of cubes $\varepsilon\square_{\mathbf{k}}$ containing the boundary of Ω. Moreover, $\varepsilon^n M(\varepsilon) \to 0$, since the boundary of Ω has Lebesgue measure 0. Thus,

$$\lim_{\varepsilon\to 0} \sum \int_{\varepsilon\square_{\mathbf{k}}\cap\Omega} \psi(x) b\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = 0. \tag{6.41}$$

As a result, the desired relation (6.37) follows by taking (6.39)–(6.41) into account. The proof for b of arbitrary sign follows by dividing b into positive and negative parts and repeating the arguments above. □

Let D be the set of functions defined as

$$D = \left\{ \phi \;:\; \phi(x,y) = \sum_{\text{finite}} \psi_i(x) b_i(y),\ \psi_i \in C_0^\infty(\Omega),\ b_i \in C_{per}^\infty(\square) \right\}. \tag{6.42}$$

Then Theorem 6.21 implies that

$$\int_\Omega \phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon \longrightarrow \int_\Omega \int_\square \phi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}x \text{ as } \varepsilon \to 0 \tag{6.43}$$

for any $\phi \in D$. It is important to obtain a wide class of functions which satisfy the mean value property (6.37). One such important class is defined below.

Definition 6.22. *Let $\mathfrak{B}$ be the class of functions $\phi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions:*

(i) The function $x \to \phi(x, y)$ is continuous for μ-almost every y.
(ii) The function $y \to \phi(x, y)$ is μ-measurable and $\square$-periodic for every $x \in \Omega$.
(iii) The function $y \to \sup_{x\in\Omega} |\phi(x, y)|$ is in $L^1_{per}(\square, \mathrm{d}\mu)$.

Theorem 6.23. *Let $\phi \in \mathfrak{B}$. Then the limit passage (6.43) holds true.*

For the proof of this theorem, we refer to Zhikov [258] and Lukkassen and Wall [179]. Note also that as follows from the counterexample in [6], property (i) of Definition 6.22 is essential for the validity of Theorem 6.23.

We are now in the position to introduce the two-scale convergence concept in variable L^p-spaces.

Definition 6.24. *We say that a bounded sequence $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}$ weakly two-scale converges to $u \in L^p(\Omega\times\square, \mathrm{d}x\times\mathrm{d}\mu) = L^p(\Omega\times\square)$ (we write $u_\varepsilon \overset{2}{\rightharpoonup} u$) if*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = \int_\Omega \int_\square u(x, y)\phi(x, y)\,\mathrm{d}\mu(y)\,\mathrm{d}x \tag{6.44}$$

for every test function ϕ of the form $\phi(x, y) = \psi(x)\, b(y)$, where $\psi \in C_0^\infty(\Omega)$ and $b \in C^\infty_{per}(\square)$.

Definition 6.25. *A bounded sequence $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}$ is said to be strongly two-scale convergent to a function $u \in L^p(\Omega \times \square, \mathrm{d}x \times \mathrm{d}\mu)$ (we write $u_\varepsilon \overset{2}{\to} u$) if*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon v_\varepsilon \,\mathrm{d}\mu_\varepsilon = \int_\Omega \int_\square u(x, y)v(x, y)\,\mathrm{d}\mu(y)\,\mathrm{d}x \tag{6.45}$$

whenever $v_\varepsilon \overset{2}{\rightharpoonup} v$ in $L^q(\Omega, \mathrm{d}\mu_\varepsilon)$ as $\varepsilon \to 0$.

We note that we have an equivalent definition of two-scale convergence if we replace the set of test functions by D (D defined as in (6.42)). Moreover, since the set D is dense in $L^q(\Omega \times \square)$, it follows that a weak two-scale limit is uniquely defined. As a direct consequence of the definition of weak two-scale convergence, we have that if $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ is a bounded sequence weakly two-scale converging to $u \in L^p(\Omega \times \square, \mathrm{d}x \times \mathrm{d}\mu)$, then $\{u_\varepsilon\}_{\varepsilon>0}$ weakly converges in $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ to $v \in L^p(\Omega)$ in the sense of Definition 6.14, where $v(x) = \int_\square u(x, y)\,\mathrm{d}\mu(y)$. This fact follows by choosing test functions independent of y in (6.44). Moreover, in view of the mean value property, we have

$$\eta(x)\, b(x/\varepsilon) = u_\varepsilon(x) \overset{2}{\rightharpoonup} \eta(x)b(y),$$

provided $\eta \in C(\overline{\Omega})$ and $b \in L^p_{per}(\square, \mathrm{d}\mu)$. Let us study the main properties of such a convergence.

Proposition 6.26. *If $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ is a bounded sequence, then it is compact with respect to the weak two-scale convergence.*

Proof. Let $\phi \in D$ be a fixed test function. By Hölder inequality and the fact that $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ is a bounded sequence, we get that

$$\left|\int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon\right| \le C \left\|\phi\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^q(\Omega, \mathrm{d}\mu_\varepsilon)} \le C\mu_\varepsilon(\Omega)^{1/q}\|\phi\|_{C(\Omega\times\square)}. \tag{6.46}$$

This means that u_ε can be identified with an element u_ε^* in the dual space D^* of D through the formula

$$\langle u_\varepsilon^*, \phi\rangle_{D^*,D} = \int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon.$$

From (6.46), it follows that

$$\|u_\varepsilon^*\|_{D^*} = \sup_{\|\phi\|_{C(\Omega\times\square)}=1} \langle u_\varepsilon^*, u_\varepsilon\rangle_{D^*,D} \le C\mu_\varepsilon(\Omega)^{1/q}.$$

Applying $\limsup_{\varepsilon\to 0}$ on both sides in this inequality and taking Theorem 6.21 into account on the right-hand side, we get $\limsup_{\varepsilon\to 0}\|u_\varepsilon^*\|_{D^*} \le C|\Omega|^{1/q}$. Further, we note that there exists a $u^* \in D^*$ such that

$$\int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = \langle u_\varepsilon^*, \phi\rangle_{D^*,D} \to \langle u^*, \phi\rangle_{D^*,D}, \quad \forall\, \phi \in D. \tag{6.47}$$

To end of the proof, we have to show that there exists an element $u \in L^p(\Omega \times \square, \mathrm{d}x \times \mathrm{d}\mu)$ such that

$$\langle u^*, \phi\rangle_{D^*,D} = \int_\Omega \int_\square u(x, y)\phi(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}x.$$

By (6.46), we have

$$\langle u_\varepsilon^*, \phi\rangle_{D^*,D} \le C \left\|\phi\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^q(\Omega, \mathrm{d}\mu_\varepsilon)}.$$

Then taking (6.47) and Theorem 6.23 into account and passing to the limit in the last inequality as $\varepsilon \to 0$, we get

$$\langle u^*, \phi\rangle_{D^*,D} \le C\|\phi\|_{L^q(\Omega\times\square)}, \quad \forall\, \phi \in D.$$

Since D is dense in $L^q(\Omega\times\square)$, it follows that for a given $\phi \in L^q(\Omega\times\square, \mathrm{d}x\times\mu)$, there exists a sequence $\{\phi_k\}_{k=1}^\infty \subset D$ such that $\phi_k \to \phi$ in $L^q(\Omega \times \square, \mathrm{d}x \times \mu)$. This means that we can define an extension $\widetilde{u}^*$ of u^* to $L^q(\Omega \times \square, \mathrm{d}x \times \mu)$ as

$$\langle \widetilde{u}^*, \phi\rangle_{(L^q(\Omega\times\square))^*, L^q(\Omega\times\square)} = \lim_{k\to\infty} \langle u^*, \phi_k\rangle_{D^*,D}.$$

Then Riesz representation theorem guarantees the existence of an $u \in L^p(\Omega \times \square, \mathrm{d}x \times \mu)$ such that

$$\langle \widetilde{u}^*, \phi \rangle_{L^p(\Omega\times\square),L^q(\Omega\times\square)} = \int_\Omega \int_\square u(x,y)\phi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}x$$

for each $\phi \in L^q(\Omega \times \square, \mathrm{d}x \times \mu)$. In particular, for $\phi \in D$ we have

$$\langle u^*, \phi \rangle_{D^*,D} = \langle \widetilde{u}^*, \phi \rangle_{L^p(\Omega\times\square),L^q(\Omega\times\square)} = \int_\Omega \int_\square u(x,y)\phi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}x$$

and this concludes the proof. □

Proposition 6.27. *If a bounded sequence* $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ *weakly two-scale converges to* $u \in L^p(\Omega \times \square, \mathrm{d}x \times \mathrm{d}\mu)$*, then*

$$\liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(\Omega,\mathrm{d}\mu_\varepsilon)} \geq \|u\|_{L^p(\Omega\times\square)} \geq \|v\|_{L^p(\Omega)},$$

where $v(x) = \int_\square u(x,y)\,\mathrm{d}\mu(y)$.

Proof. Let $\{\phi_k\}_{k=1}^\infty$ be a sequence in D such that $\phi_k \to |u|^{p-2}u$ in $L^q(\Omega \times \square)$. By Young's inequality, we have that $ab \leq |a|^p/p + |b|^q/q$ for any real numbers a and b, where $q = p/(p-1)$. Hence,

$$\int_\Omega |u_\varepsilon|^p\,\mathrm{d}\mu_\varepsilon \geq p\int_\Omega u_\varepsilon \phi_k\left(x, \frac{x}{\varepsilon}\right)\,\mathrm{d}\mu_\varepsilon - (p-1)\int_\Omega \left|\phi_k\left(x,\frac{x}{\varepsilon}\right)\right|^q\,\mathrm{d}\mu_\varepsilon.$$

Passing to the limit in this inequality as $\varepsilon \to 0$ and taking into account our assumptions, we get

$$\begin{aligned}\liminf_{\varepsilon\to 0} \|u_\varepsilon\|^p_{L^p(\Omega,\mathrm{d}\mu_\varepsilon)} &\geq p\int_\Omega \int_\square u(x,y)\phi_k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}x \\ &\quad - (p-1)\int_\Omega \int_\square |\phi_k(x,y)|^q\,\mathrm{d}\mu(y)\,\mathrm{d}x.\end{aligned}$$

Since this relation is valid for each $k \in \mathbb{N}$, we can pass to the limit as k tends to $+\infty$. As a result, we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} \|u_\varepsilon\|^p_{L^p(\Omega,\mathrm{d}\mu_\varepsilon)} &\geq p\int_\Omega \int_\square |u(x,y)|^p\,\mathrm{d}\mu(y)\,\mathrm{d}x \\ &\quad - (p-1)\int_\Omega \int_\square |u(x,y)|^p\,\mathrm{d}\mu(y)\,\mathrm{d}x = \|u\|^p_{L^p(\Omega\times\square)}.\end{aligned}$$

Finally, by Jensen's inequality, we obtain the required result

$$\begin{aligned}\|v\|^p_{L^p(\Omega)} &= \int_\Omega \left|\int_\square u(x,y)\,\mathrm{d}\mu(y)\right|^p\,\mathrm{d}x \\ &\leq \int_\Omega \int_\square |u(x,y)|^p\,\mathrm{d}\mu(y)\,\mathrm{d}x = \|u\|^p_{L^p(\Omega\times\square)}.\end{aligned}$$

□

Proposition 6.28. *Weak two-scale convergence of* $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ *to* $u \in L^p(\Omega \times \square, \mathrm{d}x \times \mathrm{d}\mu)$ *together with the relation*

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^p \,\mathrm{d}\mu_\varepsilon = \int_\Omega \int_\square |u(x,y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}x \tag{6.48}$$

is equivalent to strong two-scale convergence of $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}$ *to* u.

Proof. (i) We start by proving that weak two-scale convergence together with (6.48) imply strong two-scale convergence. Let $\{\phi_k\}_{k=1}^\infty$ be a sequence in D such that $\phi_k \to u$ in $L^p(\Omega \times \square)$. Let $\{v_\varepsilon \in L^q(\Omega, \mathrm{d}\mu_\varepsilon)\}$ be a weakly two-scale convergent sequence to a function $v \in L^q(\Omega \times \square, \mathrm{d}x \times \mathrm{d}\mu)$. Then

$$\lim_{k\to\infty} \lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon(x)\phi_k\left(x, \frac{x}{\varepsilon}\right) \,\mathrm{d}\mu_\varepsilon = \int_\Omega \int_\square v(x,y)u(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}x. \tag{6.49}$$

We also have that

$$\begin{aligned}
&\left| \int_\Omega v_\varepsilon(x)u_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon - \int_\Omega \int_\square v(x,y)u(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}x \right| \\
&\quad \le \left| \int_\Omega \left[u_\varepsilon(x) - \phi_k\left(x, \frac{x}{\varepsilon}\right)\right] v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon \right| \\
&\quad + \left| \int_\Omega \phi_k\left(x, \frac{x}{\varepsilon}\right) v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon - \int_\Omega \int_\square v(x,y)u(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}x \right|.
\end{aligned} \tag{6.50}$$

From (6.49) and (6.50), we can obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \left| \int_\Omega v_\varepsilon(x)u_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon - \int_\Omega \int_\square v(x,y)u(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}x \right| \\
&\qquad \le \limsup_{k\to\infty} \limsup_{\varepsilon\to 0} \left| \int_\Omega \left[u_\varepsilon(x) - \phi_k\left(x, \frac{x}{\varepsilon}\right)\right] v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon \right|.
\end{aligned} \tag{6.51}$$

Hence, in order to prove (6.45), we have to show that

$$\limsup_{k\to\infty} \limsup_{\varepsilon\to 0} \left| \int_\Omega \left[u_\varepsilon(x) - \phi_k\left(x, \frac{x}{\varepsilon}\right)\right] v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon \right| = 0. \tag{6.52}$$

To do so, we note that Hölder inequality and the fact that each weakly two-scale convergent sequence is bounded imply

$$\begin{aligned}
&\left| \int_\Omega \left[u_\varepsilon(x) - \phi_k\left(x, \frac{x}{\varepsilon}\right)\right] v_\varepsilon(x)\,\mathrm{d}\mu_\varepsilon \right| \\
&\qquad \le \left\| u_\varepsilon(\cdot) - \phi_k\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{L^p(\Omega, \mathrm{d}\mu_\varepsilon)} \|v_\varepsilon(x)\|_{L^q(\Omega, \mathrm{d}\mu_\varepsilon)} \\
&\qquad \le C \left\| u_\varepsilon(\cdot) - \phi_k\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{L^p(\Omega, \mathrm{d}\mu_\varepsilon)}.
\end{aligned} \tag{6.53}$$

Then using the Clarkson inequalities for $p \geq 2$ and for $p \leq 2$, respectively, we have

$$\left\|u_\varepsilon(\cdot) - \phi_k\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|^p \leq 2^p \left[\frac{1}{2}\|u_\varepsilon\|^p + \frac{1}{2}\left\|\phi_k\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|^p - \left\|\frac{u_\varepsilon(\cdot) + \phi_k\left(\cdot, \frac{\cdot}{\varepsilon}\right)}{2}\right\|^p\right],$$

$$\left\|u_\varepsilon(\cdot) - \phi_k\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|^q \leq 2^{p-1} \left[\left(\frac{1}{2}\|u_\varepsilon\|^p + \frac{1}{2}\left\|\phi_k\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|^p\right)^{1/(p-1)} - \left\|\frac{u_\varepsilon(\cdot) + \phi_k\left(\cdot, \frac{\cdot}{\varepsilon}\right)}{2}\right\|^q\right],$$

where all norms are the usual norm in $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$. By (6.48) and an application of the mean value property to the sequence $\left\{u_\varepsilon(x) + \phi_k\left(x, \frac{x}{\varepsilon}\right)\right\}$ (which weakly two-scale converges to $u + \phi_k \in L^p(\Omega \times \square)$, it follows for $p \geq 2$ that

$$\limsup_{\varepsilon \to 0} \left\|u_\varepsilon(\cdot) - \phi_k\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|^p \leq 2^p \left[\frac{1}{2}\|u\|^p_{L^p(\Omega\times\square)} + \frac{1}{2}\|\phi_k\|^p_{L^p(\Omega\times\square)} - \left\|\frac{u + \phi_k}{2}\right\|^p_{L^p(\Omega\times\square)}\right] \tag{6.54}$$

and, for $1 \leq p \leq 2$, that

$$\limsup_{\varepsilon \to 0} \left\|u_\varepsilon(\cdot) - \phi_k\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|^q \leq 2^{p-1} \left[\left(\frac{1}{2}\|u\|^p + \frac{1}{2}\|\phi_k\|^p\right)^{1/(p-1)} - \left\|\frac{u + \phi_k}{2}\right\|^q\right]. \tag{6.55}$$

In view of the choice of ϕ_k, we have that $\{(u + \phi_k)/2\}$ converges to u in $L^p(\Omega \times \square)$. By passing to lim sup as $k \to \infty$ on both sides of (6.54)) and (6.55), we come to the conclusion that

$$\limsup_{k\to\infty} \limsup_{\varepsilon \to 0} \left|\int_\Omega \left[u_\varepsilon(x) - \phi_k\left(x, \frac{x}{\varepsilon}\right)\right] v_\varepsilon(x)\, \mathrm{d}\mu_\varepsilon\right| = 0. \tag{6.56}$$

Thus, the required relation (6.45) follows from (6.53) and (6.56).

(ii) It remains to prove that strong two-scale convergence implies a weak two-scale convergence and relation (6.48). Assume that $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ strongly two-scale converges to $u \in L^p(\Omega \times \square, \mathrm{d}x \times \mathrm{d}\mu)$. By choosing $v_\varepsilon(x) = \psi(x)b(x/\varepsilon)$, where $\psi \in C_0^\infty(\Omega)$ and $b \in C^\infty_{per}(\square)$ in (6.45), we see that strong two-scale convergence implies a weak two-scale convergence. Now, we will show that strong two-scale convergence implies (6.48). In fact, let $v_\varepsilon = |u_\varepsilon|^{p-1}u_\varepsilon$. By the initial assumptions, the sequence $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded, and hence $\{v_\varepsilon \in L^q(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ is a bounded sequence as well. By Proposition 6.26, there exists a subsequence of $\{v_\varepsilon \in L^q(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ (still

denoted by ε) which weakly two-scale converges to some $v \in L^q(\Omega \times \square)$. According to Definition 6.25, we have

$$\int_\Omega |u_\varepsilon|^p \, \mathrm{d}\mu_\varepsilon = \int_\Omega u_\varepsilon v_\varepsilon \, \mathrm{d}\mu_\varepsilon \to \int_\Omega \int_\square u(x,y)v(x,y) \, \mathrm{d}\mu(y) \, \mathrm{d}x.$$

From this we see that we are done if we show that $v = |u|^{p-2}u$. In fact, the function $f(t) = |t|^{p-2}t$ is monotone, which means that $[f(t_1)-f(t_2)](t_1-t_2) \geq 0$ for every $t_1, t_2 \in \mathbb{R}^n$. Let $\phi(x,y) = \sum_{\text{finite}} \psi_k(x) b_k(y)$, where $\psi_k \in C_0^\infty(\Omega)$ and $b_k \in C_{per}^\infty(\square)$. Then

$$\int_\Omega \left[f(\phi(x, x/\varepsilon)) - f(u_\varepsilon(x)) \right] \left(\phi(x, x/\varepsilon) - u_\varepsilon(x) \right) \mathrm{d}\mu_\varepsilon \geq 0.$$

Hence, in the limit we get

$$\int_\Omega \int_\square \left[f(\phi(x,y)) - v(x,y) \right] \left(\phi(x,y) - u(x,y) \right) \mathrm{d}\mu(y) \, \mathrm{d}x \geq 0.$$

By density and continuity, this inequality holds for any $\phi \in L^p(\Omega \times \square)$. Having put $\phi = u + tw$, where $w \in L^p(\Omega \times \square)$, for $t > 0$ we get

$$\int_\Omega \int_\square \left[|u + tw|^{p-2}(u + tw) - v \right] w \, \mathrm{d}\mu(y) \, \mathrm{d}x \geq 0. \tag{6.57}$$

Whereas for $t < 0$, we obtain

$$\int_\Omega \int_\square \left[|u + tw|^{p-2}(u + tw) - v \right] w \, \mathrm{d}\mu(y) \, \mathrm{d}x \leq 0. \tag{6.58}$$

By taking (6.57) and (6.58) into account and letting t tend to 0, we find that

$$\int_\Omega \int_\square \left[|u(x,y)|^{p-2}(u(x,y) - v(x,y)) \right] w(x,y) \, \mathrm{d}\mu(y) \, \mathrm{d}x = 0, \ \forall \, w \in L^p(\Omega \times \square).$$

Thus, $v = |u|^{p-2}u$ and this concludes the proof. □

To extend the class of test functions D in Definition 6.24, we give the following motivation. Let $\mathcal{A}_q$ be the set of functions $\phi \in \mathfrak{B}$ ($\mathfrak{B}$ defined as in Definition 6.22) such that the function $y \to \sup_{x \in \Omega} |\phi(x,y)|$ is in $L^q_{per}(\square, \mathrm{d}\mu)$. It is easy to see that $\mathcal{A}_q \subset L^q(\Omega \times \square)$ and that if $\phi \in \mathcal{A}_q$, then $\phi(x,y)\psi(x)b(y) \in \mathcal{A}_q \subset \mathfrak{B}$ for any $\psi \in C_0^\infty(\Omega)$ and any $b \in C_{per}^\infty(\square)$. Moreover, if $\phi \in \mathcal{A}_q$, then $|\phi| \in \mathcal{A}_q$ and, thus, $|\phi|^q \in \mathfrak{B}$. As a result, we have the following property.

Lemma 6.29. *If $\phi \in \mathcal{A}_q$, then*

$$\phi\left(x, \frac{x}{\varepsilon}\right) \overset{2}{\to} \phi(x,y) \quad \textit{as} \quad \varepsilon \to 0. \tag{6.59}$$

Proof. Indeed, in view of the definition of the class $\mathcal{A}$ and property (6.38), we see that the sequence $\left\{\phi\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\}_{\varepsilon>0}$ as a sequence in the variable space $L^q(\Omega, \mathrm{d}\mu_\varepsilon)$ is bounded. Then, by Theorem 6.23, we have

$$\lim_{\varepsilon\to 0}\int_\Omega \phi\left(x, \frac{x}{\varepsilon}\right)\psi(x)b\left(\frac{x}{\varepsilon}\right)\,\mathrm{d}\mu_\varepsilon = \int_\Omega\int_\square \phi(x,y)\psi(x)b(y)\,\mathrm{d}\mu(y)\,\mathrm{d}x,$$
$$\forall\,\psi\in C_0^\infty(\Omega),\quad \forall\, b\in C_{per}^\infty(\square),$$
$$\lim_{\varepsilon\to 0}\int_\Omega \left|\phi\left(x, \frac{x}{\varepsilon}\right)\right|^q\,\mathrm{d}\mu_\varepsilon = \int_\Omega\int_\square |\phi(x,y)|^q\,\mathrm{d}\mu(y)\,\mathrm{d}x.$$

The first of this equalities means the weak two-scale convergence $\phi\left(x, \frac{x}{\varepsilon}\right) \overset{2}{\rightharpoonup} \phi(x,y)$. As for the second one, it gives (6.48). Hence, by Proposition 6.28, we just come to the desired conclusion. □

As an obvious consequence of this lemma and Proposition 6.28, we have the following result.

Lemma 6.30. *If a bounded sequence $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ weakly two-scale converges to $u \in L^p(\Omega\times\square, \mathrm{d}x\times\mathrm{d}\mu)$, then*

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right)\,\mathrm{d}\mu_\varepsilon = \int_\Omega\int_\square u(x,y)\phi(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}x,\quad \forall\,\phi\in\mathcal{A}_q.$$

Under some additional conditions, strong two-scale convergence can be expressed in a more convenient form.

Lemma 6.31. *Let $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ be a bounded sequence which strongly two-scale converges to $u \in L^p(\Omega\times\square, \mathrm{d}x\times\mathrm{d}\mu)$. If the limit function u belongs to the class $\mathcal{A}_p$, then*

$$\int_\Omega \left|u_\varepsilon(x) - u\left(x, \frac{x}{\varepsilon}\right)\right|^p\,\mathrm{d}\mu_\varepsilon = 0. \tag{6.60}$$

Proof. We have

$$u_\varepsilon \overset{2}{\to} u(x,y)\quad \text{by the initial assumptions,}$$
$$u\left(x, \frac{x}{\varepsilon}\right) \overset{2}{\to} u(x,y)\quad \text{by Lemma 6.29.}$$

Hence, $u_\varepsilon(x) - u\left(x, \frac{x}{\varepsilon}\right) \overset{2}{\to} 0$. Therefore, (6.60) follows by taking Proposition 6.28 into account. □

To end this section, we compare the usual strong convergence in the sense of Definition 6.17 and strong two-scale convergence.

Proposition 6.32. *If the sequence $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ converges strongly to $\overline{u} \in L^p(\Omega, \mathrm{d}\mu)$, then $u_\varepsilon \overset{2}{\to} \overline{u}$, that is, in this case, the strong two-scale limit is independent of y.*

Proof. Let $u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x,y)$. Then, in view of our assumptions, we have that $L^p(\Omega,\mathrm{d}\mu) \ni \overline{u} = \int_\square u(x,y)\,\mathrm{d}\mu$ is the strong limit of $\{u_\varepsilon \in L^p(\Omega,\mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ in the sense of Definition 6.17. Therefore, due to Lemma 6.19 and Proposition 6.27, we have

$$\int_\Omega |\overline{u}|^p\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon|^p\,\mathrm{d}\mu_\varepsilon \geq \int_\Omega\int_\square |u(x,y)|^p\,\mathrm{d}\mu(y)\,\mathrm{d}x \geq \int_\Omega |\overline{u}|^p\,\mathrm{d}x. \tag{6.61}$$

We clearly have the strict equality here; hence, $u(x,y) = \int_\square u(x,y)\,\mathrm{d}\mu$. So, $u = \overline{u}$ and to obtain the strong two-scale convergence $u_\varepsilon \overset{2}{\to} \overline{u}$, it remains only to take (6.61) and Proposition 6.28 into account. This concludes the proof. □

6.4 Variable Sobolev spaces and two-scale convergence

In this section, we present some definitions and additional facts concerning variable Sobolev spaces and two-scale convergence in such spaces. For details, we refer to Zhikov [256, 257], Bouchitté and Fragala [26] and Lukkassen and Wall [179]. As will be shown in the second part of this book, these notions are essential for the asymptotic analysis of OCPs for PDEs on reticulated domains.

Let Ω be an open domain in $\mathbb{R}^n$ and let μ be a periodic Borel (e.g., probability) measure of $\mathcal{M}_0^p(\Omega)$ such that $\int_\square \mathrm{d}\mu = 1$, where $\square = [0,1)^n$ is the cell of periodicity for μ. Let $W^{1,p}_{per}(\square,\mathrm{d}\mu)$ be the periodic Sobolev space with respect to the measure μ. Let us recall that (see Definition 2.48) the space $W^{1,p}_{per}(\square,\mathrm{d}\mu)$ is defined as the collection of the first component u of the closure in $L^p(\square,\mathrm{d}\mu)\times \mathbf{L}^p(\square,\mathrm{d}\mu)$ of the set $\big\{(u,\nabla u)\ :\ \forall\, u\in C^\infty(\overline{\square})\big\}$. We denote this closure by $W^p = W^p(\square,\mathrm{d}\mu)$. Thus, the elements of W^p are pairs $(u,\mathbf{z})$, where we denote the vector $\mathbf{z}$ by $\nabla^\mu u$ and call it a gradient or μ-gradient of u. As Example 2.50 indicates, the gradient of the $W^{1,p}_{per}(\square,\mathrm{d}\mu)$ function is not unique, in general. We recall that $\Gamma_p^\mu(u)$ denotes the set of all gradients of a fixed function $u\in W^{1,p}_{per}(\square,\mathrm{d}\mu)$, and $\Gamma_p^\mu(u)$ has the structure $\Gamma_p^\mu(u) = \nabla^\mu u + \Gamma_p^\mu(0)$, where $\nabla^\mu u$ is some gradient and $\Gamma_p^\mu(0)$ is the set of gradients of 0. In what follows, for the case $p=2$ we will use the notation $W^{1,2}_{per}(\square,\mathrm{d}\mu) = H^1_{per}(\square,\mathrm{d}\mu)$.

6.4.1 *p*-Connected measures and their properties

As we will see later, the following ergodicity property is important (see [254, 256]).

Definition 6.33. *A measure μ is said to be p-connected (or ergodic on the torus $\square$) if each function $u \in W^{1,p}_{per}(\square, \mathrm{d}\mu)$ with 0 gradient ($0 \in \Gamma^{\mu}_{p}(u)$) is constant μ-almost everywhere. In other words, if there exists a sequence $\{u_k \in C_{per}(\square)\}_{k=1}^{\infty}$ such that*

$$\int_{\square} |u_k - u|^p \, \mathrm{d}\mu \to 0 \quad \text{and} \quad \int_{\square} |\nabla u_k|^p \, \mathrm{d}\mu \to 0 \ \text{as} \ \ k \to \infty, \tag{6.62}$$

then $u \equiv \mathrm{const}$ μ-everywhere in $\square$.

To illustrate this property, we give the following two examples.

Example 6.34. Assume that a measure μ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^n$ (i.e., $\mathrm{d}\mu = w(x)\,\mathrm{d}x$, where $w \in L^1(\square)$). We show that this measure is p-connected provided $\int_{\square} w^{-1/(p-1)} \, \mathrm{d}x < \infty$. Indeed, let $\{u_k \in C_{per}(\square)\}_{k=1}^{\infty}$ be a sequence with properties (6.62). Then

$$\int_{\square} |\nabla u_k|^p \, \mathrm{d}x \le \left(\int_{\square} w^{-1/(p-1)} \, \mathrm{d}x \right)^{(p-1)/p} \left(\int_{\square} |\nabla u_k|^p w \, \mathrm{d}x \right)^{1/p} \to 0. \tag{6.63}$$

By analogy, it can be shown that $\int_{\square} |u_k - u|^p \, \mathrm{d}x \to 0$. Thus, u belongs to the usual Sobolev space $W^{1,1}(\square)$ and from (6.63), it follows that u is constant a.e. with respect to the Lebesgue measure. Hence, $u \equiv \mathrm{const}$ μ-everywhere in $\square$.

Example 6.35. Let Q be an open, connected in the usual sense set on the torus $\square$. Let χ_Q be its characteristic function. We define the periodic Borel measure μ as

$$\mathrm{d}\mu = \frac{1}{|Q \cup \square|} \chi_Q(x) \, \mathrm{d}x.$$

It is easy to see that in this case, $\Gamma^{\mu}_{p}(0) = \{0\}$. Moreover, having taken an arbitrary sequence $\{u_k \in C_{per}(\square)\}_{k=1}^{\infty}$ with properties (6.62), we have u is constant a.e. on Q with respect to the Lebesgue measure. Since Q is connected set in the usual sense, it follows that $u \equiv \mathrm{const}$ μ-everywhere in $\square$.

Definition 6.36. *We say that $a \in L^1(\square, \mathrm{d}\mu)$ is the divergence (with respect to the measure μ) of a vector function $\mathbf{b} \in \left[L^1(\square, \mathrm{d}\mu)\right]^n = \mathbf{L}^1(\square, \mathrm{d}\mu)$ in the weak sense and use the notation $a = \mathrm{div}^{\mu} \mathbf{b}$ if the following identity holds:*

$$\int_{\square} a\varphi \, \mathrm{d}\mu = -\int_{\square} \mathbf{b} \cdot \nabla\varphi \, \mathrm{d}\mu, \quad \forall\, \varphi \in C^{\infty}_{per}(\square). \tag{6.64}$$

Note that in view of the periodicity of the measure μ, (6.64) can be equivalently rewritten as

$$\int_{\mathbb{R}^n} a\varphi \, \mathrm{d}\mu = -\int_{\mathbb{R}^n} \mathbf{b} \cdot \nabla\varphi \, \mathrm{d}\mu, \quad \forall\, \varphi \in C^{\infty}_{0}(\mathbb{R}^n). \tag{6.65}$$

Then it is easy to note that each function $a \in L^1(\square, \mathrm{d}\mu)$ admitting the representation $a = \operatorname{div}^\mu \mathbf{b}$ has the mean value 0 (i.e., $\int_\square a \,\mathrm{d}\mu = 0$).

As an important property following from ergodicity of the measure μ we have the following result.

Theorem 6.37. *If the measure μ is p-connected, then the set of functions $a \in L^q(\square, \mathrm{d}\mu)$ which can be represented as $a = \operatorname{div}^\mu \mathbf{b}$, where $\mathbf{b} \in \mathbf{L}^q(\square, \mathrm{d}\mu)$, is dense in the subspace of functions in $L^q(\square, \mathrm{d}\mu)$ with mean value 0. Here, $1/p + 1/q = 1$.*

Proof. Let us denote by A the set of functions $a \in L^q(\square, \mathrm{d}\mu)$ which can be represented as $a = \operatorname{div}^\mu \mathbf{b}$ with $\mathbf{b} \in \mathbf{L}^q(\square, \mathrm{d}\mu)$. Define the annihilator $A^\perp$ as the set of functions $f \in L^p(\square, \mathrm{d}\mu)$ such that

$$\int_\square f \,\mathrm{d}\mu = 0 \quad \text{and} \quad \int_\square fa \,\mathrm{d}\mu = 0, \quad \forall\, a \in A.$$

As follows from Theorem 4.7 in [221], in order to prove that A is dense in $L^q(\square, \mathrm{d}\mu)$, it is enough to derive the equality $A^\perp = \{0\}$. For this, let us fix $f \in A^\perp$ and consider the periodic problem: Find a pair $(u, \nabla^\mu u)$ such that

$$u \in W^{1,p}_{per}(\square, \mathrm{d}\mu), \quad \nabla^\mu u \in \Gamma^\mu_p(u), \tag{6.66}$$

$$\int_\square \left(|\nabla^\mu u|^{p-2} \nabla^\mu u \cdot \nabla\varphi + |u|^{p-2} u\varphi \right) \mathrm{d}\mu = \int_\square |f|^{p-2} f\varphi \,\mathrm{d}\mu, \quad \forall\, \varphi \in C^\infty_{per}(\square). \tag{6.67}$$

First, we note that this problem is known to be solvable. Then it is obviously that $\left(|f|^{p-2} f - |u|^{p-2} u \right) \in A$, and since $f \in A^\perp$, we have

$$\int_\square \left(|f|^p - |u|^{p-2} fu \right) \mathrm{d}\mu = 0.$$

Hölder inequality implies that

$$\int_\square |f|^p \,\mathrm{d}\mu = \int_\square |u|^{p-2} fu \,\mathrm{d}\mu \le \left(\int_\square |u|^{(p-1)q} \,\mathrm{d}\mu \right)^{1/q} \left(\int_\square |f|^p \,\mathrm{d}\mu \right)^{1/p},$$

that is,

$$\int_\square |f|^p \,\mathrm{d}\mu \le \int_\square |u|^p \,\mathrm{d}\mu. \tag{6.68}$$

By the density of $C^\infty_{per}(\square)$ in $W^{1,p}_{per}(\square, \mathrm{d}\mu)$, we may choose $\varphi = u$ in (6.67). As a result, we obtain

$$\int_\square \left(|\nabla^\mu u|^p + |u|^p \right) \mathrm{d}\mu = \int_\square |f|^{p-2} fu \,\mathrm{d}\mu \le \left(\int_\square |f|^p \,\mathrm{d}\mu \right)^{1/q} \left(\int_\square |u|^p \,\mathrm{d}\mu \right)^{1/p},$$

and hence

$$\int_{\square} |u|^p \, \mathrm{d}\mu \le \int_{\square} |f|^p \, \mathrm{d}\mu \quad \text{and} \quad \int_{\square} (|\nabla^\mu u|^p + |u|^p) \, \mathrm{d}\mu \le \int_{\square} |f|^p \, \mathrm{d}\mu.$$

This together with (6.68) gives

$$\int_{\square} |u|^p \, \mathrm{d}\mu = \int_{\square} |f|^p \, \mathrm{d}\mu \quad \text{and} \quad \int_{\square} |\nabla^\mu u|^p \, \mathrm{d}\mu = 0.$$

By the p-connectedness of μ, it follows that u is constant μ-almost everywhere. This fact together with (6.67) implies that f is constant μ-almost everywhere. Since $\int_{\square} f \, \mathrm{d}\mu = 0$, we deduce that $f = 0$. The proof is complete. □

For other properties of p-connected measures, we refer to [179, 256].

6.4.2 Degenerate measures

By analogy to Sect. 2.3, we define two spaces: the space V^p_{pot} of potential vectors as the closure of the set $\{\nabla\varphi \ : \ \varphi \in C^\infty_{per}(\square)\}$ in $\mathbf{L}^p(\square, \mathrm{d}\mu)$, and the space V^q_{sol} of solenoidal vectors as the collection of vectors $\mathbf{b} \in \mathbf{L}^q(\square, \mathrm{d}\mu)$ such that

$$\int_{\square} \mathbf{b} \cdot \nabla\varphi \, \mathrm{d}\mu = 0, \quad \forall\, \varphi \in C^\infty_{per}(\square).$$

It is clear that $\mathbf{b} \in \left(V^p_{pot}\right)^\perp$ whenever $\mathbf{b} \in \mathbf{L}^q(\square, \mathrm{d}\mu)$ is a solenoidal vector. Since every norm-closed subspace of $\mathbf{L}^p(\square, \mathrm{d}\mu)$ is an annihilator of its annihilator (see [221, p. 96]), we have

$$V^q_{sol} = \left(V^p_{pot}\right)^\perp, \quad V^p_{pot} = \left(V^q_{sol}\right)^\perp. \tag{6.69}$$

In particular, for the Hilbert space case, $p = 2$, we have the orthogonal decomposition (see Sect. 2.3)

$$\mathbf{L}^2(\square, \mathrm{d}\mu) = V^2_{sol} \oplus V^2_{sol}.$$

Let $\mathbf{L}^p(\Omega, V^p_{pot})$ be defined as the closure in $L^p(\Omega \times \square)$ of the linear span of vectors $f(x)\nabla\varphi(y)$, where $f \in C^\infty_0(\Omega)$ and $\varphi \in C^\infty_{per}(\square)$. Correspondingly, $\mathbf{L}^q(\Omega, V^q_{sol})$ is defined as the closure in $\mathbf{L}^q(\Omega \times \square)$ of the linear span of vectors $f(x)\mathbf{b}(y)$, where $f \in C^\infty_0(\Omega)$ and $\mathbf{b} \in V^q_{sol}$. By (6.69), it follows that

$$\mathbf{L}^q(\Omega, V^q_{sol})^\perp = \mathbf{L}^p(\Omega, V^p_{pot}), \quad \mathbf{L}^p(\Omega, V^p_{pot})^\perp = \mathbf{L}^q(\Omega, V^q_{sol}). \tag{6.70}$$

Note that for some measures, a nonzero constant vector may be a potential vector (see the example given below). Let E be the subspace of $\mathbb{R}^n$ of such potential vectors and $E^\perp$ its orthogonal complement. We say that the measure μ is nondegenerate if $E = \{0\}$.

Example 6.38. Let $n = 2$ and let μ be the probability $\square$-periodic measures on $\mathbb{R}^2$, concentrated on the interval $I = [0,1) \times \{1/2\}$ and proportional there to 1D Lebesgue measure. Let $g = g(x_2)$ be a smooth 1-periodic function such that $g(1/2) = 0$ and $g'(1/2) = 1$. For every $k \in \mathbb{N}$, we define the test functions $\varphi_k \in C^\infty_{per}(\square)$ as $\varphi_k(x_1, x_2) = \nu g(x_2)$. Here, $\nu = \text{const}$. Then $\nabla\varphi_k \in \mathbf{L}^p(\square)$ $\forall\, k \in \mathbb{N}$ and

$$\int_\square \left|\frac{\partial\varphi_k}{\partial x_1}\right|^p \mathrm{d}\mu = \int_0^1 \left|\frac{\partial\varphi_k(x_1, 1/2)}{\partial x_1}\right|^p \mathrm{d}x_1 = 0, \quad \forall\, k \in \mathbb{N},$$
$$\int_\square \left|\frac{\partial\varphi_k}{\partial x_2} - \nu\right|^p \mathrm{d}\mu = \int_0^1 \left|\frac{\partial\varphi_k(x_1, 1/2)}{\partial x_2} - \nu\right|^p \mathrm{d}x_1 = 0, \quad \forall\, k \in \mathbb{N}.$$

Hence, the constant vector $(0, \nu) \in V^p_{pot}$. Thus, μ is an example of a degenerate Borel measure.

Now, we formulate some results following from nondegeneration of the measure μ (their proof can be found in [254]).

Theorem 6.39. *Let $m : \mathbb{R}^n \to \mathbb{R}$ be the functional given by*

$$m(\xi) = \min_{\mathbf{v} \in V^p_{pot}} \int_\square |\xi + \mathbf{v}|^p \,\mathrm{d}\mu. \tag{6.71}$$

Then the following assertions hold true:

(i) m is a convex functional.
(ii) $m(\xi + \eta) = m(\xi)$ for any $\xi \in \mathbb{R}^n$ and $\eta \in E$.
(iii) There exists a constant $C > 0$ such that $m(\xi) \geq C|\xi|^p$ for $\xi \in E^\perp$ (i.e., m is coercive on $E^\perp$).

Theorem 6.40. *A vector $\mathbf{b}$ belongs to V^q_{sol} if and only if its mean value belongs to $E^\perp$.*

6.4.3 Two-scale convergence in variable Sobolev spaces

Let $\{u_\varepsilon \in C^\infty_0(\Omega)\}_{\varepsilon>0}$ be a sequence of smooth functions such that

$$\{u_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in} \quad L^p(\Omega, \mathrm{d}\mu_\varepsilon), \tag{6.72}$$
$$\{\nabla u_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in} \quad \mathbf{L}^p(\Omega, \mathrm{d}\mu_\varepsilon), \tag{6.73}$$
$$u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x, y), \quad \nabla u_\varepsilon \overset{2}{\rightharpoonup} \mathbf{p}\,(x, y). \tag{6.74}$$

We are going to discuss the following questions: When is the two-scale limit $u(x, y)$ independent of y? Is $u(x, y) = u(x)$ in the Sobolev space $W^{1,p}(\Omega)$? What is the relation between $\nabla^\mu u$ and $\mathbf{p}\,(x, y)$? As we will see later, the p-connectedness of the measure μ is essential for the answers to these questions.

Theorem 6.41. *Let μ be a p-connected nondegenerate measure and let*

$$\{u_\varepsilon \in C_0^\infty(\Omega)\}_{\varepsilon>0}$$

be a sequence with properties (6.72)–(6.74). Then the weak two-scale limit u is independent of y and belongs to the Sobolev space $W_0^{1,p}(\Omega)$. Moreover, in this case,

$$\mathbf{p}(x,y) = \nabla^\mu u(x) + \mathbf{v}(x,y), \quad \text{where } \mathbf{v} \in \mathbf{L}^p(\Omega, V_{pot}^p). \tag{6.75}$$

Proof. Let functions $\mathbf{b} \in \mathbf{L}_{per}^q(\square, \mathrm{d}\mu)$ and $a \in L_{per}^q(\square, \mathrm{d}\mu)$ be such that $a = \operatorname{div}^\mu \mathbf{b}$. Then (6.65) implies that

$$\varepsilon \int_\Omega \nabla\varphi(x) \cdot \mathbf{b}\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = -\int_\Omega \varphi(x) a\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon, \quad \varphi \in C_0^\infty(\Omega). \tag{6.76}$$

On the other hand, for $\psi \in C_0^\infty(\Omega)$, the partial differentiation gives

$$\begin{aligned}\varepsilon \int_\Omega \nabla\left(\psi(x) u_\varepsilon(x)\right) \cdot \mathbf{b}\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon \\ = \varepsilon \int_\Omega u_\varepsilon \nabla\psi(x) \cdot \mathbf{b}\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon + \varepsilon \int_\Omega \psi(x) \nabla u_\varepsilon(x) \cdot \mathbf{b}\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon.\end{aligned}$$

This together with (6.76) means that

$$\begin{aligned}-\int_\Omega \psi(x) u_\varepsilon(x) a\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon \\ = \varepsilon \int_\Omega u_\varepsilon \nabla\psi(x) \cdot \mathbf{b}\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon + \varepsilon \int_\Omega \psi(x) \nabla u_\varepsilon(x) \cdot \mathbf{b}\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon.\end{aligned}$$

By our assumptions, the right-hand side tends to 0 as $\varepsilon \to 0$ since $\{u_\varepsilon\}_{\varepsilon>0}$ and $\{\nabla u_\varepsilon\}_{\varepsilon>0}$ are weakly two-scale convergent sequences. Thus, by passing to the limit, we obtain

$$\int_\Omega \int_\square \psi(x) u(x,y) a(y) \,\mathrm{d}\mu(y)\,\mathrm{d}x = 0. \tag{6.77}$$

We note that $\displaystyle\int_\square a(y)\,\mathrm{d}\mu = 0$ and by Theorem 6.37 the set of functions $a \in L^q(\square, \mathrm{d}\mu)$ which can be represented as $a = \operatorname{div}^\mu \mathbf{b}$, where $\mathbf{b} \in \mathbf{L}^q(\square, \mathrm{d}\mu)$, is dense in the subspace of functions in $L^q(\square, \mathrm{d}\mu)$ with mean value 0. Hence, in view of (6.77), we just conclude that u is independent of y.

Next, we prove that $u \in W_0^{1,p}(\Omega)$ and that $p(x,y) = \nabla^\mu u(x) + \mathbf{v}(x,y)$, where $\mathbf{v} \in \mathbf{L}^p(\Omega, V_{pot}^p)$. For $\mathbf{b} \in V_{sol}^q$ with $\displaystyle\int_\square \mathbf{b}\,\mathrm{d}\mu = \eta = (\eta_1, \ldots, \eta_n)$ and $\phi \in C^\infty(\overline{\Omega})$, we have the identity

$$0 = \int_\Omega \nabla(\phi u_\varepsilon) \cdot \mathbf{b}\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon = \int_\Omega u_\varepsilon \nabla\phi \cdot \mathbf{b}\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon + \int_\Omega \phi \nabla u_\varepsilon \cdot \mathbf{b}\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon.$$

By passing to the limit in the weak two-scale sense, we get

$$0 = \int_\Omega \int_\square u(x) \nabla\phi \cdot \mathbf{b}(y) \,\mathrm{d}\mu(y) \,\mathrm{d}x + \int_\Omega \int_\square \phi \mathbf{p}\,(x,y) \cdot \mathbf{b}(y) \,\mathrm{d}\mu(y) \,\mathrm{d}x. \quad (6.78)$$

Thus,

$$\int_\Omega u(x) \nabla\phi \cdot \eta \,\mathrm{d}x = -\int_\Omega v_{\mathbf{b}}(x)\, \phi \,\mathrm{d}x \quad \forall\, \phi \in C^\infty(\overline{\Omega}),$$

where $v_{\mathbf{b}}(x) = \displaystyle\int_\square \mathbf{p}\,(x,y) \cdot \mathbf{b}(y) \,\mathrm{d}\mu(y)$.

We see that $\eta \cdot \nabla^\mu u \in L^p(\Omega)$ in the sense of distributions and that $\eta \cdot \nabla^\mu u = v_{\mathbf{b}}(x)$. We now use the fact that μ is a nondegenerate measure. In this case, for each $\eta \in \mathbb{R}^n$ there exists a vector $\mathbf{b}^k \in V^q_{sol}$ such that

$$\int_\Omega u(x) \partial\phi/\partial x_k \,\mathrm{d}x = -\int_\Omega v_{\mathbf{b}^k}(x)\, \phi \,\mathrm{d}x, \quad \forall\, \phi \in C^\infty(\overline{\Omega}),\ k = 1, \dots, n. \quad (6.79)$$

Thus, the distributional partial derivatives $\partial u/\partial x_k = v_{\mathbf{b}^k}$ of u are in $L^p(\Omega)$, that is, $u \in W^{1,p}(\Omega)$. Moreover, (6.79) together with the integration by parts formula gives that $u \in W^{1,p}_0(\Omega)$ for Ω with a Lipschitz boundary. As a result, (6.78) can now be rewritten as

$$\begin{aligned}\int_\Omega \int_\square \phi(x) \mathbf{p}\,(x,y) \cdot \mathbf{b}(y) \,\mathrm{d}\mu(y) \,\mathrm{d}x &= -\int_\Omega \int_\square u(x) \nabla\phi(x) \cdot \mathbf{b}(y) \,\mathrm{d}\mu(y) \,\mathrm{d}x \\ &= \int_\Omega \int_\square \phi \nabla^\mu u(x) \cdot \mathbf{b}(y) \,\mathrm{d}\mu(y) \,\mathrm{d}x.\end{aligned}$$

Hence,

$$\int_\Omega \int_\square [\mathbf{p}\,(x,y) - \nabla^\mu u(x)] \cdot \phi(x) \mathbf{b}(y) \,\mathrm{d}\mu(y) \,\mathrm{d}x = 0.$$

Since $\mathbf{L}^q(\Omega, V^q_{sol})$ is defined as closure in $\mathbf{L}^q(\Omega \times \square)$ of the linear span of vectors $f(x)\mathbf{b}(y)$, where $f \in C^\infty_0(\Omega)$ and $\mathbf{b} \in V^q_{sol}$, we obtain that

$$\int_\Omega \int_\square [\mathbf{p}\,(x,y) - \nabla^\mu u(x)] \cdot \mathbf{w}(x,y) \,\mathrm{d}\mu(y) \,\mathrm{d}x = 0, \quad \forall\, \mathbf{w} \in \mathbf{L}^q(\Omega, V^q_{sol}).$$

To conclude the proof, it remains only to note that from (6.70) we have that $[\mathbf{p}\,(x,y) - \nabla^\mu u(x)]$ belongs to $\mathbf{L}^p(\Omega, V^p_{pot})$ or, in other words,

$$\mathbf{p}\,(x,y) = \nabla^\mu u(x) + \mathbf{v}(x,y),$$

where $\mathbf{v} \in \mathbf{L}^p(\Omega, V^p_{pot})$. □

As an obvious consequence of this theorem, we have the following result.

Corollary 6.42. *Let μ be a p-connected non-degenerate measure and assume that for a sequence $\{u_\varepsilon \in C_0^\infty(\Omega)\}_{\varepsilon>0}$, the following conditions hold:*

(i) $u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x,y)$ *as* $\varepsilon \to 0$.
(ii) $\varepsilon \|\nabla u_\varepsilon\|_{\mathbf{L}^2(\Omega,\mathrm{d}\mu_\varepsilon)} \to 0$ *as* $\varepsilon \to 0$.

Then the weak two-scale limit $u(x,y)$ is independent of y.

Note that for the results given above (see Theorem 6.41 and its corollary) the following supposition is essential: μ is a p-connected nondegenerate periodic Borel measure on $\mathbb{R}^n$. In order to analyze the situation without these conditions, we introduce another (dual) definition of Sobolev spaces connected with the Borel measure μ (see [260] for the case $p = 2$).

Definition 6.43. *We say that a function $u \in L^p(\square, \mathrm{d}\mu)$ belongs to the space $\widehat{W}^{1,p}_{per}(\square, \mathrm{d}\mu)$ and a vector $\mathbf{v} \in V^p_{pot}$ is its gradient if*

$$\int_\square ua \,\mathrm{d}\mu = -\int_\square \mathbf{v}\cdot\mathbf{b} \,\mathrm{d}\mu \tag{6.80}$$

provided a vector $\mathbf{b} \in \mathbf{L}^q(\square, \mathrm{d}\mu)$ and a function $a \in L^q(\square, \mathrm{d}\mu)$ are connected by the relation $\mathrm{div}^\mu \mathbf{b} = a$ with respect to the measure μ (see (6.64)).

It is clear that the set of pairs $(u, \mathbf{v})$ such that $u \in \widehat{W}^{1,p}_{per}(\square, \mathrm{d}\mu)$ is a closed subset of $L^p(\square, \mathrm{d}\mu) \times \mathbf{L}^p(\square, \mathrm{d}\mu)$. Let us show that this space can be viewed as the dual definition of the Sobolev space $W^{1,p}_{per}(\square, \mathrm{d}\mu)$.

Theorem 6.44. *The spaces $W^{1,p}_{per}(\square, \mathrm{d}\mu)$ and $\widehat{W}^{1,p}_{per}(\square, \mathrm{d}\mu)$ coincide.*

Proof. Let $(u, \nabla^\mu u)$ be any pair such that

$$u \in W^{1,p}_{per}(\square, \mathrm{d}\mu), \quad \nabla^\mu u \in \mathit{\Gamma}^\mu_p(u).$$

Then, there exists a sequence $\{\varphi_k \in C_{per}(\square)\}_{k=1}^\infty$ satisfying

$$\int_\square |\varphi_k - u|^p \,\mathrm{d}\mu \to 0 \quad \text{and} \quad \int_\square |\nabla\varphi_k - \nabla^\mu u|^p \,\mathrm{d}\mu \to 0 \text{ as } k \to \infty. \tag{6.81}$$

Hence, (6.81) and (6.64) imply

$$\int_\square \mathbf{b}\cdot\nabla\varphi_k \,\mathrm{d}\mu = -\int_\square a\varphi_k \,\mathrm{d}\mu, \quad \int_\square \mathbf{b}\cdot\nabla^\mu u \,\mathrm{d}\mu = -\int_\square au \,\mathrm{d}\mu$$

(i.e., $u \in \widehat{W}^{1,p}_{per}(\square, \mathrm{d}\mu)$ and $\nabla^\mu u$ is its gradient in the sense of Definition 6.43). Assume that $W^{1,p}_{per}(\square, \mathrm{d}\mu) \neq \widehat{W}^{1,p}_{per}(\square, \mathrm{d}\mu)$. Find a non-zero pair $(u, \mathbf{v})$ such

that $u \in \widehat{W}^{1,p}_{per}(\square, \mathrm{d}\mu)$, (6.80) holds true, and

$$\int_{\square} \left[|u|^{p-2} u\varphi + |\mathbf{v}|^{p-2}\mathbf{v} \cdot \nabla\varphi \right] \mathrm{d}\mu = 0, \quad \forall\, \varphi \in C^{\infty}_{per}(\square).$$

Then, by Definition 6.36, this means that $\operatorname{div}(|\mathbf{v}|^{p-2}\mathbf{v}) = |u|^{p-2}u$. Hence, setting in (6.80) $\mathbf{b} = |\mathbf{v}|^{p-2}\mathbf{v}$ and $a = |u|^{p-2}u$, we obtain

$$\int_{\square} |u|^p \,\mathrm{d}\mu = -\int_{\square} |\mathbf{v}|^p \,\mathrm{d}\mu. \tag{6.82}$$

Since (6.82) implies that $u = 0$ and $\mathbf{v} = 0$, this concludes the proof. □

As a result, Corollary 6.42 can be generalized as follows.

Theorem 6.45. *Let $\{u_\varepsilon \in C^{\infty}_0(\Omega)\}_{\varepsilon>0}$ be a sequence such that*

$$\{u_\varepsilon\}_{\varepsilon>0} \quad \textit{is bounded in} \quad L^p(\Omega, \mathrm{d}\mu_\varepsilon),$$

$$\{\varepsilon\nabla u_\varepsilon\}_{\varepsilon>0} \quad \textit{is bounded in} \quad \mathbf{L}^p(\Omega, \mathrm{d}\mu_\varepsilon).$$

Then (up to considering subsequences) we have the following:

(i) $u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x,y) \in L^p(\Omega, W^{1,p}_{per}(\square, \mathrm{d}\mu))$ *as* $\varepsilon \to 0$.
(ii) $\varepsilon\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla^{\mu}_{y} u(x,y)$ *as* $\varepsilon \to 0$.

6.5 Approximation of singular measures by smoothing and its application

Due to the dual definition of Sobolev spaces (see Definition 6.43), we can approximate singular measures by smoothing. Let $K(x)$ be a non-negative C^{∞}_0-function such that $\int_{\mathbb{R}^n} K(x)\,\mathrm{d}x = 1$ and $K(x) = K(-x)$ for all $x \in \mathbb{R}^n$. For an arbitrary Radon measure $\mu \in \mathcal{M}_b(\mathbb{R}^n)$, we set

$$\mathrm{d}\mu^\delta \equiv \mathrm{d}\mathcal{K}^\delta(\mu) = \rho^\delta(x)\,\mathrm{d}x, \tag{6.83}$$

where

$$\rho^\delta(x) = \delta^{-n} \int_{\mathbb{R}^n} K\left(\frac{x-y}{\delta}\right) \mathrm{d}\mu(y). \tag{6.84}$$

Note that the measures μ^δ locally weakly converge in $\mathbb{R}^n$ to the measure μ. Indeed, for an arbitrary continuous function φ with compact support, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n}\varphi(x)\,\mathrm{d}\mu^\delta-\int_{\mathbb{R}^n}\varphi(x)\,\mathrm{d}\mu\Big|\\
&\qquad=\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\delta^{-n}\varphi(x)K\left(\frac{x-y}{\delta}\right)\mathrm{d}\mu(y)\,\mathrm{d}x\right.\\
&\qquad\left.-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\delta^{-n}\varphi(y)K\left(\frac{x-y}{\delta}\right)\mathrm{d}\mu(y)\,\mathrm{d}x\right|\\
&\qquad\le\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\delta^{-n}|\varphi(x)-\varphi(y)|K\left(\frac{x-y}{\delta}\right)\mathrm{d}\mu(y)\,\mathrm{d}x\\
&\qquad\le\max_{|x-y|\le C_K\delta}|\varphi(x)-\varphi(y)|\int_{\mathbb{R}^n}\chi_{\operatorname{spt}\varphi}(y)\,\mathrm{d}\mu(y)\\
&\qquad\le C\max_{|x-y|\le C_K\delta}|\varphi(x)-\varphi(y)|
\end{aligned}$$

as $\delta\to 0$, where the constant C_K is chosen in such a way that $K(y)=0$ for $|y|>C_K$. The right-hand side of the last formula converges to 0 as $\delta\to 0$ since $\varphi(x)$ is uniformly continuous.

We also introduce an ordinary smoothing operator by the equality

$$\varphi^\delta(x)=\left(\widetilde{\mathcal{K}}^\delta\varphi\right)(x)=\delta^{-n}\int_{\mathbb{R}^n}K\left(\frac{y}{\delta}\right)\varphi(x-y)\,\mathrm{d}y. \tag{6.85}$$

It is easy to verify that in this case, we have the equality

$$\int_{\mathbb{R}^n}\varphi^\delta(x)\,\mathrm{d}\mu(x)=\int_{\mathbb{R}^n}\varphi(x)\,\mathrm{d}\mu^\delta(x). \tag{6.86}$$

The following assertion allows us to define a special smoothing operator associated with the measure μ.

Lemma 6.46. *For every $v\in L^2(\mathbb{R}^n,\mathrm{d}\mu)$, there is $v_\delta\in L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta)$ such that*

$$\int_{\mathbb{R}^n}v_\delta(x)\varphi(x)\,\mathrm{d}\mu^\delta(x)=\int_{\mathbb{R}^n}v(x)\varphi^\delta(x)\,\mathrm{d}\mu(x) \tag{6.87}$$

for all $\varphi\in C_0(\mathbb{R}^n)$. Moreover,

$$v_\delta\to v\quad in\quad L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta)\quad as\quad\delta\to 0.$$

Proof. Let us denote

$$F_\delta(\varphi)=\int_{\mathbb{R}^n}v(x)\varphi^\delta(x)\,\mathrm{d}\mu(x),\quad C(v)=\int_{\mathbb{R}^n}v^2(x)\,\mathrm{d}\mu.$$

Then for all $\varphi\in C_0(\mathbb{R}^n)$, we have

$$F_\delta^2(\varphi)\le C(v)\int_{\mathbb{R}^n}\left(\varphi^\delta(x)\right)^2\,\mathrm{d}\mu.$$

Jensen's inequality and property $\int_{\mathbb{R}^n} K(x)\,\mathrm{d}x = 1$ and $K(x) = K(-x)$ imply

$$\int_{\mathbb{R}^n} \bigl(\varphi^\delta(x)\bigr)^2\,\mathrm{d}\mu \le \int_{\mathbb{R}^n} \bigl(\varphi^2(x)\bigr)^\delta\,\mathrm{d}\mu.$$

Hence, by (6.86), we get

$$F_\delta^2(\varphi) \le C(v)\int_{\mathbb{R}^n} \bigl(\varphi^2(x)\bigr)^\delta\,\mathrm{d}\mu = C(v)\int_{\mathbb{R}^n} \varphi^2(x)\,\mathrm{d}\mu^\delta(x).$$

Therefore, the linear functional $F_\delta(\varphi)$ is bounded on the space $L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)$ and, by the Riesz representation theorem, $F_\delta(\varphi)$ can be represented in the form

$$F_\delta(\varphi) = \int_{\mathbb{R}^n} v_\delta(x)\varphi(x)\,\mathrm{d}\mu^\delta(x), \quad \text{where} \quad v_\delta \in L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta).$$

Moreover, in this case, we have

$$\int_{\mathbb{R}^n} v_\delta^2\,\mathrm{d}\mu^\delta \le \int_{\mathbb{R}^n} v^2\varphi(x)\,\mathrm{d}\mu. \tag{6.88}$$

Since for every $\varphi \in C_0(\mathbb{R}^n)$, the functions $\varphi^\delta \in C_0^\infty(\mathbb{R}^n)$ uniformly converge to φ by properties of the classical smoothing, it follows that

$$\int_{\mathbb{R}^n} v_\delta(x)\varphi(x)\,\mathrm{d}\mu^\delta(x) = \int_{\mathbb{R}^n} v\varphi^\delta\,\mathrm{d}\mu \to \int_{\mathbb{R}^n} v\varphi\,\mathrm{d}\mu \quad \text{as} \quad \delta \to 0$$

(i.e., $v_\delta \rightharpoonup v$ in $L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)$). The strong convergence in $L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)$ immediately follows from (6.88) and Lemma 6.19. This concludes the proof. □

In the following assertions, the measure μ^δ is also defined by (6.83) and (6.84), but the functions $g_\delta \in L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)$, $v_\delta \in \bigl[L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)\bigr]^n$, and $u_\delta \in H^1(\mathbb{R}^n, \mathrm{d}\mu^\delta)$ are arbitrary and are not related to the smoothing of any fixed function.

Lemma 6.47. *Let $g = \operatorname{div}^\mu \mathbf{v}$. Then there are sequences*

$$\bigl\{g_\delta \in L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)\bigr\}_{\delta\to 0} \quad \textit{and} \quad \bigl\{\mathbf{v}_\delta \in \mathbf{L}^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)\bigr\}_{\delta\to 0}$$

such that

$$\operatorname{div}^{\mu^\delta} \mathbf{v}_\delta = g_\delta \tag{6.89}$$

and

$$g_\delta \to g \quad \textit{in} \quad L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta) \quad \textit{as} \quad \delta \to 0, \tag{6.90}$$

$$\mathbf{v}_\delta \to \mathbf{v} \quad \textit{in} \quad \mathbf{L}^2(\mathbb{R}^n, \mathrm{d}\mu^\delta) \quad \textit{as} \quad \delta \to 0. \tag{6.91}$$

Proof. Let us show that it suffices to take the same g_δ and $\mathbf{v}_\delta$ as in Lemma 6.46. As a result, the strong convergence in (6.90) and (6.91) is guaranteed by this lemma. To do so, we check (6.89). Indeed, in this case, for every $\varphi \in C_0^\infty(\mathbb{R}^n)$, by (6.87), we have

$$\int_{\mathbb{R}^n} \mathbf{v}_\delta \cdot \nabla\varphi \, \mathrm{d}\mu^\delta = \int_{\mathbb{R}^n} \mathbf{v} \cdot (\varphi^\delta) \, \mathrm{d}\mu = \int_{\mathbb{R}^n} g\varphi^\delta \, \mathrm{d}\mu = \int_{\mathbb{R}^n} g_\delta \varphi \, \mathrm{d}\mu^\delta.$$

□

Theorem 6.48. *Let $u_\delta \in H^1(\mathbb{R}^n, \mathrm{d}\mu^\delta)$ for every $\delta > 0$ and let*

$$u_\delta \rightharpoonup u \quad \text{in} \quad L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta) \quad \text{as} \quad \delta \to 0, \tag{6.92}$$

$$\nabla^{\mu^\delta} u_\delta \rightharpoonup \mathbf{z} \quad \text{in} \quad \mathbf{L}^2(\mathbb{R}^n, \mathrm{d}\mu^\delta) \quad \text{as} \quad \delta \to 0. \tag{6.93}$$

Then $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ and $\mathbf{z} = \nabla^\mu u$.

Proof. For functions g and $\mathbf{v}$ such that $g = \operatorname{div}^\mu \mathbf{v}$, we use the approximations $\{g_\delta \in L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)\}_{\delta\to 0}$ and $\{\mathbf{v}_\delta \in \mathbf{L}^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)\}_{\delta\to 0}$ constructed in Lemma 6.46, so that $\operatorname{div}^{\mu^\delta} \mathbf{v}_\delta = g_\delta$. Then

$$\int_{\mathbb{R}^n} g_\delta u_\delta \, \mathrm{d}\mu^\delta = \int_{\mathbb{R}^n} \mathbf{v}_\delta \cdot \nabla^{\mu^\delta} u_\delta \, \mathrm{d}\mu^\delta.$$

Passing to the limit in this equality as $\delta \to 0$ and taking into account the properties of strong convergence of g_δ and $\mathbf{v}_\delta$, we come to the relation

$$\int_{\mathbb{R}^n} gu \, \mathrm{d}\mu = \int_{\mathbb{R}^n} \mathbf{v} \cdot \mathbf{z} \, \mathrm{d}\mu.$$

To end of this proof, it remains to use the dual definition of Sobolev spaces (see Definition 6.43). As a result, we have $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ and $\mathbf{z} = \nabla^\mu u$. □

As a natural application of the above assertions, we provide the following important result on the convergence and approximation of solutions of elliptic equations.

Let μ be a Radon measure in $\mathbb{R}^n$ and let μ^δ be a smoothed measure given by (6.83). Let $A(\cdot) = [a_{ij}(\cdot)]_{i,j=1,\dots,n}$ be a function with values in the space of symmetric $n \times n$ matrices satisfying the uniform ellipticity condition

$$\alpha|\xi|^2_{\mathbb{R}^n} \le (A(x)\xi, \xi)_{\mathbb{R}^n} \le \alpha^{-1}|\xi|^2_{\mathbb{R}^n}, \quad \alpha > 0, \ \xi \in \mathbb{R}^n, \tag{6.94}$$

for all $x \in \mathbb{R}^n$. Let f be a given element of $L^2(\mathbb{R}^n, \mathrm{d}\mu)$ and let $\lambda > 0$. We consider the elliptic equation in $\mathbb{R}^n$

$$-\operatorname{div}^\mu \Big(A(x) \nabla^\mu u \Big) + \lambda u = f \quad \text{in} \quad L^2(\mathbb{R}^n, \mathrm{d}\mu) \tag{6.95}$$

and the family of approximating equations of the form

$$-\operatorname{div}^{\mu^\delta}\left(A_\delta(x)\nabla^{\mu^\delta}u\right)+\lambda u=f_\delta \quad \text{in} \quad L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta), \tag{6.96}$$

where A_δ and f_δ are defined by the rule (6.87).

The main result of this section we formulate as follows.

Theorem 6.49. *Assume that the approximations* $\{A_\delta\}_{\delta\to 0}$ *and* $\{f_\delta\}_{\delta\to 0}$ *are such that*

$$\begin{gathered}A_\delta\in\left[L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta)\right]^{n\times n},\quad f_\delta\in L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta),\\ \alpha|\xi|^2_{\mathbb{R}^n}\le(A_\delta(x)\xi,\xi)_{\mathbb{R}^n}\le\alpha^{-1}|\xi|^2_{\mathbb{R}^n},\quad\forall\,\xi\in\mathbb{R}^n,\ \forall\,x\in\mathbb{R}^n,\\ A_\delta\to A\quad in\quad\left[L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta)\right]^{n\times n},\\ f_\delta\to f\quad in\quad L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta).\end{gathered}$$

Then the solutions u_δ *of (6.96) strongly converge in the variable* $L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta)$ *to the solution* u *of (6.95) as* $\delta\to 0$.

Proof. Since each of the pairs $\left(u_\delta,\nabla^{\mu^\delta}u_\delta\right)$ satisfies the estimate

$$\|u_\delta\|_{L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta)}+\|\nabla^{\mu^\delta}u_\delta\|_{(L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta))^n}\le C,$$

it follows that (see Proposition 6.15) for some subsequence $\delta_k\to 0$, we have weak convergence

$$\left(u_\delta,\nabla^{\mu^\delta}u_\delta\right)\rightharpoonup(u_0,\mathbf{z}_0)\quad\text{in}\quad\left[L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta)\right]^{n+1}.$$

By Theorem 6.48, we deduce $u_0\in H^1(\mathbb{R}^n,\mathrm{d}\mu)$ and $\mathbf{z}_0=\nabla^\mu u_0$. It is easy to verify that

$$A_\delta(x)\nabla^{\mu^\delta}u_\delta\rightharpoonup A(x)\nabla^\mu u_0\quad\text{in}\quad\left[L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta)\right]^n.$$

Then, passing to the limit in the integral identity

$$\int_{\mathbb{R}^n}A_\delta(x)\nabla^{\mu^\delta}u_\delta\cdot\nabla\varphi\,\mathrm{d}\mu^\delta+\lambda\int_{\mathbb{R}^n}u_\delta\varphi\,\mathrm{d}\mu^\delta=\int_{\mathbb{R}^n}f_\delta\varphi\,\mathrm{d}\mu^\delta,\quad\forall\,\varphi\in C_0^\infty(\mathbb{R}^n),$$

we conclude that $(u_0,\nabla^\mu u_0)$ is a solution to (6.95). Strong convergence $u_\delta\to u_0$ in $L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta)$ can be proved by using the trick based on the energy convergence. Taking $\left(u_\delta,\nabla^{\mu^\delta}u_\delta\right)$ for a test function in the integral identity for (6.96) and taking into account the strong convergence of f_δ to f in $L^2(\mathbb{R}^n,\mathrm{d}\mu^\delta)$, we obtain the chain of equalities

$$\begin{aligned}\lim_{\delta\to0}\int_{\mathbb{R}^n}&\left(A_\delta(x)\nabla^{\mu^\delta}u_\delta\cdot\nabla^{\mu^\delta}u_\delta+\lambda u_\delta^2\right)\mathrm{d}\mu^\delta\\ &=\lim_{\delta\to0}\int_{\mathbb{R}^n}f_\delta u_\delta\,\mathrm{d}\mu^\delta=\int_{\mathbb{R}^n}fu_0\,\mathrm{d}\mu\\ &=\int_{\mathbb{R}^n}\left(A(x)\nabla^\mu u_0\cdot\nabla^\mu u_0+\lambda u_0^2\right)\mathrm{d}\mu.\end{aligned}\tag{6.97}$$

However, both quadratic terms on the left-hand side of this equality are lower semi-continuous with respect to weak convergence in $L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)$ (see Proposition 6.16). Hence,

$$\liminf_{\delta\to 0} \int_{\mathbb{R}^n} A_\delta(x)\nabla^{\mu^\delta} u_\delta \cdot \nabla^{\mu^\delta} u_\delta \,\mathrm{d}\mu^\delta \geq \int_{\mathbb{R}^n} A(x)\nabla^{\mu} u_0 \cdot \nabla^{\mu} u_0 \,\mathrm{d}\mu,$$

$$\liminf_{\delta\to 0} \int_{\mathbb{R}^n} \lambda u_\delta^2 \,\mathrm{d}\mu^\delta \geq \int_{\mathbb{R}^n} \lambda u_0^2 \,\mathrm{d}\mu.$$

Therefore, (6.97) can be valid only if the two last inequalities are, in fact, equalities. Strong convergence $u_\delta \to u_0$ in $L^2(\mathbb{R}^n, \mathrm{d}\mu^\delta)$ immediately follows from this and Lemma 6.19. □

6.6 Two-scale convergence with respect to a variable measure

In the previous sections, we supposed that μ was a non-negative periodic fixed Borel measure on $\mathbb{R}^n$ such that $\int_\square \mathrm{d}\mu = 1$. Having defined the scaling measure μ_ε by (6.36), we introduced the concept of convergence in the variable spaces $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ and two-scale convergence. However, a typical feature of thin structures is the fact that their geometry depends on two parameters mutually related to one another – namely, ε defines the periodicity cell and εh is the thickness of constituting elements of the structure (see Chaps. 9 and 10). Moreover, for the thin structures the parameters ε and $h = h(\varepsilon)$ are usually related by the supposition $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. In view of this, we begin this section with the following assumptions: Let μ^h be a $\square$-periodic normalized Borel measure depending on the parameter h, that is, $\int_\square \mathrm{d}\mu^h = 1$ for all h and $\mu^h \in \mathcal{M}(\Omega)$, where Ω is an open bounded domain in $\mathbb{R}^n$. Hereinafter, we suppose that the sequence $\{\mu^h\} \subset \mathcal{M}(\Omega)$ converges weakly to a Borel measure μ, that is,

$$\mu^h \rightharpoonup \mu \Longleftrightarrow \lim_{h\to 0} \int_\square \psi \,\mathrm{d}\mu^h = \int_\square \psi \,\mathrm{d}\mu, \quad \forall\, \psi \in C^\infty_{per}(\square).$$

For every fixed h, we define $\varepsilon\square$-periodic scaling measure μ^h_ε as follows:

$$\mu^h_\varepsilon(B) = \varepsilon^n \mu^h(\varepsilon^{-1}B) \text{ for any Borel set } \ B \subset \mathbb{R}^n. \tag{6.98}$$

Further, we assume that there is a relation $h = h(\varepsilon)$ between the parameters ε and h such that $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$. Then

$$\int_{\varepsilon\square} \mathrm{d}\mu^{h(\varepsilon)}_\varepsilon = \varepsilon^n \int_{\varepsilon\square} \mathrm{d}\mu^{h(\varepsilon)}(x/\varepsilon) = \varepsilon^n \int_\square \mathrm{d}\mu^{h(\varepsilon)} = \varepsilon^n.$$

This leads us to the following conclusion.

Lemma 6.50. *Assume that an open bounded domain $\Omega \subset \mathbb{R}^n$ is measurable in the Jordan sense. Then the measure $\mu_\varepsilon^{h(\varepsilon)}$ converges weakly in $\mathcal{M}(\Omega)$ to the Lebesgue measure on $\mathbb{R}^n$ as ε tends to 0: $\mathrm{d}\mu_\varepsilon^{h(\varepsilon)} \rightharpoonup \mathrm{d}x$, that is,*

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi\,\mathrm{d}\mu_\varepsilon^{h(\varepsilon)} = \int_\Omega \varphi\,\mathrm{d}x, \quad \forall\,\varphi \in C_0(\Omega). \tag{6.99}$$

Proof. Let $\square_{\mathbf{i}} = \square + \mathbf{i}$, where $\mathbf{i}$ is a vector in $\mathbb{R}^n$ with integer components. Then $\{\varepsilon\square_{\mathbf{i}}\}$ is a partition of $\mathbb{R}^n$ for every fixed $\varepsilon > 0$ and

$$\int_\Omega \varphi(x)\,\mathrm{d}\mu_\varepsilon^{h(\varepsilon)} = \sum\int_{\varepsilon\square_{\mathbf{i}}} \varphi(x)\,\mathrm{d}\mu_\varepsilon^{h(\varepsilon)} + \sum\int_{\varepsilon\square_{\mathbf{i}}\cap\Omega} \varphi(x)\,\mathrm{d}\mu_\varepsilon^{h(\varepsilon)} \tag{6.100}$$

for every $\varphi \in C_0(\Omega)$, where the first sum is taken over all $\mathbf{i}$ such that $\varepsilon\square_{\mathbf{i}}$ is inside Ω and the second sum is over all $\mathbf{i}$ such that $\varepsilon\square_{\mathbf{i}}$ and $\partial\Omega$ have common points. Let us first consider the first sum in (6.100). Since $\varphi \in C_0(\Omega)$, there exist points $x_{\mathbf{i}} \in \varepsilon\square_{\mathbf{i}}$ such that

$$\int_{\varepsilon\square_{\mathbf{i}}} \varphi(x)\,\mathrm{d}\mu_\varepsilon^{h(\varepsilon)} = \varphi(x_{\mathbf{i}})\int_{\varepsilon\square_{\mathbf{i}}} \mathrm{d}\mu_\varepsilon^{h(\varepsilon)} = \varphi(x_{\mathbf{i}})\varepsilon^n\int_{\square}\mathrm{d}\mu^h = \varepsilon^n\varphi(x_{\mathbf{i}}).$$

This together with the fact that $\sum\varepsilon^n\varphi(x_{\mathbf{i}})$ is the construction of a Riemann sum implies that

$$\lim_{\varepsilon\to 0}\sum\varepsilon^n\varphi(x_{\mathbf{i}}) \to \int_\Omega \varphi(x)\,\mathrm{d}x. \tag{6.101}$$

Let us consider the second sum in (6.100). We have

$$\left|\sum\int_{\varepsilon\square_{\mathbf{i}}\cap\Omega} \varphi(x)\,\mathrm{d}\mu_\varepsilon^{h(\varepsilon)}\right| \le \max_{x\in\Omega}|\varphi(x)|\varepsilon^n M(\varepsilon),$$

where $M(\varepsilon)$ is the number of cubes $\varepsilon\square_{\mathbf{i}}$ containing the boundary of Ω. Since $\varepsilon^n M(\varepsilon) \to 0$ by the Jordan measurability property of Ω, we conclude that

$$\lim_{\varepsilon\to 0}\sum\int_{\varepsilon\square_{\mathbf{i}}\cap\Omega} \varphi(x)\,\mathrm{d}\mu_\varepsilon^{h(\varepsilon)} = 0. \tag{6.102}$$

Thus, (6.99) immediately follows by taking (6.100)–(6.102) into account. □

Two-scale convergence with respect to a measure – in particular, a variable measure μ_ε^h – was introduced and studied by Zhikov [256, 257, 258] and was also considered by Bouchitté and Fragala [26]. In the works of Zhikov and Pastukhova [211, 213, 265], this convergence is crucial for studying homogenization problems of elasticity for thin structures. It is easy to see that for a sequence $\{u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon\to 0}$, we have $u_\varepsilon \overset{2}{\rightharpoonup} u(x,y) \in L^p(\Omega\times\square, \mathrm{d}x\times\mathrm{d}\mu)$ provided that $\mu_\varepsilon = \mu_\varepsilon^{h(\varepsilon)}$. The strong two-scale convergence in this case can be defined in a similar manner. However, as indicated in [258], the main properties of two-scale convergence (see Lemmas 6.29–6.31 and Propositions 6.26–6.28) with respect to the variable two-parametric measure μ_ε^h can be established in a sharp form. Indeed, to begin, we consider the mean value property in two-parametric spaces $L^p(\square, \mathrm{d}\mu^h)$.

Theorem 6.51. *If a sequence* $\{a^h \in L^p(\square, \mathrm{d}\mu^h)\}$ *is such that* $a^h \rightharpoonup a$ *in* $L^p(\square, \mathrm{d}\mu^h)$ *as* $h \to 0$, *then*

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi(x) a^{h(\varepsilon)} \left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon^{h(\varepsilon)} = \int_\Omega \int_\square \varphi(x) a(y) \,\mathrm{d}\mu(y)\,\mathrm{d}x \tag{6.103}$$

for any continuous bounded function φ.

Proof. We start with the case when $a^h \geq 0$ for each $h \to 0$. Then closely following the proof of Theorem 6.21 and using the fact that

$$\lim_{h\to 0} \int_\square a^h \,\mathrm{d}\mu^h = \int_\square a \,\mathrm{d}\mu \text{ (by the weak convergence } a^h \rightharpoonup a),$$

we see that (6.40) can be rewritten in the form

$$\begin{aligned} \lim_{\varepsilon\to 0} \sum \int_{\varepsilon\square_{\mathbf{k}}} \varphi(x) a^h \left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon^h &= \lim_{h\to 0} \int_\Omega \int_\square \varphi(x) a^h(y) \,\mathrm{d}\mu^h(y)\,\mathrm{d}x \\ &= \int_\Omega \int_\square \varphi(x) a(y) \,\mathrm{d}\mu(y)\,\mathrm{d}x. \end{aligned} \tag{6.104}$$

So, proceeding as in the proof of Theorem 6.21, we come to the desired relation (6.103).

As for the general case, we can represent each of a^h in the form $a^h = a_1^h - a_2^h$, where $\{a_i^h \geq 0\}$ are uniformly bounded sequences in $L^p(\square, \mathrm{d}\mu^h)$. Having assumed that $a_1^h \rightharpoonup a_1$, $a_2^h \rightharpoonup a_2$ in $L^p(\square, \mathrm{d}\mu^h)$, where $a = a_1 - a_2$ and applying the previous arguments, we obtain (6.103). □

Definition 6.52. *We say that a bounded sequence* $\{u_{\varepsilon,h} \in L^p(\Omega, \mathrm{d}\mu_\varepsilon^h)\}$, *that is,*

$$\sup_{\varepsilon\to 0} \|u_{\varepsilon,h}\|_{L^p(\Omega, \mathrm{d}\mu_\varepsilon^h)} < +\infty,$$

weakly two-scale converges to a function $u(x,y) \in L^p(\Omega \times \square, \mathrm{d}x \times \mathrm{d}\mu)$ *if*

$$\lim_{\varepsilon\to 0} \int_\Omega u_{\varepsilon,h}(x) \varphi(x) b\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon^h = \int_\Omega \int_\square u(x,y) \varphi(x) b(y) \,\mathrm{d}\mu(y)\,\mathrm{d}x \tag{6.105}$$

for every test function $\varphi \in C_0^\infty(\Omega)$ *and* $b \in C_{per}^\infty(\square)$.

Definition 6.53. *We say that a sequence* $\{u_{\varepsilon,h} \in L^p(\Omega, \mathrm{d}\mu_\varepsilon^h)\}$ *strongly two-scale converges to a function* $u(x,y) \in L^p(\Omega \times \square, \mathrm{d}x \times \mathrm{d}\mu)$ *if*

$$\lim_{\varepsilon\to 0} \int_\Omega u_{\varepsilon,h}(x) v_{\varepsilon,h}(x) \,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \int_\square u(x,y) v(x,y) \,\mathrm{d}\mu(y)\,\mathrm{d}x \tag{6.106}$$

whenever $v_{\varepsilon,h}(x) \overset{2}{\rightharpoonup} v(x,y)$.

Let us list here certain general properties of the two-scale convergence, which are the obvious consequence of the results of Sect. 6.3.

(i) Any sequence bounded in $L^p(\Omega, \mathrm{d}\mu_\varepsilon^h)$ contains a weakly two-scale convergent subsequence.

(ii) If $u_{\varepsilon,h}(x) \overset{2}{\rightharpoonup} u(x,y)$, then

$$\liminf_{\varepsilon\to 0} \int_\Omega |u_{\varepsilon,h}|^p \,\mathrm{d}\mu_\varepsilon^h \geq \int_\Omega \int_\square |u(x,y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}x.$$

(iii) Strong two-scale convergence $u_{\varepsilon,h}(x) \overset{2}{\to} u(x,y)$ is equivalent to

$$u_{\varepsilon,h}(x) \overset{2}{\rightharpoonup} u(x,y) \text{ and}$$
$$\lim_{\varepsilon\to 0} \int_\Omega |u_{\varepsilon,h}|^p \,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \int_\square |u(x,y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}x. \tag{6.107}$$

(iv) If $u_{\varepsilon,h}(x) \overset{2}{\rightharpoonup} u(x,y)$, then $u_{\varepsilon,h} \rightharpoonup \int_\square u(x,y)\,\mathrm{d}\mu$ in $L^p(\Omega, \mathrm{d}\mu_\varepsilon^h)$.

However, this collection can be extended by some additional properties of two-scale convergence in $L^p(\square, \mathrm{d}\mu_\varepsilon^h)$.

Proposition 6.54. *If $c^h \rightharpoonup c$ in $L^p(\square, \mathrm{d}\mu^h)$, then $c^h(x/\varepsilon) \overset{2}{\rightharpoonup} c(y)$.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ and $b \in C_{per}^\infty(\square)$ be any fixed functions. Then, in view of Definition 6.14, we have $c^h b \rightharpoonup cb$ in $L^p(\square, \mathrm{d}\mu^h)$. Hence, by the mean value property (see (6.103)), we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi(x) c^h\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}\right) \varphi(x) \,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \int_\square c(y) b(y) \varphi(x) \,\mathrm{d}\mu(y)\,\mathrm{d}x.$$

It remains to take Definition 6.52 into account. □

Proposition 6.55. *If $c^h \to c$ in $L^p(\square, \mathrm{d}\mu^h)$, then $c^h(x/\varepsilon) \overset{2}{\to} c(y)$.*

Proof. By the previous proposition, we have $c^h(x/\varepsilon) \overset{2}{\rightharpoonup} c(y)$. So, it remans to verify the equality

$$\lim_{\varepsilon\to 0} \int_\Omega \left| c^h\left(\frac{x}{\varepsilon}\right)\right|^p \,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \int_\square |c(y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}x.$$

However, this fact immediately follows from (6.103) since

$$\lim_{h\to 0} \int_\square |c^h|^p \,\mathrm{d}\mu^h = \int_\square |c|^p \,\mathrm{d}\mu,$$

by the property of strong convergence $c^h \to c$ in $L^p(\square, \mathrm{d}\mu^h)$. □

To conclude, we note that the class of test functions in Definition 6.52 is minimal. However, as follows from the proposition given below, this class can be essentially extended with preservation of the convergence in (6.105).

Proposition 6.56. *Let* $u_{\varepsilon,h}(x) \overset{2}{\rightharpoonup} u(x,y)$ *in the sense of Definition 6.52 and let* $b^h \to b$ *in* $L^q(\square, \mathrm{d}\mu^h)$, $1/p + 1/q = 1$. *Then*

$$\lim_{\varepsilon\to 0} \int_\Omega u_{\varepsilon,h}(x)\varphi(x) b^h\left(\frac{x}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon^h = \int_\Omega \int_\square u(x,y)\varphi(x) b(y)\, \mathrm{d}\mu(y)\, \mathrm{d}x, \quad \forall\, \varphi \in C_0^\infty(\Omega). \quad (6.108)$$

The proof of this result can be found in [258].

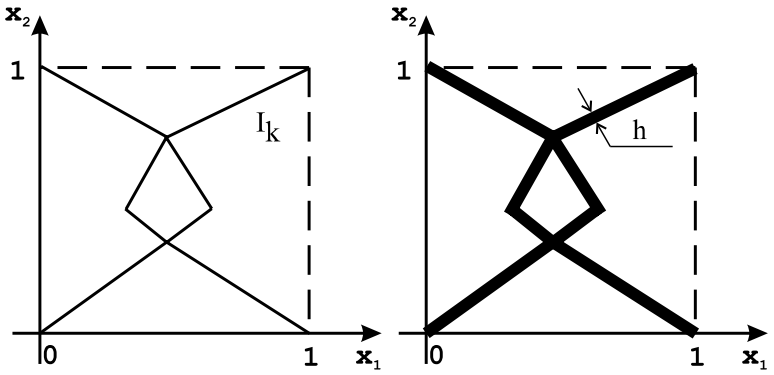
6.7 Some properties of the strong convergence in spaces $L^2(\square, \mathrm{d}\mu^h)$ associated with thin periodic structures

To illustrate some specific properties of the strong convergence in variable spaces $L^p(\square, \mathrm{d}\mu^h)$, we consider a particular case of the scaling measure μ_ε^h associated with periodic thin structures in the plane. This highly contrasting medium depends on two small related parameters ε and h which control the size of a periodicity cell and thickness of the reinforcement. We assume that the periodic thin structure is contained in an open bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. Let $\mathcal{F}$ be a periodic graph in the plane that consists of segments. Let $\square = [0,1)^2$ be its periodicity cell. We call this graph a singular (or an infinitely thin) grid. We define a thin grid $\mathcal{F}^h$ applying the procedure of h-thickening to the graph $\mathcal{F}$. It means that $\mathcal{F}^h$ can be obtained if any component I of $\mathcal{F}$ is replaced by a strip I^h of width h whose median coincides with I. Sometimes it is convenient to include into a thin grid $\mathcal{F}^h$ disks of radius $h/2$ centered at nodes of the grid $\mathcal{F}$. The procedure of h-thickening becomes simpler if the grid $\mathcal{F}$ is composed by several families of parallel straight lines. In this case, we obtain the grid $\mathcal{F}^h$ by replacing all of the lines l of the grid $\mathcal{F}$ by the infinite strip of width h with median l (see Fig. 6.1).

Denote by λ a measure concentrated on $\mathcal{F}$ and proportional to the linear Lebesgue measure. Denote by λ^h a measure concentrated on $\mathcal{F}^h$ and proportional to the planar Lebesgue measure. It is assumed that the measures λ and λ^h are normed by the following condition: $\int_\square \mathrm{d}\lambda = \int_\square \mathrm{d}\lambda^h = 1$. Clearly, $\lambda^h \rightharpoonup \lambda$ as $h \to 0$, that is,

$$\lim_{h\to 0} \int_\square \varphi\, \mathrm{d}\lambda^h = \int_\square \varphi\, \mathrm{d}\lambda \quad \text{for any function } \varphi \in C_{per}^\infty(\square).$$

The homothetic contraction of the plane by a factor of ε^{-1} takes the grids $\mathcal{F}^h$ and $\mathcal{F}$ to $\mathcal{F}_\varepsilon^h$ and $\mathcal{F}_\varepsilon$, respectively. In particular, $\mathcal{F}_\varepsilon^h = \varepsilon\mathcal{F}^h = \{\varepsilon x \ : \ x \in \mathcal{F}^h\}$. Note that in this case, $\mathcal{F}_\varepsilon^h$ consists of strips whose thickness is εh. Each of the grids $\mathcal{F}_\varepsilon^h$ and $\mathcal{F}_\varepsilon$ can be associated with ε-periodic measure λ_ε^h and λ_ε, respectively, such that



(a) The cell of the singular graph $\mathcal{F}$ (b) The cell of the thin grid $\mathcal{F}^h$

Fig. 6.1. The procedure of h-thickening for the graph $\mathcal{F}$

$$\lambda_\varepsilon^h(B) = \varepsilon^2 \lambda^h(B) \quad \text{and} \quad \lambda_\varepsilon(B) = \varepsilon^2 \lambda(B) \qquad \text{for any planar Borel set} \quad B.$$

As usual, we relate parameters h and ε assuming that $h = h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since $\lambda_\varepsilon^h(\varepsilon\square) = \lambda_\varepsilon(\square) = \varepsilon^2$, it follows that (see Lemma 6.50) the measures λ_ε^h and λ_ε converge weakly to the planar Lebesgue measure as $\varepsilon \to 0$, that is,

$$\int_{\mathbb{R}^2} \varphi \, \mathrm{d}\lambda_\varepsilon^h \to \int_{\mathbb{R}^2} \varphi \, \mathrm{d}x, \quad \int_{\mathbb{R}^2} \varphi \, \mathrm{d}\lambda_\varepsilon \to \int_{\mathbb{R}^2} \varphi \, \mathrm{d}x, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^2).$$

We define a thin structure $\Omega_{\varepsilon,h}$ as $\Omega_{\varepsilon,h} = \Omega \cap \mathcal{F}_\varepsilon^h$. Following in many aspects Pastuchova (see [213]), we give some additional properties of two-scale convergence variable spaces $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$.

Lemma 6.57. *For any periodic functions $\varphi \in C_{per}^\infty(\square)$, the Poincaré inequality*

$$\int_\square \varphi^2 \, \mathrm{d}\lambda^h \le C \int_\square |\nabla\varphi|^2 \, \mathrm{d}x, \quad \int_\square \varphi \, \mathrm{d}\lambda^h = 0, \ \forall\, h > 0, \tag{6.109}$$

holds true, where the constant C depends only on the geometry of the grid $\mathcal{F}$.

Proof. Let $I^h = (0,h) \times (0,1)$ be any strip in the square $\square$. Then the following inequality (see [213]) is obvious:

$$\frac{1}{h} \int_{I^h} \varphi^2 \, \mathrm{d}x \le C_1 \int_\square \left(\varphi^2 + |\nabla\varphi|^2\right) \mathrm{d}x, \quad \varphi \in C_{per}^\infty(\square), \ C_1 = C_1(\square). \tag{6.110}$$

This inequality combined with the standard Poincaré inequality

$$\int_\square \varphi^2 \, \mathrm{d}x \le C_2 \int_\square |\nabla\varphi|^2 \, \mathrm{d}x, \quad \varphi \in C_{per}^\infty(\square), \quad \int_\square \varphi \, \mathrm{d}x = 0$$

implies that

$$\frac{1}{h}\int_{I^h}\varphi^2\,\mathrm{d}x \le C_3\int_{\square}|\nabla\varphi|^2\,\mathrm{d}x,\quad \varphi\in C^\infty_{per}(\square),\quad \int_{\square}\varphi\,\mathrm{d}x=0 \tag{6.111}$$

uniformly in h.

Since the set $\mathcal{F}^h\cap\square$ can be presented as a collection of strips in $\square$, from (6.111) we deduce that

$$\int_{\square}\varphi^2\,\mathrm{d}\lambda^h \le C_3\int_{\square}|\nabla\varphi|^2\,\mathrm{d}x,\quad \varphi\in C^\infty_{per}(\square),\quad \int_{\square}\varphi\,\mathrm{d}x=0,$$

where C_3 depends only on the geometry of the set $\mathcal{F}^h\cap\square$. If we set $\varphi=v-m$, where $m=\displaystyle\int_{\square}v\,\mathrm{d}x$, then

$$\int_{\square}(v-m)^2\,\mathrm{d}\lambda^h \le C\int_{\square}|\nabla v|^2\,\mathrm{d}x,\quad v\in C^\infty_{per}(\square).$$

This proves the Poincaré inequality (6.109). □

Lemma 6.58. *There exists a constant C_0 depending only on the geometry of the grid $\mathcal{F}$ such that for any compactly supported function $v\in C_0^\infty(\mathbb{R}^2)$ the following inequality holds true:*

$$\int_{\mathbb{R}^2}v^2\,\mathrm{d}\lambda^h_\varepsilon \le C_0\int_{\mathbb{R}^2}\left(v^2+\varepsilon^2|\nabla v|^2\right)\,\mathrm{d}x,\quad v\in C_0^\infty(\mathbb{R}^2). \tag{6.112}$$

Proof. In view of (6.110) and the fact that the measure λ^h can be represented as the weighed sum of planar Lebesgue measures supported on different strips I^h in $\square$, we have

$$\int_{\square}\varphi^2\,\mathrm{d}\lambda^h \le C_0\int_{\square}\left(\varphi^2+|\nabla\varphi|^2\right)\,\mathrm{d}x,\quad \varphi\in C^\infty(\overline{\square}),\ C_0=C_0(\mathcal{F}). \tag{6.113}$$

After a homothetic contraction, the latter is transformed into the following inequality for the square $\varepsilon\square$:

$$\int_{\varepsilon\square}\varphi^2\,\mathrm{d}\lambda^h_\varepsilon \le C_0\int_{\varepsilon\square}\left(\varphi^2+\varepsilon^2|\nabla\varphi|^2\right)\,\mathrm{d}x,\quad \varphi\in C^\infty(\varepsilon\overline{\square}).$$

Decomposing the plane into disjoint semiopen squares, $\mathbb{R}^2=\bigcup_i\varepsilon\square_i$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}v^2\,\mathrm{d}\lambda^h_\varepsilon &= \sum_i\int_{\varepsilon\square}\varphi^2\,\mathrm{d}\lambda^h_\varepsilon \le C_0\sum_i\int_{\varepsilon\square}\left(\varphi^2+\varepsilon^2|\nabla\varphi|^2\right)\,\mathrm{d}x\\ &\le C_0\int_{\mathbb{R}^2}\left(\varphi^2+\varepsilon^2|\nabla\varphi|^2\right)\,\mathrm{d}x,\quad \forall\, v\in C_0^\infty(\mathbb{R}^2).\end{aligned}$$

Thus, (6.112) is valid. □

Let us apply Lemmas 6.57 and 6.58 to establish some specific properties of convergence in the space $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$.

Proposition 6.59. *Let v be any element of $H_0^1(\Omega)$. Then*

$$\int_\Omega v^2 \,\mathrm{d}\lambda_\varepsilon^h \to \int_\Omega v^2 \,\mathrm{d}x \quad \text{as} \quad \varepsilon \to 0. \tag{6.114}$$

Proof. Indeed, in the case when $v \in C_0^\infty(\Omega)$, (6.114) is the direct consequence of the weak convergence $\mathrm{d}\lambda_\varepsilon^h \rightharpoonup \mathrm{d}x$. For an arbitrary $v \in H_0^1(\Omega)$, we have the following estimate:

$$\begin{aligned}\left|\int_\Omega v^2 \,\mathrm{d}\lambda_\varepsilon^h - \int_\Omega v^2 \,\mathrm{d}x\right| &\le \int_\Omega \left|v^2 - v_\delta^2\right| \,\mathrm{d}\lambda_\varepsilon^h \\ &\quad + \left|\int_\Omega v_\delta^2 \,\mathrm{d}\lambda_\varepsilon^h - \int_\Omega v_\delta^2 \,\mathrm{d}x\right| + \left|\int_\Omega v_\delta^2 \,\mathrm{d}x - \int_\Omega v^2 \,\mathrm{d}x\right|,\end{aligned}$$

where an element $v_\delta \in C_0^\infty(\Omega)$ is such that $\|v - v_\delta\|_{H_0^1(\Omega)} < \delta$.

Using (6.112) and Hölder inequality, we get

$$\begin{aligned}\int_\Omega \left|v^2 - v_\delta^2\right| \,\mathrm{d}\lambda_\varepsilon^h &\le \|v + v_\delta\|_{L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)} \|v - v_\delta\|_{L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)} \\ &\le C\|v - v_\delta\|_{H^1(\Omega)} \le C_1\delta.\end{aligned}$$

Moreover, by weak convergence $\mathrm{d}\lambda_\varepsilon^h \rightharpoonup \mathrm{d}x$, we conclude

$$\left|\int_\Omega v_\delta^2 \,\mathrm{d}\lambda_\varepsilon^h - \int_\Omega v_\delta^2 \,\mathrm{d}x\right| \le \delta \quad \text{for} \quad \varepsilon \quad \text{small enough.}$$

Since $\left|\int_\Omega v_\delta^2 \,\mathrm{d}x - \int_\Omega v^2 \,\mathrm{d}x\right| < C\delta$ by the estimate $\|v - v_\delta\|_{H_0^1(\Omega)} < \delta$, it follows that

$$\limsup_{\varepsilon \to 0} \left|\int_\Omega v^2 \,\mathrm{d}\lambda_\varepsilon^h - \int_\Omega v^2 \,\mathrm{d}x\right| \le C\delta$$

for an arbitrary $\delta > 0$, where the constant C does not depend on δ. This completes the proof of (6.114). □

Proposition 6.60. *Let $\left\{v_\varepsilon \in H_0^1(\Omega)\right\}_{\varepsilon > 0}$ be a sequence such that*

$$v_\varepsilon \rightharpoonup v \quad \text{in} \quad H_0^1(\Omega) \ \text{as} \ \varepsilon \to 0. \tag{6.115}$$

Then

$$v_\varepsilon \to v \quad \text{in} \quad L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h) \quad \text{as} \quad \varepsilon \to 0. \tag{6.116}$$

Proof. By strong convergence in $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$ (see Proposition 6.16) and the property of lower semicontinuity (see Lemma 6.19), to establish convergence (6.116) it is enough to show that

$$\lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^h=\int_\Omega v\varphi\,\mathrm{d}x,\quad \forall\,\varphi\in C_0^\infty(\Omega),\tag{6.117}$$

$$\lim_{\varepsilon\to 0}\int_\Omega |v_\varepsilon|^2\,\mathrm{d}\lambda_\varepsilon^h\le\int_\Omega v^2\,\mathrm{d}x.\tag{6.118}$$

Let v_δ be an element of $C_0^\infty(\Omega)$ such that $\|v-v_\delta\|_{H_0^1(\Omega)}<\delta$ for a given $\delta>0$. Then

$$\begin{aligned}\int_\Omega v_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^h-\int_\Omega v\varphi\,\mathrm{d}x&=\int_\Omega (v_\varepsilon-v)\varphi\,\mathrm{d}\lambda_\varepsilon^h+\int_\Omega (v-v_\delta)\varphi\,\mathrm{d}\lambda_\varepsilon^h\\&\quad+\left(\int_\Omega v_\delta\varphi\,\mathrm{d}\lambda_\varepsilon^h-\int_\Omega v_\delta\varphi\,\mathrm{d}x\right)\\&\quad+\int_\Omega (v_\delta-v)\varphi\,\mathrm{d}x.\end{aligned}\tag{6.119}$$

Applying Hölder inequality and (6.112), we can estimate the second and fourth terms in (6.119) as follows:

$$\left|\int_\Omega (v-v_\delta)\varphi\,\mathrm{d}\lambda_\varepsilon^h\right|\le C_\delta\delta,\quad \left|\int_\Omega (v_\delta-v)\varphi\,\mathrm{d}x\right|\le C_\delta\delta.$$

It follows from (6.115) and the facts that

$$\begin{aligned}\int_\Omega |v_\varepsilon-v|^2\,\mathrm{d}\lambda_\varepsilon^h&\le C_0\int_{\mathbb{R}^2}\left((v_\varepsilon-v)^2+\varepsilon^2|\nabla(v_\varepsilon-v)|^2\right)\mathrm{d}x\\&=C_0\int_{\mathbb{R}^2}(v_\varepsilon-v)^2\,\mathrm{d}x+O(\varepsilon^2)=o(1),\\\int_\Omega v_\delta\varphi\,\mathrm{d}\lambda_\varepsilon^h&\longrightarrow\int_\Omega v_\delta\varphi\,\mathrm{d}x\ \text{due to the convergence}\ \ \mathrm{d}\lambda_\varepsilon^h\rightharpoonup\mathrm{d}x\end{aligned}\tag{6.120}$$

that the limits of the first and third terms in (6.119) as $\varepsilon\to 0$ are equal to 0. Hence,

$$\limsup_{\varepsilon\to 0}\left(\int_\Omega v_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^h-\int_\Omega v\varphi\,\mathrm{d}x\right)\le C_\delta\delta.$$

Since $\delta>0$ is an arbitrary value, (6.117) is proved. As for the validity of (6.117), we apply (6.120) and (6.114). One gets

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega |v_\varepsilon|^2\,\mathrm{d}\lambda_\varepsilon^h&\le\lim_{\varepsilon\to 0}\int_\Omega |v_\varepsilon^2-v^2|\,\mathrm{d}\lambda_\varepsilon^h+\lim_{\varepsilon\to 0}\int_\Omega v^2\,\mathrm{d}\lambda_\varepsilon^h\\&\le C\lim_{\varepsilon\to 0}\|v_\varepsilon-v\|_{L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)}+\int_\Omega v^2\,\mathrm{d}x=\int_\Omega v^2\,\mathrm{d}x.\end{aligned}$$

This concludes the proof. □

We are now in the position to establish the compactness principle for the strong convergence in $L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)$.

Theorem 6.61. *Let* $\{v_{\varepsilon,h} \in C_0^\infty(\Omega)\}_{\varepsilon\to 0}$ *be a sequence such that*

$$\sup_{\varepsilon\to 0} \|v_{\varepsilon,h}\|_{L^2(\Omega)} < +\infty, \quad \sup_{\varepsilon\to 0} \|\nabla v_{\varepsilon,h}\|_{\mathbf{L}^2(\Omega)} < +\infty.$$

Then this sequence is relatively compact with respect to the strong convergence in $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$.

Proof. Due to the Banach–Alaoglu theorem, there exist a subsequence of $\{v_{\varepsilon,h} \in C_0^\infty(\Omega)\}_{\varepsilon\to 0}$ (still indexed by ε) and an element $v \in H_0^1(\Omega)$ such that $v_\varepsilon \rightharpoonup v$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$. Then by Proposition 6.60, we come to the required conclusion. □

To end this section, we cite a couple of results which turn out to be useful for applications.

Lemma 6.62. *If we replace* $H_0^1(\Omega)$ *by* $H^1(\Omega)$ *in (6.114) and Proposition 6.60, then the conclusions of Propositions 6.59 and 6.60 remain valid.*

Proof. Taking into account Lemma 1.1 in [258], we have that the weak convergence of measures $\mathrm{d}\lambda_\varepsilon^h \rightharpoonup \mathrm{d}x$ possesses the property

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi \, \mathrm{d}\lambda_\varepsilon^h = \int_\Omega \varphi \, \mathrm{d}x, \quad \forall\, \varphi \in C(\overline{\Omega}). \tag{6.121}$$

It follows that it is enough to consider $v_\delta \in C^\infty(\overline{\Omega})$ to deduce (6.114) for $v \in H^1(\Omega)$. The remaining reasoning is preserved. The lemma is proved. □

Lemma 6.63. *If* $v_\varepsilon \to v$ *in* $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$ *and* $v \in H^1(\Omega)$, *then*

$$\lim_{\varepsilon\to 0} \int_\Omega |v_\varepsilon - v|^2 \, \mathrm{d}\lambda_\varepsilon^h = 0.$$

Proof. We note that due to Lemma 6.62, the still sequence $\{g_\varepsilon = v \in H^1(\Omega)\}$ is strongly convergent to v in $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$. Therefore, $(v_\varepsilon - v) \to 0$ in $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$. Hence, to obtain the required conclusion, it remains only to use the criterium of strong convergence in $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$ (see Lemma 6.19). □

6.8 On approximative properties of Hilbert spaces with respect to a periodic Borel measure μ^h

In this section, we discuss the approximation properties connecting a periodic Borel measure μ^h with its weak-limit measure μ as $h \to 0$. In the case $p = 2$, Zhikov showed that (see [257]) these properties are necessary for the averaging of a wide class of variational problems in $W_0^{1,2}(\Omega, \mathrm{d}\mu_\varepsilon^h)$ with two small parameters arising in thin periodic structures with thickness tending to 0.

To make this consideration more precise, let us assume that μ is a periodic Borel measure on $\mathbb{R}^n$, $\square = [0,1)^n$ is its periodicity cell and $\int_\square \mathrm{d}\mu = 1$.

We associate with the measure μ the Sobolev space $W^{1,2}_{per}(\square, \mathrm{d}\mu)$ defined in Definition 2.48 as the collection of the first components u of the closure in $L^p(\square, \mathrm{d}\mu) \times \mathbf{L}^p(\square, \mathrm{d}\mu)$ of the set $\big\{(u, \nabla u) \ : \ \forall\, u \in C^\infty(\overline{\square})\big\}$. Hereinafter, we will use the following standard notation for this space: $H^1_{per}(\square, \mathrm{d}\mu)$. Let μ^h be a $\square$-periodic normalized Borel measure depending on the parameter h such that $\int_\square \mathrm{d}\mu^h = 1$ for all h and suppose μ^h converges weakly to the Borel measure μ as $h \to 0$.

Now, we formulate the approximation conditions necessary for the asymptotic analysis of the variational problems with two small parameters.

Definition 6.64. *We say that the family of $\square$-periodic normalized Borel measures $\big\{\mu^h\big\}_{h>0}$ with limit property $\mu^h \rightharpoonup \mu$ as $h \to 0$ possesses the approximative conditions if the following hold:*

(a) For any function $a \in C^\infty_{per}(\square)$ with 0 mean value

$$\left(\langle a \rangle = \int_\square a \, \mathrm{d}\mu = 0\right),$$

there are a function $a^h \in L^2(\square, \mathrm{d}\mu^h)$ and a vector $\mathbf{b}^h \in \mathbf{L}^2(\square, \mathrm{d}\mu^h)$ such that

$$\mathrm{div}^{\mu^h} \mathbf{b}^h = a^h \iff \int_\square a^h \varphi \, \mathrm{d}\mu^h = -\int_\square \mathbf{b}^h \cdot \nabla\varphi \, \mathrm{d}\mu^h \quad \forall\, \varphi \in C^\infty_{per}(\square),$$
$$a^h \longrightarrow a \ \ in \ \ L^2(\square, \mathrm{d}\mu^h), \quad \mathbf{b}^h \longrightarrow \mathbf{b} \ \ in \ \ \mathbf{L}^2(\square, \mathrm{d}\mu^h).$$

(b) For any vector $\mathbf{b} \in V^2_{sol}(\square, \mathrm{d}\mu)$, there is a vector $\mathbf{b}^h \in V^2_{sol}(\square, \mathrm{d}\mu^h)$ such that $\mathbf{b}^h \to \mathbf{b}$ in $\mathbf{L}^2(\square, \mathrm{d}\mu^h)$.

It should be emphasized that the main point of these approximative conditions is to apply the strong convergence in variable spaces $L^2(\square, \mathrm{d}\mu^h)$ and $\mathbf{L}^2(\square, \mathrm{d}\mu^h)$. Note also that the approximative properties are closely connected with the possibility of passing to the limit in the variable Sobolev space $H^1_{per}(\square, \mathrm{d}\mu^h)$. This means that if $u^h \in H^1_{per}(\square, \mathrm{d}\mu^h)$ for all $h > 0$ and

$$u^h \rightharpoonup u \ \ \text{in} \ \ L^2(\square, \mathrm{d}\mu^h), \quad \nabla^{\mu^h} u^h \rightharpoonup \mathbf{v} \ \ \text{in} \ \ \mathbf{L}^2(\square, \mathrm{d}\mu^h), \tag{6.122}$$

then

$$u \in H^1_{per}(\square, \mathrm{d}\mu) \ \ \text{and} \ \ \mathbf{v} = \nabla^\mu u. \tag{6.123}$$

As was shown in [257] if the limit measure μ satisfies the Poincaré inequality

$$\int_\square \varphi^2 \, \mathrm{d}\mu \le C \int_\square |\nabla\varphi|^2 \, \mathrm{d}\mu, \quad \varphi \in C^\infty_{per}(\square), \ \int_\square \varphi \, \mathrm{d}\mu = 0,$$

then the approximate conditions (a) and (b) guarantee the passage to the limit in the Sobolev space $H^1_{per}(\square, \mathrm{d}\mu^h) = W^{1,2}_{per}(\square, \mathrm{d}\mu^h)$.

The main result of this section can be stated as follows (see [230]):

Theorem 6.65. *Assume that the Poincaré inequality*

$$\int_{\square} |\varphi|^2 \,\mathrm{d}\mu^h \le C \int_{\square} |\nabla\varphi|^2 \,\mathrm{d}\mu^h, \quad \varphi \in C^\infty_{per}(\square), \ \int_{\square} \varphi \,\mathrm{d}\mu^h = 0 \tag{6.124}$$

holds true uniformly with respect to the parameter h and that the passage to the limit in the variable Sobolev space $H^1_{per}(\square, \mathrm{d}\mu^h)$ is possible. Then the approximative conditions (a) and (b) are valid.

Proof. Let us prove (a). For any function $a \in C^\infty_{per}(\square)$ such that $\int_{\square} a \,\mathrm{d}\mu = 0$, we take the sequence $\left\{ a^h = a - \int_{\square} a \,\mathrm{d}\mu^h \right\}_{h>0}$. Then, taking into account the weak convergence $\mu^h \rightharpoonup \mu$ and Lemma 6.62, we have

$$\begin{aligned} \lim_{h\to 0} \int_{\square} a^h \varphi \,\mathrm{d}\mu^h &= \lim_{h\to 0} \int_{\square} a\varphi \,\mathrm{d}\mu^h \\ &\quad - \lim_{h\to 0} \int_{\square} a \,\mathrm{d}\mu^h \lim_{h\to 0} \int_{\square} \varphi \,\mu^h \\ &= \int_{\square} a\varphi \,\mathrm{d}\mu, \quad \forall\, \varphi \in C^\infty_{per}(\square), \end{aligned}$$

$$\|a^h\|^2_{L^2(\square,\mathrm{d}\mu^h)} = \|a\|^2_{L^2(\square,\mathrm{d}\mu^h)} - \left(\int_{\square} a \,\mathrm{d}\mu^h \right)^2 \to \|a\|^2_{L^2(\square,\mathrm{d}\mu)}.$$

Hence, due to Lemma 6.19, we conclude that $a^h \to a$ strongly in $L^2(\Omega, \mathrm{d}\mu^h)$.

Let us consider the following parameterized periodic problem: Find a pair $(u^h, \nabla^{\mu^h} u^h)$ such that

$$u^h \in H^1_{per}(\square, \mathrm{d}\mu^h), \quad \nabla^{\mu^h} u^h \in \Gamma_2^{\mu^h}(u), \tag{6.125}$$

$$\mathrm{div}^{\mu^h} \left(\nabla u^h \right) = a^h. \tag{6.126}$$

The variational formulation of this problem is

$$\int_{\square} \nabla^{\mu^h} u^h \cdot \nabla\varphi \,\mathrm{d}\mu^h = \int_{\square} a^h \varphi \,\mathrm{d}\mu^h, \quad \forall\, \varphi \in C^\infty_{per}(\square). \tag{6.127}$$

It is well known that this problem is solvable [261]. On the other hand, the uniform Poincaré inequality (6.124) implies that $\left\{u^h\right\}_{h>0}$ and $\left\{\nabla^{\mu^h} u^h\right\}_{h>0}$ are bounded in $L^2(\square, \mathrm{d}\mu^h)$ and $\mathbf{L}^2(\square, \mathrm{d}\mu^h)$, respectively. Without loss of generality, we can assume that there exists $u \in H^1_{per}(\square, \mathrm{d}\mu)$ such that conditions (6.122) and (6.123) hold true. Let us show that $\nabla^{\mu^h} u^h \to \nabla^{\mu} u$ strongly in $\mathbf{L}^2(\square, \mathrm{d}\mu^h)$.

For this, we pass to the limit in (6.127). As a result, we obtain

$$\int_{\square} \nabla u \cdot \nabla\varphi \,\mathrm{d}\mu = \int_{\square} a\varphi \,\mathrm{d}\mu, \quad \forall\, \varphi \in C^\infty_{per}(\square). \tag{6.128}$$

Now, setting the test function $\varphi = u^h$ in (6.127) and using the strong convergence $a^h \to a$, we get

$$\lim_{h\to 0} \int_{\square} \nabla^{\mu} u^h \cdot \nabla u^h \, \mathrm{d}\mu^h = \int_{\square} au \, \mathrm{d}\mu.$$

If we compare this identity with (6.128) for $\varphi = u$, we obtain

$$\lim_{h\to 0} \int_{\square} |\nabla^{\mu^h} u^h|^2 \mathrm{d}\mu^h = \int_{\square} |\nabla^{\mu} u|^2 \, \mathrm{d}\mu.$$

This together with (6.122) implies the strong convergence $\nabla^{\mu^h} u^h \to \nabla^{\mu} u$ in $\mathbf{L}^2(\square, \mathrm{d}\mu^h)$. Setting $\mathbf{b}^h = \nabla^{\mu^h} u^h$ and $\mathbf{b} = \nabla^{\mu} u$, we deduce the validity of the approximative property (a).

Now we prove (b). Let $\mathbf{b} \in V^2_{sol}(\square, \mathrm{d}\mu)$ be any fixed vector and let $\{\tilde{\mathbf{b}}^h \in \mathbf{L}^2(\square, \mathrm{d}\mu^h)\}_{h>0}$ be a sequence such that $\tilde{\mathbf{b}}^h \to \mathbf{b}$ in $\mathbf{L}^2(\square, \mathrm{d}\mu^h)$. For every $h > 0$, we consider the variational problem

$$u^h \in H^1_{per}(\square, \mathrm{d}\mu^h), \quad \mathrm{div}^{\mu^h} \nabla u^h = \mathrm{div}^{\mu^h} \tilde{\mathbf{b}}^h.$$

Then the integral identity

$$\int_{\square} \nabla^{\mu^h} u^h \cdot \nabla \varphi \, \mathrm{d}\mu^h = \int_{\square} \tilde{\mathbf{b}}^h \cdot \nabla \varphi \, \mathrm{d}\mu^h, \quad \forall\, \varphi \in C^{\infty}_{per}(\square) \tag{6.129}$$

and the uniform Poincaré inequality imply that $\{u^h\}_{h>0}$ and $\{\nabla^{\mu^h} u^h\}_{h>0}$ are bounded sequences in $L^2(\square, \mathrm{d}\mu^h)$ and $\mathbf{L}^2(\square, \mathrm{d}\mu^h)$, respectively. By the initial suppositions,

$$\nabla^{\mu^h} u^h \rightharpoonup \nabla^{\mu} u \ \text{ in } \ \mathbf{L}^2(\square, \mathrm{d}\mu^h) \quad \text{and} \quad u \in H^1_{per}(\square, \mathrm{d}\mu).$$

Therefore, in view of the fact that $\mathbf{b} \in V^2_{sol}(\square, \mathrm{d}\mu)$ and $\nabla^{\mu} u \in V^2_{pot}(\square, \mathrm{d}\mu)$, we obtain

$$\begin{aligned} \lim_{h\to 0} \int_{\square} \nabla^{\mu^h} u^h \cdot \nabla^{\mu^h} u^h \, \mathrm{d}\mu^h &= \lim_{h\to 0} \int_{\square} \tilde{\mathbf{b}}^h \cdot \nabla^{\mu^h} u^h \, \mathrm{d}\mu^h \\ &= \int_{\square} \mathbf{b} \cdot \nabla^{\mu} u \, \mathrm{d}\mu = 0. \end{aligned} \tag{6.130}$$

To obtain the desired property (b) it remains to set $\mathbf{b}^h = \tilde{\mathbf{b}}^h - \nabla^{\mu^h} u^h$. Indeed, in this case, we have the following:

(i) $\displaystyle\int_{\square} \mathbf{b}^h \cdot \nabla \varphi \, \mathrm{d}\mu^h = 0$ by (6.129). Hence, $\mathbf{b}^h \in V^2_{sol}(\square, \mathrm{d}\mu)$.

(ii) Since $\nabla^{\mu^h} u^h \rightharpoonup \nabla^{\mu} u$ in $\mathbf{L}^2(\square, \mathrm{d}\mu^h)$ and

$$0 = \lim_{h\to 0} \int_{\square} |\nabla^{\mu^h} u^h|^2 \,\mathrm{d}\mu^h \geq \int_{\square} |\nabla^{\mu} u|^2 \,\mathrm{d}\mu$$

(by (6.130)), it follows that $\nabla^{\mu} u = 0$. Hence, $\mathbf{b}^h \rightharpoonup \mathbf{b}$ in $\mathbf{L}^2(\square, \mathrm{d}\mu^h)$. Moreover, due to the equality

$$\begin{aligned}\lim_{h\to 0} \|\mathbf{b}^h\|^2_{\mathbf{L}^2(\square,\mathrm{d}\mu^h)} &= \lim_{h\to 0} \|\tilde{\mathbf{b}}^h\|^2_{\mathbf{L}^2(\square,\mathrm{d}\mu^h)} + 2 \lim_{h\to 0} \int_{\square} \tilde{\mathbf{b}}^h \cdot \nabla^{\mu^h} u^h \,\mathrm{d}\mu^h \\ &+ \lim_{h\to 0} \|u^h\|^2_{\mathbf{L}^2(\square,\mathrm{d}\mu^h)} = \lim_{h\to 0} \|\tilde{\mathbf{b}}^h\|^2_{\mathbf{L}^2(\square,\mathrm{d}\mu^h)} \\ &= \|\mathbf{b}\|^2_{\mathbf{L}^2(\square,\mathrm{d}\mu)},\end{aligned}$$

we conclude that $\mathbf{b}^h \to \mathbf{b}$ in $\mathbf{L}^2(\square, \mathrm{d}\mu^h)$.

This concludes the proof. □

6.9 The homothetic mean value property on periodically perforated domains

In this section, we study the limit properties of one class of periodic functions defined on ε-periodically perforated domain as ε tends to 0. The principal feature is the fact that the holes have a critical size with respect to ε-sized mesh of periodicity. We will use these results in Chap. 12 to the construction of suboptimal controls for a class of boundary OCPs in ε-periodically perforated domains with small holes.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open domain and let ε be a small positive parameter. We define a perforated domain Ω_ε as follows: Let $Y = [-1/2, +1/2)^n$; Q and K be compact subsets of Y such that $0 \in \mathrm{int} K \cap \partial Q$,

$$\begin{aligned}\Theta_\varepsilon &= \{\mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n \ : \ (\varepsilon Y + \varepsilon\,\mathbf{k}) \cap \Omega \neq \emptyset\}, \\ Y_\varepsilon &= \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \{\varepsilon(Y + \mathbf{k})\}, \qquad T_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \left\{\varepsilon^{n/(n-1)} Q + \varepsilon\,\mathbf{k}\right\}, \\ S_\varepsilon &= \begin{cases} \varepsilon^{n/(n-2)} K \cap \partial(\varepsilon^{n/(n-1)} Q), & n \geq 3, \\ \exp(-1/\varepsilon^2) K \cap \partial(\varepsilon^{n/(n-1)} Q), & n = 2, \end{cases} \\ \Gamma_\varepsilon^D &= \left[\bigcup_{\mathbf{k}\in\Theta_\varepsilon} \{S_\varepsilon + \varepsilon\,\mathbf{k}\}\right] \cap \overline{\Omega}, \qquad \Gamma_\varepsilon^N = \left[\partial T_\varepsilon \setminus \Gamma_\varepsilon^D\right] \cap \overline{\Omega}.\end{aligned}$$

Then we set $\Omega_\varepsilon = \Omega \setminus T_\varepsilon$. As was mentioned earlier, the principal feature of this perforated domain is the fact that the size of the holes $Q_\varepsilon + \varepsilon\,\mathbf{k}$ tends to 0 as $\varepsilon \to 0$ and their boundaries Γ_ε^D, Γ_ε^N are not proportional to the size of the periodicity cell εY.

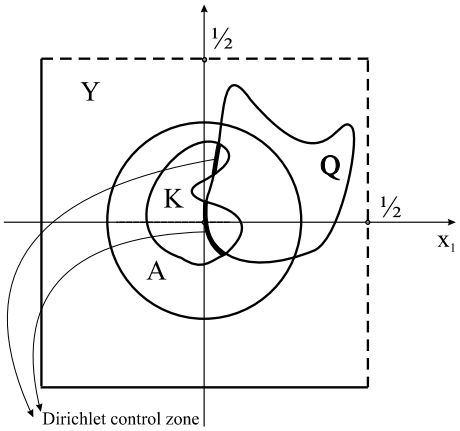


Fig. 6.2. Example of perforation scheme

Our aim is to study the limit properties of the classes of periodic functions

$$\left\{\int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\,\lambda(\varepsilon)}\right)\,\mathrm{d}\mu_\varepsilon^{\lambda,h}\right\}, \qquad \left\{\int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\,h(\varepsilon)}\right)\,\mathrm{d}\nu_\varepsilon^{\lambda,h}\right\}$$

as ε tends to 0 and to obtain an explicit form of these limits. Here,

$$0 < \lambda << h < 1, \quad \lim_{\varepsilon\to 0} h(\varepsilon) = 0, \quad \lim_{\varepsilon\to 0} \lambda(\varepsilon) = 0,$$

and $\mu_\varepsilon^{\lambda,h}$ and $\nu_\varepsilon^{\lambda,h}$ are some singular periodic Borel measures concentrated on the manifolds

$$\Gamma^{\lambda,h} = \lambda K \cap \partial(h\,Q) \quad \text{and} \quad \Lambda^{\lambda,h} = \partial(h\,Q) \setminus \Gamma^{\lambda,h},$$

respectively.

Throughout this section, we suppose that Ω is a measurable set in the sense of Jordan; the small parameter ε varies in a strictly decreasing sequence of positive numbers which converges to 0; the compact set Q has Lipschitz boundary ∂Q, $\operatorname{int} Q$ is a strongly connected set, $Q \subset \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1 \geq 0\}$ and its boundary ∂Q contains the origin; $A = B(\mathbf{0}, r_0)$ is an open ball centered at the origin with a radius $r_0 < 1/2$, so that $A \subset\subset Y$ and $K \subset\subset A$ (see Fig. 6.2 for a 2D example). For any subset $E \subset \Omega$, we denote by $|E|$ its n-dimensional Lebesgue measure $\mathcal{L}_n(E)$, whereas $|\partial E|_H$ denotes the $(n-1)$-dimensional Hausdorff measure of the manifold ∂E on $\mathbb{R}^n$. We always suppose that the sets $K \cap \partial Q^\varsigma$ and $\partial Q \setminus (K \cap \partial Q^\varsigma)$ have nonzero capacity for any $\varsigma > 1$, where $Q^\varsigma = \{\varsigma x, \ \forall\, x = (x_1, \ldots, x_n) \in Q\}$ is the homothetic stretching of Q by a factor of ς. Hence, $|K \cap \partial Q^\varsigma|_H \neq 0$ for all $\varsigma > 1$.

6.9.1 A measure approach to the description of the sets Ω_ε

We will describe of the geometry of the perforated domain Ω_ε in terms of singular periodic Borel measures on $\mathbb{R}^n$. Let us denote by K^λ and Q^h the homothetic contractions of the sets K and Q by factors λ^{-1} and h^{-1}, respectively. In what follows, it is always assumed that $0 < \lambda << h < 1$. Let the sets $\Gamma^{\lambda,h}$ and $\Lambda^{\lambda,h}$ be defined as follows:

$$\Gamma^{\lambda,h} = K^\lambda \cap \partial Q^h, \quad \Lambda^{\lambda,h} = \partial Q^h \setminus \Gamma^{\lambda,h}. \tag{6.131}$$

Let $\mu^{\lambda,h}$ and $\nu^{\lambda,h}$ be the normalized periodic Borel measures on $\mathbb{R}^n$ with the periodicity cell Y such that $\mu^{\lambda,h}$ is concentrated on $\Gamma^{\lambda,h}$, $\nu^{\lambda,h}$ is concentrated on $\Lambda^{\lambda,h}$ and both of these measures are proportional to the $(n-1)$-dimensional Hausdorff measure. Since these measures are concentrated and uniformly distributed on the corresponding sets, it follows that $\mu^{\lambda,h}(Y \setminus \Gamma^{\lambda,h}) = 0$.

For any function $\varphi \in C^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{\Gamma^{\lambda,h}} \varphi \, \mathrm{d}\mathcal{H}^{n-1} &= |\Gamma^{\lambda,h}|_H \int_Y \varphi \, \mathrm{d}\mu^{\lambda,h} \\ &= \lambda^{n-1} |K \cap \partial Q^{h/\lambda}|_H \int_Y \varphi \, \mathrm{d}\mu^{\lambda,h},\end{aligned} \tag{6.132}$$

$$\begin{aligned}\int_{\Lambda^{\lambda,h}} \varphi \, \mathrm{d}\mathcal{H}^{n-1} &= |\Lambda^{\lambda,h}|_H \int_Y \varphi \, \mathrm{d}\nu^{\lambda,h} \\ &= \left(|\partial Q^h|_H - |\Gamma^{\lambda,h}|_H\right) \int_Y \varphi \, \mathrm{d}\nu^{\lambda,h} \\ &= \left(h^{n-1}|\partial Q|_H - \lambda^{n-1}|K \cap \partial Q^{h/\lambda}|_H\right) \int_Y \varphi \, \mathrm{d}\nu^{\lambda,h}.\end{aligned} \tag{6.133}$$

Let δ be the periodic Dirac measure on $\mathbb{R}^n$ concentrated at nodes of the grid $\mathbb{Z}^n$ and such that $\delta(\mathbf{k}) = 1$ for all $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$. It is clear that the measures $\mu^{\lambda,h}$ and $\nu^{\lambda,h}$ converge weakly to δ as $\max(\lambda, h) \to 0$ (in symbols, $\mu^{\lambda,h} \rightharpoonup \delta$, $\nu^{\lambda,h} \rightharpoonup \delta$), that is,

$$\left.\begin{aligned}\lim_{\max(\lambda,h)\to 0} \int_Y \varphi \, \mathrm{d}\mu^{\lambda,h} = \int_Y \varphi \, \mathrm{d}\delta = \varphi(0), \\ \lim_{\max(\lambda,h)\to 0} \int_Y \varphi \, \mathrm{d}\nu^{\lambda,h} = \int_Y \varphi \, \mathrm{d}\delta = \varphi(0)\end{aligned}\right\}$$

for every $\varphi \in C^\infty_{per}(Y)$.

We also introduce the scaling measures $\mu_\varepsilon^{\lambda,h}$ and $\nu_\varepsilon^{\lambda,h}$ by setting

$$\mu_\varepsilon^{\lambda,h}(B) = \varepsilon^n \mu^{\lambda,h}(\varepsilon^{-1}B), \quad \nu_\varepsilon^{\lambda,h}(B) = \varepsilon^n \nu^{\lambda,h}(\varepsilon^{-1}B) \tag{6.134}$$

for every Borel set $B \subset \mathbb{R}^n$ and relate the parameters λ, h and ε by the rule

$$h(\varepsilon) = \varepsilon^{n/(n-1)}, \quad \lambda(\varepsilon) = \begin{cases} \varepsilon^{n/(n-2)} & \text{if } n \geq 3, \\ \exp(-\varepsilon^{-2}) & \text{if } n = 2. \end{cases} \tag{6.135}$$

Then

$$\int_{\varepsilon Y} \mathrm{d}\mu_\varepsilon^{\lambda,h} = \varepsilon^n \int_Y \mathrm{d}\mu^{\lambda,h} = \varepsilon^n, \quad \int_{\varepsilon Y} \mathrm{d}\nu_\varepsilon^{\lambda,h} = \varepsilon^n \int_Y \mathrm{d}\nu^{\lambda,h} = \varepsilon^n.$$

This means that the measures $\mu_\varepsilon^{\lambda,h}$ and $\nu_\varepsilon^{\lambda,h}$ weakly converge to the Lebesgue measure:

$$\mathrm{d}\mu_\varepsilon^{\lambda,h} \rightharpoonup \mathrm{d}x, \qquad \mathrm{d}\nu_\varepsilon^{\lambda,h} \rightharpoonup \mathrm{d}x,$$

that is, for every $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu_\varepsilon^{\lambda,h} = \int_{\mathbb{R}^n} \varphi \, \mathrm{d}x, \quad \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\nu_\varepsilon^{\lambda,h} = \int_{\mathbb{R}^n} \varphi \, \mathrm{d}x. \tag{6.136}$$

Note that, in this case, we have the representation

$$\begin{aligned} \Gamma_\varepsilon^D &= \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \left[K^{\lambda(\varepsilon)} \cap \partial Q^{h(\varepsilon)} + \varepsilon \mathbf{k} \right], \\ \Gamma_\varepsilon^N &= \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \left[\partial Q^{h(\varepsilon)} \setminus (K^{\lambda(\varepsilon)} \cap \partial Q^{h(\varepsilon)}) + \varepsilon \mathbf{k} \right]. \end{aligned}$$

At the end of this subsection, we cite some auxiliary results that we feel are interesting per se. Let

$$\sigma(\varepsilon) = \left(h(\varepsilon)^{n-1} |\partial Q|_H - \lambda(\varepsilon)^{n-1} |K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H \right), \tag{6.137}$$

$$\varsigma(\varepsilon) = h(\varepsilon)/\lambda(\varepsilon) = \left\{ \begin{array}{ll} \exp\left(\dfrac{-n}{n^2 - 3n + 2} \ln \varepsilon \right) & \text{if } \ n \geq 3, \\ \varepsilon^2 \exp\left(\dfrac{1}{\varepsilon^2} \right) & \text{if } \ n = 2. \end{array} \right\}$$

Then

$$\varsigma(\varepsilon) \in (1, +\infty), \ \forall\, \varepsilon > 0, \ \text{ and } \ \lim_{\varepsilon\to 0} \varsigma(\varepsilon) = +\infty.$$

We are interested in the limit behavior of the sequence $\{|K \cap \partial Q^\varsigma(\varepsilon)|_H\}$ as $\varepsilon \to 0$. We recall that each of the sets $Q^\varsigma(\varepsilon) = \{\varsigma(\varepsilon)x, \ \forall\, x = (x_1, \ldots, x_n) \in Q\}$ is the homothetic stretching of Q by a factor of $\varsigma(\varepsilon)$.

Proposition 6.66. *There exist an open cone*

$$\Lambda \subset \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1 > 0\}$$

such that

$$\lim_{\varepsilon\to 0} |K \cap \partial Q^\varsigma(\varepsilon)|_H = |K \cap \partial \Lambda|_H.$$

Proof. Indeed, by initial assumptions, the origin is a Lipschitz point of the boundary ∂Q and $\operatorname{int} Q$ is a strongly connected set in classical sense. Hence, there is a neighborhood $\mathcal{U}(0)$ such that $\mathcal{U}(0) \cap \operatorname{int} Q$ is a convex set [53, 106]. Hence, the set

$$\Lambda = \{x \in \mathbb{R}^n \ : \ x \in t\,[\mathcal{U}(0) \cap \operatorname{int} Q] \ \ \forall\, t \in (0, +\infty)\}$$

is a nonempty open cone.

Assume that the origin is a Lipschitz point of ∂Q, but does not belong to a smooth part of the boundary ∂Q. Then the inclusion $K \cap \Lambda \subset K \cap Q^{\varsigma(\varepsilon)}$ holds true for ε small enough, and it immediately implies the existence of a value $\varepsilon_0 > 0$ such that

$$|K \cap \partial\Lambda|_H = |K \cap \partial Q^{\varsigma}(\varepsilon)|_H, \quad \forall\, \varepsilon < \varepsilon_0.$$

If a part of the boundary ∂Q containing the origin is smooth, it follows that there is a neighborhood $\mathcal{U}(0)$ of the origin such that $\mathcal{U}(0) \cap \partial Q$ is a graph of a smooth function whose epigraph contains $\mathcal{U}(0) \cap Q$. Thus, we may always suppose that there is a function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}_{\geq}$ such that $\Psi \in C_0^{\infty}(\mathbb{R}^{n-1})$ and $x_1 = \Psi(x_2, \ldots, x_n)$ for every $x = (x_1, x_2, \ldots, x_n) \in \mathcal{U}(0) \cap \partial Q$.

Let $\Lambda = \{x \in \mathbb{R}^n \ : \ x_1 > 0\}$. Then $\partial\Lambda = \{x \in \mathbb{R}^n \ : \ x_1 = 0\}$ and, taking into account the homothetic transform, we have that

$$x = (x_1, x_2, \ldots, x_n) \in K \cap \varsigma(\varepsilon)\partial Q$$

for small enough ε if and only if $x \in \varsigma(\varepsilon)\,(\mathcal{U}(0) \cap \partial Q)$. Hence,

$$x_1 = \frac{1}{\varsigma(\varepsilon)}\Psi(\varsigma(\varepsilon)x_2, \ldots, \varsigma(\varepsilon)x_n) \quad \text{ for every } \quad x \in K \cap \varsigma(\varepsilon)\partial Q.$$

Since $\lim_{\varepsilon \to 0} \varsigma^{-1}(\varepsilon)\Psi(\varsigma(\varepsilon)x_2, \ldots, \varsigma(\varepsilon)x_n) = 0$ by definition of the Hausdorff measure $\mathcal{H}^{n-1}$, we immediately obtain the required result. □

Remark 6.67. As follows from the proof of the last proposition, the cone Λ can be recovered in an explicit form for the case when the origin belongs to a smooth part of the boundary ∂Q. Moreover, as an evident consequence of Proposition 6.66 and (6.137), we have

$$\lim_{\varepsilon \to 0} \sigma(\varepsilon)/\varepsilon^n = |\partial Q|_H. \tag{6.138}$$

In a similar way, the following statement can be proved.

Proposition 6.68. *Let* $\{\rho'_\varepsilon \in \mathbb{R}\}_{\varepsilon > 0}$ *be a numerical sequence such that*

$$\rho'_\varepsilon = |A \setminus Q^{\varsigma(\varepsilon)}|/|A|, \quad \forall\, \varepsilon > 0. \tag{6.139}$$

Then $\{\rho'_\varepsilon\}_{\varepsilon > 0}$ *is the monotone sequence and there exists a value* $\rho^* \in [1/2, 1)$ *such that* $\lim_{\varepsilon \to 0} \rho'_\varepsilon = \rho^*$.

Remark 6.69. The example in Fig. 6.3 indicates the case when the origin in not a Lipschitz's point of ∂Q, and hence ρ^* can be equal to 1.

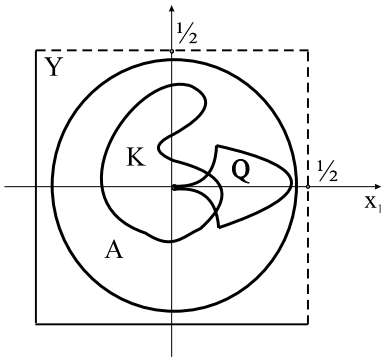


Fig. 6.3. Example of the set Q with non-Lipschitz boundary at the origin

6.9.2 The homothetic mean value property

To begin, let us recall the main types of convergence in variable spaces with respect to the family of periodic Borel measures $\mu_\varepsilon^{\lambda,h}$ that was introduced in (6.134), where the parameters $\lambda = \lambda(\varepsilon)$ and $h = h(\varepsilon)$ are defined by (6.135). Let $\left\{v_\varepsilon^{\lambda,h} \in L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h}\right\}$ be a bounded sequence, that is,

$$\limsup_{\varepsilon\to 0} \int_\Omega (v_\varepsilon^{\lambda,h})^2 \,\mathrm{d}\mu_\varepsilon^{\lambda,h} < +\infty.$$

1. The weak convergence $v_\varepsilon^{\lambda,h} \rightharpoonup v$ in $L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})$ means that

$$v \in L^2(\Omega) \text{ and } \lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon^{\lambda,h}\varphi \,\mathrm{d}\mu_\varepsilon^{\lambda,h} = \int_\Omega v\varphi \,\mathrm{d}x \text{ for any } \varphi \in C_0^\infty(\Omega).$$

2. The strong convergence $v_\varepsilon^{\lambda,h} \to v$ in $L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})$ means that

$$v \in L^2(\Omega) \text{ and } \lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon^{\lambda,h} z_\varepsilon^{\lambda,h} \,\mathrm{d}\mu_\varepsilon^{\lambda,h} = \int_\Omega vz \,\mathrm{d}x$$

   if $z_\varepsilon^{\lambda,h} \rightharpoonup z$ in $L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})$.
3. If a sequence $\left\{b^{\lambda,h} \in L^2(Y, \mathrm{d}\mu^{\lambda,h})\right\}$ is bounded, then the weak convergence $b^{\lambda,h} \rightharpoonup b$ in $L^2(Y, \mathrm{d}\mu^{\lambda,h})$ means that $b \in L^2(Y, \mathrm{d}\delta)$ and

$$\lim_{\max(\lambda,h)\to 0} \int_Y b^{\lambda,h}\psi \,\mathrm{d}\mu^{\lambda,h} = \int_Y b\psi \,\mathrm{d}\delta \quad \text{for any} \quad \psi \in C_{per}^\infty(Y),$$

   where $\mathrm{d}\delta$ is the Y-periodic Dirac measure concentrated at the origin.

We are now in a position to state the main result of this section.

Theorem 6.70. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a Y-periodic function such that $g \in L^2(\partial Q, \mathrm{d}\mathcal{H}^{n-1})$. Then*

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\,\mathrm{d}\nu_\varepsilon^{\lambda,h} = \left(\frac{1}{|\partial Q|_H}\int_{\partial Q} g\,\mathrm{d}\mathcal{H}^{n-1}\right)\int_\Omega \varphi \mathrm{d}x$$

for any $\varphi \in C(\overline{\Omega})$. *In particular,*

$$\lim_{\varepsilon\to 0}\int_\Omega g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\,\mathrm{d}\nu_\varepsilon^{\lambda,h} = |\Omega|\,\left(\frac{1}{|\partial Q|_H}\int_{\partial Q} g\,\mathrm{d}\mathcal{H}^{n-1}\right).$$

Proof. It is evident that we can restrict our attention to the case when $g \geq 0$. Let us partition the set Ω into cubes εY with edges ε and denote these cubes by the symbols εY^j. Then

$$\begin{aligned}\int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\,\mathrm{d}\nu_\varepsilon^{\lambda,h} &= \sum\int_{\varepsilon Y^j} \varphi(x)\, g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\,\mathrm{d}\nu_\varepsilon^{\lambda,h}\\ &+ \sum\int_{\Omega\cap\varepsilon Y^j} \varphi(x)\, g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\,\mathrm{d}\nu_\varepsilon^{\lambda,h}\\ &= \sum \varphi(x_j^{\lambda,h})\int_{\varepsilon Y^j} g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\,\mathrm{d}\nu_\varepsilon^{\lambda,h}\\ &+ \sum\int_{\Omega\cap\varepsilon Y^j} \varphi(x)\, g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\,\mathrm{d}\nu_\varepsilon^{\lambda,h}, \qquad (6.140)\end{aligned}$$

where $x_j^{\lambda,h}$ is a point of the cube εY^j and the second sum is calculated over the set of "boundary" cubes. Taking into account the definition of the scaling measure $\nu_\varepsilon^{\lambda,h}$ and Y-periodicity of g, we have

$$\begin{aligned}\int_{\varepsilon Y^j} g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\,\mathrm{d}\nu_\varepsilon^{\lambda,h} &= \varepsilon^n\int_{\varepsilon Y} g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\,\mathrm{d}\nu^{\lambda(\varepsilon),h(\varepsilon)}\left(\frac{x}{\varepsilon}\right)\\ &= \frac{\varepsilon^n}{\sigma(\varepsilon)}\int_{\Lambda^{\lambda(\varepsilon),h(\varepsilon)}} g\left(\frac{x}{h(\varepsilon)}\right)\,\mathrm{d}\mathcal{H}^{n-1}, \qquad (6.141)\end{aligned}$$

where the set $\Lambda^{\lambda(\varepsilon),h(\varepsilon)}$ is defined by (6.131).

Since $\Lambda^{\lambda(\varepsilon),h(\varepsilon)} = \partial Q^{h(\varepsilon)} \setminus \Gamma^{\lambda(\varepsilon),h(\varepsilon)}$, it follows that

$$\int_{\Lambda^{\lambda(\varepsilon),h(\varepsilon)}} g\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial Q^{h(\varepsilon)}} g\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\Gamma^{\lambda(\varepsilon),h(\varepsilon)}} g\,\mathrm{d}\mathcal{H}^{n-1}.$$

Then due to the definition of the homothetic contraction and (6.135), we have

$$\int_{\partial Q^{h(\varepsilon)}} g\left(\frac{x}{h(\varepsilon)}\right)\,\mathrm{d}\mathcal{H}^{n-1} = h^{n-1}(\varepsilon)\int_{\partial Q} g\,\mathrm{d}\mathcal{H}^{n-1} = \varepsilon^n\int_{\partial Q} g\,\mathrm{d}\mathcal{H}^{n-1}, \quad (6.142)$$

$$\int_{\Gamma^{\lambda(\varepsilon),h(\varepsilon)}} g\left(\frac{x}{h(\varepsilon)}\right)\,\mathrm{d}\mathcal{H}^{n-1} = h^{n-1}(\varepsilon)\int_{h^{-1}(\varepsilon)\Gamma^{\lambda(\varepsilon),h(\varepsilon)}} g\,\mathrm{d}\mathcal{H}^{n-1}. \quad (6.143)$$

Using the definition of the set $\Gamma^{\lambda(\varepsilon),h(\varepsilon)}$ (see (6.131)), we obtain

$$h^{-1}(\varepsilon)\Gamma^{\lambda(\varepsilon),h(\varepsilon)} = K^{\lambda(\varepsilon)/h(\varepsilon)} \cap \partial Q = \lambda(\varepsilon)\left(K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}\right).$$

In view of Proposition 6.66, we have

$$\lim_{\varepsilon\to 0}\left|\lambda(\varepsilon)\left(K\cap\partial Q^{h(\varepsilon)/\lambda(\varepsilon)}\right)\right|_H=|K\cap\partial\Lambda|_H\lim_{\varepsilon\to 0}\lambda^{n-1}(\varepsilon)=0.$$

Thus, combining (6.141)–(6.143), we conclude

$$\int_{\varepsilon Y^j} g\left(\frac{x}{\varepsilon\,h(\varepsilon)}\right)\mathrm{d}\nu_\varepsilon^{\lambda,h}=\frac{\varepsilon^n}{\sigma(\varepsilon)}\varepsilon^n\left(\int_{\partial Q} g\,\mathrm{d}\mathcal{H}^{n-1}+J(\varepsilon)\right),\tag{6.144}$$

$$J(\varepsilon)\le\|g\|_{L^2(\partial Q,d\mathcal{H}^{n-1})}|K^{\lambda(\varepsilon)/h(\varepsilon)}\cap\partial Q|_H^{1/2}\overset{\varepsilon\to 0}{\longrightarrow}0.\tag{6.145}$$

Substituting (6.144) and (6.145) into (6.140), we have

$$\begin{aligned}\left|\int_\Omega\varphi(x)g\left(\frac{x}{\varepsilon\,h(\varepsilon)}\right)\mathrm{d}\nu_\varepsilon^{\lambda,h}-\frac{\varepsilon^n}{\sigma(\varepsilon)}\left(\int_{\partial Q} g\,\mathrm{d}\mathcal{H}^{n-1}\right)\sum\varphi(x_j^{\lambda,h})\,\varepsilon^n\right|&\\ \le\frac{\varepsilon^n}{\sigma(\varepsilon)}J(\varepsilon)\sum\varphi(x_j^{\lambda,h})\,\varepsilon^n&\\ +\frac{\varepsilon^n}{\sigma(\varepsilon)}\left(\int_{\partial Q} g\,\mathrm{d}\mathcal{H}^{n-1}+J(\varepsilon)\right)\sup_{x\in\Omega}|\varphi|\,\varepsilon^n D(\varepsilon),&\end{aligned}\tag{6.146}$$

where $D(\varepsilon)$ is a quantity of "boundary" cubes and $\varepsilon^n D(\varepsilon)\to 0$ by Jordan's measurability property of the set $\partial\Omega$.

Since

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum\varphi(x_j^{\lambda,h})\,\varepsilon^n=\int_\Omega\varphi\,\mathrm{d}x&\quad\text{by construction of Riemann sum,}\\ \lim_{\varepsilon\to 0}\frac{\varepsilon^n}{\sigma(\varepsilon)}=|\partial Q|_H^{-1}&\quad\text{by (6.138),}\\ \lim_{\varepsilon\to 0}J(\varepsilon)=0&\quad\text{by (6.145),}\\ \lim_{\varepsilon\to 0}\varepsilon^n D(\varepsilon)=0&\quad\text{by Jordan's measurability property of }\partial\Omega,\end{aligned}$$

it follows that estimate (6.146) immediately leads us to the required result. □

In a similar way, the following result can be proved.

Theorem 6.71. *Let Λ be the cone defined in Proposition 6.66 and let $g:\mathbb{R}^n\to\mathbb{R}$ be a Y-periodic such that $g\in H^1(K)$. Then*

$$\lim_{\varepsilon\to 0}\int_\Omega\varphi(x)g\left(\frac{x}{\varepsilon\,\lambda(\varepsilon)}\right)\mathrm{d}\mu_\varepsilon^{\lambda,h}=\left(\frac{1}{|K\cap\partial\Lambda|_H}\int_{K\cap\partial\Lambda} g\,\mathrm{d}\mathcal{H}^{n-1}\right)\int_\Omega\varphi\,\mathrm{d}x$$

for any $\varphi\in C(\overline{\Omega})$. In particular,

$$\lim_{\varepsilon\to 0}\int_\Omega g\left(\frac{x}{\varepsilon\,\lambda(\varepsilon)}\right)\mathrm{d}\mu_\varepsilon^{\lambda,h}=|\Omega|\left(\frac{1}{|K\cap\partial\Lambda|_H}\int_{K\cap\partial\Lambda} g\,\mathrm{d}\mathcal{H}^{n-1}\right).$$

Remark 6.72. Theorems 6.70 and 6.71 provide examples of bounded sequences in variable spaces whose weak limits can be recovered in an explicit form.

To illustrate the application of these results, let us consider the following example. Let Ω be a bounded open subset of $\mathbb{R}^2$ and let $Y = [-1/2, +1/2)^2$. We set the following: K is the cube with the center at the origin and sides $1/2$, Q is the circle with the center $(1/6, 0)$ in radius $1/6$, $h(\varepsilon) = \varepsilon^2$, and $\lambda(\varepsilon) = \exp(-\varepsilon^{-2})$. Let $v : Y \to \mathbb{R}$ be a function such that

$$v(x,y) = x^2 + y^2 \ \text{ if } \ (x,y) \in K, \quad v(x,y) = 0 \ \text{ in } \ Y \setminus K.$$

For every positive $\alpha \in (0,1]$, let us introduce the Y-periodic function $V_\alpha \in L^1_{per}(Y)$ as follows:

$$V_\alpha(x,y) = \begin{cases} v\left(\dfrac{x}{\alpha}, \dfrac{y}{\alpha}\right) & \text{if } (x,y) \in Y^\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then the support of this function satisfies the inclusion $Y \cap \operatorname{supp} V_\alpha \subseteq K^\alpha$. Now, we consider the following sequence of Y-periodic functions:

$$\left\{ g_\varepsilon \in L^1_{per}(Y) \cap H^1(K) \right\}, \quad \text{where } \ g_\varepsilon(x,y) = V_{\lambda(\varepsilon)}\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \quad \forall\, \varepsilon > 0.$$

The problem is to find

$$\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon^D} \varphi(x,y) g_\varepsilon(x,y) \mathrm{d}\mathcal{H}^1, \quad \forall\, \varphi \in C(\overline{\Omega}),$$

where $S_\varepsilon = \exp(-1/\varepsilon^2) K \cap \partial(\varepsilon^2 Q)$,

$$\Gamma_\varepsilon^D = \left[\bigcup_{\mathbf{k} \in \Theta_\varepsilon} \{ S_\varepsilon + \varepsilon\, \mathbf{k} \} \right] \cap \overline{\Omega}.$$

Since

$$\begin{aligned} \Lambda &= \left\{ (x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 > 0 \right\}, \\ K \cap \partial\Lambda &= \left\{ (0, x_2) \ : \ x_2 \in (-1/4, +1/4) \right\}, \end{aligned}$$

it follows that (see Theorem 6.71)

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon^D} \varphi(x,y) g_\varepsilon(x,y)\, \mathrm{d}\mathcal{H}^1 &= \lim_{\varepsilon \to 0} \int_\Omega \varphi(x,y) V_{\lambda(\varepsilon)}\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \mathrm{d}\mu_\varepsilon^{\lambda,h} \\ &= \left(\frac{1}{|K \cap \partial\Lambda|_H} \int_{K \cap \partial\Lambda} (x^2 + y^2) \mathrm{d}\mathcal{H}^1 \right) \int_\Omega \varphi \,\mathrm{d}x\,\mathrm{d}y \\ &= 2 \int_{-1/4}^{1/4} y^2 \mathrm{d}y \int_\Omega \varphi \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{48} \int_\Omega \varphi \,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

7

Convergence of Sets in Variable Spaces

The objective of this chapter is to give a presentation of the mathematical construction and tools that can be used to study problems for which the knowledge of "the limit set" in some appropriate sense is important. We discuss different analytical frameworks and concepts that can be formed on the basis of set convergence in fixed and varying spaces. In this context, we introduce different notions of limits for sequences of nonempty sets and give some applications to the theory of thin periodic and reticulated structures.

7.1 Set convergence in $\mathbb{R}^n$ via the limit properties of characteristic functions

The main object of our concern in this section is a family of subsets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of a finite-dimensional Euclidean space with nonempty boundary and interior. We assume that each of the sets Ω_ε is Lebesgue measurable and can be associated with its characteristic function

$$\chi_{\Omega_\varepsilon}(x) = 1 \ \text{ if } \ x \in \Omega_\varepsilon \ \text{ and } \ \chi_{\Omega_\varepsilon}(x) = 0 \ \text{ if } \ x \notin \Omega_\varepsilon.$$

In most applications, it is interesting to consider only domains Ω_ε that are contained in a fixed bounded measurable subset D of $\mathbb{R}^n$. In what follows, we associate with a subset D the set

$$\mathbf{X}(D) = \{\chi_\Omega \ : \ \forall\, \Omega \ \text{ measurable in } \ D\}\,. \tag{7.1}$$

Clearly, $\mathbf{X}(D) \subset L^\infty(D)$, and for all $p \geq 1$, $\mathbf{X}(D) \subset L^p(D)$. This induces a complete metric space structure on $\mathbf{X}(D)$ via the bijection

$$[\Omega] \mapsto \chi_\Omega \in \mathbf{X}(D) \subset L^p(D),$$

where by $[\Omega]$ we denote the equivalence classes of Lebesgue measurable subsets Ω. Recall that two sets Ω_1 and Ω_2 are said to be equivalent if the set $\{x \ : \ \chi_{\Omega_1}(x) \neq \chi_{\Omega_2}(x)\}$ has zero Lebesgue measure $\mathcal{L}_n$.

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_7,
© Springer Science+Business Media, LLC 2011

Theorem 7.1. *Let $1 \leq p < \infty$ be an integer. Then for a nonempty measurable subset D of $\mathbb{R}^n$, $\mathbf{X}(D)$ is closed and bounded in $L^p(D)$ and*

$$\rho_D([\Omega_1], [\Omega_2]) = \|\chi_{\Omega_1} - \chi_{\Omega_2}\|_{L^p(D)}$$

defines a complete metric structure on the set of equivalence classes of measurable subsets of D.

Proof. Let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of Lebesgue measurable subsets of D such that $\{\chi_{\Omega_k}\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^p(D)$. It converges to some function f in $L^p(D)$ and there exists a subsequence such that $\chi_{\Omega_k}(x) \to f(x)$ in D except for a subset Z of zero Lebesgue measure. Hence,

$$0 = \chi_{\Omega_k}(x)(1 - \chi_{\Omega_k}(x)) \to f(x)(1 - f(x)).$$

Define the set

$$\Omega = \{x \in D \setminus Z \ : \ f(x) = 1\}.$$

Clearly, Ω is measurable and $\chi_\Omega = f$ on $D \setminus Z$ since $f(x)(1 - f(x)) = 0$ on $D \setminus Z$. Hence, $f = \chi_\Omega$ a.e. on D, $\chi_{\Omega_k} \to \chi_\Omega$, and $\chi_\Omega \in \mathbf{X}(D)$. □

Remark 7.2. As an obvious consequence of this theorem, we have the following result: For $p = 1$ and Ω_1 and Ω_2 in D, the metric $\rho_D([\Omega_1], [\Omega_2])$ is the Lebesgue measure of the symmetric difference

$$\Omega_1 \triangle \Omega_2 = (\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1).$$

In other words, it means that the distance $\mu(A, B)$ between two open sets $A \subset D$ and $B \subset D$ defined as

$$\mu(A, B) = \mathcal{L}_n(A \triangle B), \tag{7.2}$$

coincides with the well known Ekeland metric in $L^\infty(D)$ applied to the characteristic functions:

$$d_E(\chi_A, \chi_B) = \mathcal{L}_n\{x \in D \mid \chi_A(x) \neq \chi_B(x)\} = \mu(A, B). \tag{7.3}$$

However, relations (7.2) and (7.3) are defined up to sets of measure 0 (see Fig. 7.1).

The next question concerns the comparison between the topologies on $\mathbf{X}(D)$ induced by $L^p(D)$ for different values $p \in [1, \infty)$. From Delfour and Zolésio (see [91]), we cite the following result.

Theorem 7.3. *Let D be a nonempty bounded measurable subset of $\mathbb{R}^n$. Then the topologies induced by $L^p(D)$ on $\mathbf{X}(D)$ are equivalent for $1 \leq p < \infty$.*

Proof. For a bounded domain D, any $p \geq 1$, $1/p+1/q = 1$, and any functions χ and $\overline{\chi}$ in $\mathbf{X}(D)$, we have

$$\|\chi - \overline{\chi}\|_{L^1(D)} = \int_D |\chi(x) - \overline{\chi}(x)|\,\mathrm{d}x \leq \|\chi - \overline{\chi}\|_{L^p(D)}|D|^{1/q}$$

since $\chi(x)$ and $\overline{\chi}(x)$ are either 0 or 1 in D. On the other hand,

$$\|\chi - \overline{\chi}\|^p_{L^p(D)} = \int_D |\chi(x) - \overline{\chi}(x)|^p\,\mathrm{d}x = \int_D |\chi(x) - \overline{\chi}(x)|\,\mathrm{d}x = \|\chi - \overline{\chi}\|_{L^1(D)}.$$

Therefore, for any $\varepsilon > 0$, selecting $\delta = \varepsilon^p$, we have

$$\forall\,\overline{\chi} \ \text{ such that } \ \|\chi - \overline{\chi}\|_{L^1(D)} < \delta, \quad \|\chi - \overline{\chi}\|_{L^p(D)} < \varepsilon.$$

Hence, the metrics on $\mathbf{X}(D)$ for $L^1(D)$ and $L^p(D)$ are equivalent. □

In view of this property, we introduce the following notion.

Definition 7.4. *Let D be a nonempty bounded measurable subset of $\mathbb{R}^n$. We say that a sequence of subsets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ in D strongly χ-converges to some $\Omega \subseteq D$ as $\varepsilon \to 0$ if the corresponding sequence of characteristic functions $\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0} \subset \mathbf{X}(D)$ converges strongly to χ_Ω in $L^p(D)$ for some p, $1 \leq p < \infty$.*

We have the following approximating result.

Theorem 7.5. *Let Ω be an arbitrary Lebesgue measurable subset of $D \subset \mathbb{R}^n$. Then there exists a sequence $\{\Omega_k\}_{k=1}^\infty$ of open C^∞-domains in $\mathbb{R}^n$ such that*

$$\chi_{\Omega_k} \to \chi_\Omega \quad \text{in} \ \ L^1(D).$$

Proof. The construction of the family of C^∞ domains $\{\Omega_k\}_{k=1}^\infty$ can be found in many places (see, for instance, Giusti [115, Sect. 1.14]). Associate with the characteristic function $\chi = \chi_\Omega$ the sequence of convolutions $f_k = \chi * \rho_k$ for a sequence $\{\rho_k\}_{k=1}^\infty$ of mollifiers (see [115]). By construction, we have $0 \leq f_k \leq 1$ for all $k \in \mathbb{N}$. For t $(0 < t < 1)$, define the sets

$$F_{kt} = \{x \in \mathbb{R}^n \ : \ f_k(x) > t\}.$$

By definition,

$$f_k - \chi > t \ \text{ in } \ F_{kt} \setminus \Omega, \quad \chi - f_k > 1 - t \ \text{ in } \ \Omega \setminus F_{kt},$$

and

$$|\chi - \chi_{F_{kt}}| \leq \frac{1}{\min\{t, 1-t\}}\,|\chi - f_k| \ \text{ a.e. in } \ \mathbb{R}^n.$$

By Sard's theorem (see Theorem 4.3 in [91]), the set F_{kt} is a C^∞-domain for almost all t in $(0,1)$. Fix α $(0 < \alpha < 1/2)$ and choose a sequence $\{t_k\}_{k=1}^\infty$ such that $\alpha \leq t_k \leq 1 - \alpha$ for all k, and define $\Omega_k = F_{kt_k}$. As a result,

$$|\chi_\Omega - \chi_{\Omega_k}| \le \frac{1}{\min\{t_k, 1-t_k\}} |\chi_\Omega - f_k| \le \frac{1}{\alpha} |\chi_\Omega - f_k| \quad \text{a.e. in } R^n.$$

Therefore, for a bounded measurable subset D of $\mathbb{R}^n$, we have

$$\|\chi_\Omega - \chi_{\Omega_k}\|_{L^1(D)} \le \alpha^{-1} \|\chi_\Omega - f_k\|_{L^1(D)},$$

which goes to 0 as k tends to infinity. The proof is complete. □

It should be stressed here that even if the set $\mathbf{X}(D)$ is closed and bounded in $L^p(D)$, it is not strongly compact. Hence, the strong χ-convergence of the sequences $\{\Omega_\varepsilon\}_{\varepsilon>0}$ in D needs to be relaxed to a weaker convergence. To this end, we note that for $1 < p < \infty$, the closed convex hull $\overline{\text{co}}\,\mathbf{X}(D)$ is weakly compact in the reflexive Banach space $L^p(D)$. In fact, for the set $\overline{\text{co}}\,\mathbf{X}(D)$, we have the representation

$$\overline{\text{co}}\,\mathbf{X}(D) = \{\chi \in L^p(D) \ : \ \chi(x) \in [0,1] \ \text{ a.e. in } \ D\}. \tag{7.4}$$

Indeed, by definition,

$$\overline{\text{co}}\,\mathbf{X}(D) \subseteq \{\chi \in L^p(D) \ : \ \chi(x) \in [0,1] \ \text{ a.e. in } \ D\}.$$

On the other hand, any χ which belongs to the right-hand side of (7.4) can be approximated by a sequence of convex combinations of elements of $\mathbf{X}(D)$. Choose

$$\chi_k = \sum_{m=1}^{k} \frac{1}{k} \chi_{B_{km}}, \quad B_{km} = \left\{x \ : \ \chi(x) \ge \frac{m}{k}\right\}$$

for which $|\chi_k(x) - \chi(x)| < 1/k$. However, the elements of the set $\overline{\text{co}}\,\mathbf{X}(D)$ are not necessary characteristic functions of a subdomain of D, that is, the identity

$$\chi(x)(1-\chi(x)) = 0 \ \text{ a.e. in } \ D$$

is not necessary satisfied. This fact leads to apparent paradoxes. That is why Definition 7.4 cannot be relaxed to the notion of weak χ-convergence of subsets.

Before we introduce the appropriate terminology, we first give a few basic results (see [91] for the proof)

Lemma 7.6. *Let D be a bounded open subset of $\mathbb{R}^n$, G a bounded subset of $\mathbb{R}$, and*

$$\mathcal{G} = \{g : D \to \mathbb{R} \ : \ g \ \text{ is measurable and } \ g(x) \in G \ \text{ a.e. in } \ D\}.$$

Then the following hold:

(i) For any p, $1 \le p < \infty$, and any sequence $\{g_k\}_{k=1}^{\infty} \subset \mathcal{G}$, the following statements are equivalent:
(a) $\{g_k\}_{k=1}^{\infty}$ weakly-$*$ converges in $L^\infty(D)$,

(b) $\{g_k\}_{k=1}^{\infty}$ *weakly converges in* $L^p(D)$,
(c) $\{g_k\}_{k=1}^{\infty}$ *converges in* $\mathcal{D}'(D)$, *where* $\mathcal{D}'(D)$ *is the space of scalar distributions on* D.

(ii) If G *is bounded, closed, and convex, then* $\mathcal{G}$ *is convex and compact with respect to the weak-* topology in* $L^\infty(D)$, *the weak topology in* $L^p(D)$, *and the convergence in* $\mathcal{D}'(D)$.

Due to these results, we adopt the following terminology.

Definition 7.7. *We say that a sequence of characteristic functions* $\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ *is weakly convergent in* $\mathbf{X}(D)$ *if it converges for some topology between the weak-* topology of* $L^\infty(D)$ *and the topology of* $\mathcal{D}'(D)$.

It is interesting to observe that working with the weak convergence in $\mathbf{X}(D)$ only makes sense when the limit element of $\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ is not a characteristic function.

Theorem 7.8. *Let* D *be a bounded open subset of* $\mathbb{R}^n$ *and let* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *be a sequence of subsets in* D. *Let* χ *be element of* $\mathbf{X}(D)$ *such that* $\chi_{\Omega_\varepsilon} \rightharpoonup \chi$ *in* $L^2(D)$. *Then there exists a subset* $\Omega \subset D$ *such that the sequence* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *strongly* χ*-converges to* Ω *in the sense of Definition 7.4.*

Proof. As follows from Theorem 7.3, it is sufficient to prove that $\chi_{\Omega_\varepsilon} \to \chi_\Omega$ strongly in $L^p(D)$ for some p, $1 \le p < \infty$, where we set $\chi_\Omega = \chi$. Let $p = 2$. Then the strong $L^2(D)$-convergence follows from the the fact that

$$\int_D |\chi_{\Omega_\varepsilon}|^2 \,\mathrm{d}x = \int_D \chi_{\Omega_\varepsilon} \,\mathrm{d}x \to \int_D \chi_\Omega \,\mathrm{d}x = \int_D |\chi_\Omega|^2 \,\mathrm{d}x$$

implies

$$\begin{aligned}\int_D |\chi_{\Omega_\varepsilon} - \chi_\Omega|^2 \,\mathrm{d}x &= \int_D \left(|\chi_{\Omega_\varepsilon}|^2 - 2\chi_{\Omega_\varepsilon}\chi_\Omega + |\chi_\Omega|^2\right) \mathrm{d}x \\ &\to \int_D \left(|\chi_\Omega|^2 - 2\chi_\Omega\chi_\Omega + |\chi_\Omega|^2\right) \mathrm{d}x = 0.\end{aligned}$$

This proof is complete. □

Let us note that working with the weak convergence in $\mathbf{X}(D)$ creates new phenomena and difficulties. For instance, when a characteristic function is present in the coefficients of the highest-order term of a differential equation, the weak convergence of a sequence of characteristic functions $\{\chi_k\}_{k=1}^{\infty}$ to some element χ of $\overline{\mathrm{co}}\,\mathbf{X}(D)$,

$$\chi_k \rightharpoonup \chi \ \text{ in } \ L^2(D) \implies \chi_k \overset{*}{\rightharpoonup} \chi \ \text{ in } \ L^\infty(D),$$

does not imply the weak convergence in $H^1(D)$ of the sequence $\{y(\chi_k)\}_{k=1}^{\infty}$ of solutions to the solution of the differential equation corresponding to $y(\chi)$, that is,

$$y(\chi_k) \rightharpoonup y(\chi) \quad \text{weakly in} \quad H^1(D).$$

Since D is supposed to be a smooth bounded domain, if follows that compactness of the injection of $H^1(D)$ into $L^2(D)$ would have implied strong convergence in $L^2(D)$:

$$y(\chi_k) \to y(\chi) \quad \text{in} \quad L^2(D).$$

This fact was pointed out in 1971 by Murat [193] in the following example, which will be rewritten to emphasize the role of the characteristic function in the asymptotic analysis of boundary value problems.

Example 7.9. Let us consider the following problem:

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(g\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) + gy = 0 \quad \text{in} \quad D = (0,1), \\ y(0) = 1, \quad y(1) = 2, \end{cases} \tag{7.5}$$

where

$$\mathcal{G} = \{g = g_1(x)\chi + g_2(x)(1-\chi) \ : \ \chi \in \mathbf{X}(D)\}$$

with

$$g_1(x) = 1 - \sqrt{\frac{1}{2} - \frac{x^2}{6}}, \quad g_2(x) = 1 + \sqrt{\frac{1}{2} - \frac{x^2}{6}}.$$

This is equivalent to

$$\mathcal{G} = \{g \in L^\infty(0,1) \ : \ g(x) \in \{g_1(x), g_2(x)\} \quad \text{a.e. in} \quad (0,1)\}.$$

Associate with each integer $p \geq 1$ the function g^p,

$$g^p(x) = \begin{cases} 1 - \sqrt{\dfrac{1}{2} - \dfrac{x^2}{6}} & \text{if } \dfrac{m}{p} < x \leq \dfrac{2m+1}{2p}, \\ 1 + \sqrt{\dfrac{1}{2} - \dfrac{x^2}{6}} & \text{if } \dfrac{2m+1}{2p} < x \leq \dfrac{m+1}{p}, \end{cases} \qquad 0 \leq m \leq p-1,$$

and the corresponding function χ_p ($g^p = g_1\chi_p + g_2(1-\chi_p)$),

$$\chi_p(x) = \begin{cases} 1 \ \text{ if } \dfrac{m}{p} < x \leq \dfrac{2m+1}{2p}, \\ 0 \ \text{ if } \dfrac{2m+1}{2p} < x \leq \dfrac{m+1}{p}, \end{cases} \qquad 0 \leq m \leq p-1.$$

Murat showed that for each p, the following statements hold true: $g^p \in \mathcal{G}$, $\chi_p \in \mathbf{X}(D)$,

$$g^p \overset{*}{\rightharpoonup} g_\infty = 1 \quad \left(\text{resp.}, \ \chi_p \overset{*}{\rightharpoonup} \frac{1}{2}\right) \quad \text{in} \quad L^\infty(0,1),$$

$$\frac{1}{g^p} \overset{*}{\rightharpoonup} \frac{1}{2}\left[\frac{1}{g_1} + \frac{1}{g_2}\right] = \frac{1}{1/2 + x^2/6} \quad \text{in} \quad L^\infty(0,1).$$

Moreover,

$$y_p \rightharpoonup y \quad \text{in} \quad H^1(0,1),$$

where y_p denotes the solution of (7.5) corresponding to $g = g^p$ and y denotes the solution of the boundary value problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\left(\dfrac{1}{2}+\dfrac{x^2}{6}\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)+y=0 \quad \text{in} \quad D=(0,1), \\ y(0)=1, \quad y(1)=2, \end{cases} \tag{7.6}$$

Let us define the function

$$g_H(x) = \frac{1}{2} + \frac{x^2}{6},$$

which corresponds to

$$\chi_H(x) = \frac{1}{2}\left[1+\sqrt{\frac{1}{2}-\frac{x^2}{6}}\right] \in \overline{\mathrm{co}}\,\mathbf{X}(D).$$

Notice that g_H appears in the second-order term and g_∞ appears in the zeroth-order term in (7.6):

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(g_H\frac{\mathrm{d}y}{\mathrm{d}x}\right)+g_\infty y=0 \quad \text{in} \quad (0,1). \tag{7.7}$$

It is easy to check that

$$y(x) = 1 + x^2 \quad \text{in} \quad (0,1),$$

which is not equal to the solution y_∞ of (7.5) for the weak-$*$ limit $g_\infty = 1$:

$$y_\infty(x) = \frac{2(\mathrm{e}^x - \mathrm{e}^{-x}) + \mathrm{e}^{1-x} - \mathrm{e}^{-(1-x)}}{\mathrm{e} - \mathrm{e}^{-1}}.$$

The above example uses space-varying coefficients $g_1(x)$ and $g_2(x)$. However, it is still valid for two constants $g_1 > 0$ and $g_2 > 0$. In that case, it is easy to show that

$$\begin{aligned} g_\infty &= \frac{g_1+g_2}{2}, & \frac{1}{g_H} &= \frac{1}{2}\left[\frac{1}{g_1}+\frac{1}{g_2}\right], \\ \chi_\infty &= \frac{1}{2}, & \chi_H &= \frac{g_2}{g_1+g_2}, \end{aligned}$$

and the solution y of the boundary value problem (7.7) is given by

$$y(x) = \frac{2\sinh cx + \sinh c(1-x)}{\sinh c},$$

where

$$c = \frac{g_1 + g_2}{2\sqrt{g_1 g_2}} \geq 1,$$

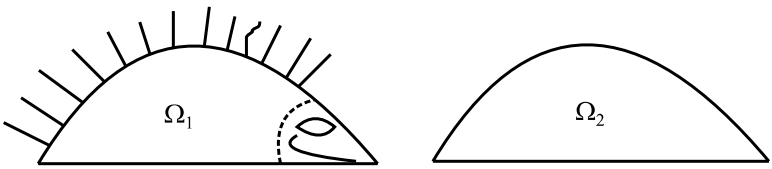
and the solution y_∞ is given by

$$y_\infty(x) = \frac{2\sinh x + \sinh(1-x)}{\sinh 1}.$$

Thus, for $g_1 \neq g_2$, or equivalently, $c > 1$, we have $y \neq y_\infty$. In fact, using the same sequence of χ_ps, the sequence $\{y_p = y(\chi_p)\}_{p=1}^{\infty}$ of the solutions of (7.5) weakly converges to $y_H = y(\chi_H)$, which is different from the solution $y_\infty = y(\chi_\infty)$ for $g_1 \neq g_2$. This is readily seen by noticing that since χ_∞ and χ_H are constant,

$$\begin{aligned} -\Delta(g_H y_H) = \chi_\infty f = -\Delta(g_\infty y_\infty) \\ \Longrightarrow\ g_H y_H = g_\infty y_\infty \\ \Longrightarrow\ y_H = \frac{(g_1+g_2)^2}{4g_1 g_2} y_\infty \neq y_\infty \text{ for } g_1 \neq g_2. \end{aligned}$$

Remark 7.10. Since the strong and weak χ-convergence are defined to within the equivalence classes $[\Omega]$ of (Lebesgue) measurable subsets Ω of $\mathbb{R}^n$, it should be noted that such a type of convergence is rather hard. As an illustration of what we meant by "hard convergence", consider two sets: the smiling hedgehog and the expressionless circular segment in Fig. 7.1. The circular segment Ω_2 is obtained by adding missing points and curves "inside" of Ω_1 and removing the needles "outside" Ω_1. Hence, these sets belong to the same equivalence class. However, from a geometrical point of view, we have absolutely different junction structures.



(a) A smiling hedgehog (b) A circular segment

Fig. 7.1. Examples of sets belonging to the same equivalence class

7.2 On some limit properties of characteristic functions for nonperiodically perforated domains

Much of this section will be devoted to the study of some specific properties of the weak convergence in $\mathbf{X}(D)$ with respect to the weak-$*$ topology of $L^\infty(D)$. Attention will be focused on the case when the limit element of

$\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0} \subset \mathbf{X}(D)$ is not a characteristic function. We will follow results obtained by Kesavan and Saint Jean Paulin [135, 136].

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain and let $S_\varepsilon \subset \Omega$ be a closed subset for each $\varepsilon > 0$. We set $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$ and call it the perforated domain. Let $\chi_{\Omega_\varepsilon}$ be the characteristic function of the perforated domain Ω_ε.

Following Braine, Damlamian, and Donato [32], we say that $\{S_\varepsilon\}_{\varepsilon>0}$ is an admissible family of holes in Ω if the following conditions are fulfilled:

(H1) There exists, for each $\varepsilon > 0$, an extension operator

$$P_\varepsilon : V_\varepsilon \to H_0^1(\Omega),$$

where

$$V_\varepsilon = \left\{ u \in H^1(\Omega_\varepsilon) \ : \ u = 0 \ \text{ on } \ \partial\Omega \right\}, \tag{7.8}$$

such that for every $u \in V_\varepsilon$,

$$P_\varepsilon u\big|_{\Omega_\varepsilon} = u, \quad \|\nabla P_\varepsilon u\|_{\mathbf{L}^2(\Omega)} \le C \|\nabla u\|_{\mathbf{L}^2(\Omega_\varepsilon)}. \tag{7.9}$$

Here, as usual, the symbol C denotes a positive constant independent of ε.

(H2) Every weak-$*$ limit point in $L^\infty(\Omega)$ of the sequence $\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ is positive a.e. on Ω.

(H3) If $\chi_{\Omega_\varepsilon} \overset{*}{\rightharpoonup} \chi_0$ in $L^\infty(\Omega)$, then $\chi_0^{-1} \in L^\infty(\Omega)$.

In addition to the extension operator P_ε described above, we always have the trivial extension by 0 of a function $f \in L^p(\Omega_\varepsilon)$. We denote this extension by $\widetilde{f}$.

Cioranescu and Saint Jean Paulin [67] have shown that when Ω is perforated periodically, hypotheses (H1)–(H3) are satisfied. In particular, χ_0 will be a positive constant. However, we present below an example where hypotheses (H2) and (H3) hold true, whereas hypothesis (H1) is not fulfilled.

Example 7.11. Let us consider a planar, thick multi-structure Ω_ε, which consists of the junction body

$$\Omega_0 = \{x = (x_1, x_2) \in \mathbb{R}^2 : \ 0 < x_1 < a, \quad 0 < x_2 < \gamma(x_1)\},$$

and a large number N of the thin rods $G_\varepsilon = \cup_{j=0}^{N-1} G_\varepsilon^j$,

$$G_\varepsilon^j = \{x \in \mathbb{R}^2 : \ |x_1 - \varepsilon\,(j + 1/2)| < \varepsilon h/2, \ x_2 \in (-l, 0]\}, \ j = 0, 1, \ldots, N-1,$$

situated along the boundary $\Gamma_0 = \{x_2 = 0, 0 < x_1 < a\}$, that is, $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon$ (see Fig. 7.2). Here, $\gamma \in C^1([0, a])$, $0 < \gamma_0 = \min_{[0,a]} \gamma$, the fixed number $h \in (0, 1)$ and N is a large positive integer; therefore, $\varepsilon = a/N$ is a small discrete parameter, which characterizes the distance between the rods and their thickness. Let Ω be the interior of $\overline{\Omega_0} \cup \overline{D}$, where $D = (0, a) \times (-l, 0)$ is a rectangle that is filled up by the thin rods in the limit passage.

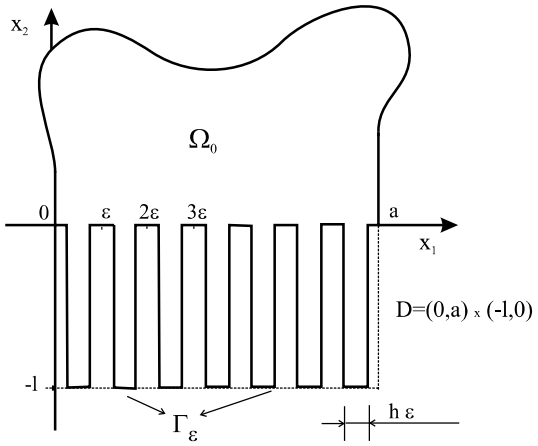


Fig. 7.2. Thick multi-structure Ω_ε

Let $\chi_{\Omega_\varepsilon}$ be the characteristic function of the domain Ω_ε. Let us define a 1-periodic function $\chi_h(\cdot)$ as follows:

$$\chi_h(t) = \begin{cases} 1 \text{ if } |t - 1/2| < h/2, \\ 0 \text{ if } h/2 \le |t - 1/2| \le 1/2. \end{cases}$$

Since

$$\chi_h(x_1/\varepsilon) \rightharpoonup \int_0^1 \chi_h(t)\,\mathrm{d}t = h \quad \text{weakly-* in} \quad L^\infty(0,a) \quad \text{as} \quad \varepsilon \to 0$$

(see the mean value property of periodic functions (2.35)), it follows that for any element g of $L^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega g\chi_{\Omega_\varepsilon}\,\mathrm{d}x &= \int_{\Omega_0} g\,\mathrm{d}x + \int_0^a \chi_h(x_1/\varepsilon) \int_{-l}^0 g(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &\to \int_{\Omega_0} g\,\mathrm{d}x + h\int_D g\,\mathrm{d}x = \int_\Omega \chi_0 g\,\mathrm{d}x, \quad \varepsilon \to 0. \end{aligned}$$

Thus, $\chi_{\Omega_\varepsilon} \overset{*}{\rightharpoonup} \chi_0$ in $L^\infty(\Omega)$ and, clearly, $\chi_0(x) = 1$ if $x \in \Omega_0 \cup \Gamma$ and $\chi_0(x) = h$ if $x \in D$. So, hypotheses (H2) and (H3) are satisfied. However, one of the principal futures of domain Ω_ε (see also Fig. 7.3 for a 3D example) is the fact that hypothesis (H1) fails (see [187]).

Proposition 7.12. *Let $1 \le p < \infty$. Assume that hypotheses (H2) and (H3) are satisfied. Let $f_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ be functions in $L^p(\Omega_\varepsilon)$ such that $\widetilde{f_\varepsilon} \rightharpoonup f_0$ in $L^p(\Omega)$. Then, we have*

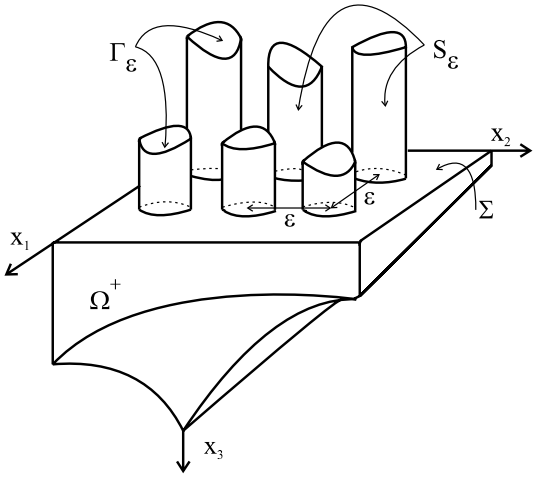


Fig. 7.3. 3D thick multi-structure

$$\liminf_{\varepsilon\to 0}\int_{\Omega}|\widetilde{f}_{\varepsilon}|^{p}\,\mathrm{d}x\geq\int_{\Omega}\frac{|f_0|^{p}}{\chi_0^{p-1}}\,\mathrm{d}x. \tag{7.10}$$

Proof. Inequality (7.10) is obvious if $p=1$ by virtue of the lower semicontinuity of the norm in $L^p(\Omega)$ with respect to the weak convergence. To prove (7.10) in the general case, we consider the convex functional $\Phi : L^p(\Omega)\to\mathbb{R}$ defined by

$$\Phi(\varphi)=\int_{\Omega}|\varphi|^{p}\,\mathrm{d}x.$$

Note that hypothesis (H3) implies the inclusion $(\chi_{\Omega_\varepsilon}/\chi_0)f_0\in L^p(\Omega)$. Then, taking into account the well-known inequality for convex differentiable functionals (see, e.g., [16])

$$\Phi(\varphi)-\Phi(\psi)\geq\langle\mathcal{D}\Phi(\psi),\varphi-\psi\rangle_{L^q(\Omega),L^p(\Omega)},\quad\forall\varphi,\psi\in L^p(\Omega),$$

we have

$$\Phi\left(\widetilde{f}_{\varepsilon}\right)-\Phi\left(\frac{\chi_{\Omega_\varepsilon}}{\chi_0}f_0\right)\geq p\int_{\Omega}\left|\frac{\chi_{\Omega_\varepsilon}}{\chi_0}f_0\right|^{p-2}\frac{\chi_{\Omega_\varepsilon}}{\chi_0}f_0\left(\widetilde{f}_{\varepsilon}-\frac{\chi_{\Omega_\varepsilon}}{\chi_0}f_0\right)\mathrm{d}x.$$

However, $\chi_{\Omega_\varepsilon}\widetilde{f}_{\varepsilon}=\widetilde{f}_{\varepsilon}$ and $\chi^r_{\Omega_\varepsilon}=\chi_{\Omega_\varepsilon}$ for all $r>0$. Hence, the right-hand side of the above inequality becomes

$$p\int_{\Omega}\frac{f_0|f_0|^{p-2}}{\chi_0^{p-1}}\left(\widetilde{f}_{\varepsilon}-\frac{\chi_{\Omega_\varepsilon}}{\chi_0}f_0\right)\mathrm{d}x,$$

which tends to 0 as $\varepsilon\to 0$ by the weak convergence. Further, by the definition of Φ, we have

$$\Phi\left(\frac{\chi_{\Omega_\varepsilon}}{\chi_0} f_0\right) = \int_\Omega \chi_{\Omega_\varepsilon} \frac{|f_0|^p}{\chi_0^p}\,\mathrm{d}x \to \int_\Omega \frac{|f_0|^p}{\chi_0^{p-1}}\,\mathrm{d}x.$$

Hence, (7.10) is an immediate consequence of these observations. □

Remark 7.13. Let $\{f_\varepsilon : \Omega \to \mathbb{R}\}_{\varepsilon>0}$ be a bounded sequence in $L^p(\Omega)$ such that

$$\widetilde{f}_\varepsilon = \chi_{\Omega_\varepsilon} f_\varepsilon \rightharpoonup f_0 \quad \text{in} \quad L^p(\Omega).$$

Then, by the lower semicontinuity of the L^p-norm with respect to the weak convergence, we have

$$\liminf_{\varepsilon\to 0} \int_\Omega |\widetilde{f}_\varepsilon|^p\,\mathrm{d}x = \liminf_{\varepsilon\to 0} \int_\Omega \chi_{\Omega_\varepsilon} |f_\varepsilon|^p\,\mathrm{d}x \geq \int_\Omega |f_0|^p\,\mathrm{d}x. \tag{7.11}$$

However, as follows from Proposition 7.12, the fact that the functions $\widetilde{f}_\varepsilon = \chi_{\Omega_\varepsilon} f_\varepsilon$ vanish on the holes (i.e., where $\chi_{\Omega_\varepsilon}$ vanishes) gives the improved estimate (7.11).

On the other hand, inequality (7.10) has a simple explanation. Let us consider the sequence $\{f_\varepsilon : \Omega \to \mathbb{R}\}_{\varepsilon>0}$ in the variable space $L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ $(1 \leq p < \infty)$, where the Radon measure μ_ε is defined as $\mathrm{d}\mu_\varepsilon = \chi_{\Omega_\varepsilon}\mathrm{d}x$ for every $\varepsilon > 0$. In view of supposition (H3), we conclude that

$$\mathrm{d}\mu_\varepsilon \rightharpoonup \chi_0\,\mathrm{d}x \quad \text{in the sense of Definition 2.29,}$$
$$f_\varepsilon \rightharpoonup f_* \quad \text{in variable space} \quad L^p(\Omega, \mathrm{d}\mu_\varepsilon) \quad \text{in the sense of Definition 6.14.}$$

Then Proposition 6.16 implies

$$\liminf_{\varepsilon\to 0} \int_\Omega |\widetilde{f}_\varepsilon|^p\,\mathrm{d}x = \liminf_{\varepsilon\to 0} \int_\Omega |f_\varepsilon|^p\,\mathrm{d}\mu_\varepsilon \geq \int_\Omega |f_*|^p \chi_0\,\mathrm{d}x. \tag{7.12}$$

However, due to hypothesis (H3) and the definition of the weak convergence in the variable spaces (see (6.25)), we have $f_* = f_0/\chi_0$. Combining this fact with (7.12), we obtain the estimate (7.10).

Proposition 7.14. *Assume that hypotheses (H2) and (H3) are satisfied. Let $f_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ be functions in $L^\infty(\Omega_\varepsilon)$ such that $\widetilde{f}_\varepsilon \overset{*}{\rightharpoonup} f_0$ in $L^\infty(\Omega)$. Then, we have*

$$\liminf_{\varepsilon\to 0} \left\|\widetilde{f}_\varepsilon\right\|_{L^\infty(\Omega)} \geq \left\|\frac{f_0}{\chi_0}\right\|_{L^\infty(\Omega)}. \tag{7.13}$$

Proof. Since for any $1 < p < \infty$,

$$\|\widetilde{f}_\varepsilon\|_{L^p(\Omega)} \leq \left\|\widetilde{f}_\varepsilon\right\|_{L^\infty(\Omega)} |\Omega|^{1/p},$$

it follows that

$$\liminf_{\varepsilon\to 0} \|\widetilde{f}_\varepsilon\|_{L^p(\Omega)} \leq |\Omega|^{1/p} \liminf_{\varepsilon\to 0} \left\|\widetilde{f}_\varepsilon\right\|_{L^\infty(\Omega)},$$

and hence,

$$\lim_{p\to\infty} \liminf_{\varepsilon\to 0} \|\widetilde{f}_\varepsilon\|_{L^p(\Omega)} \le \liminf_{\varepsilon\to 0} \left\|\widetilde{f}_\varepsilon\right\|_{L^\infty(\Omega)}. \tag{7.14}$$

On the other hand, by (7.10) we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} \|\widetilde{f}_\varepsilon\|_{L^p(\Omega)} &\ge \left(\int_\Omega \frac{|f_0|^p}{\chi_0^{p-1}}\,\mathrm{d}x\right)^{1/p} \\ &= \left(\int_\Omega \left|\frac{f_0}{\chi_0}\right|^p \chi_0\,\mathrm{d}x\right)^{1/p} \ge \alpha^{1/p}\left\|\frac{f_0}{\chi_0}\right\|_{L^p(\Omega)},\end{aligned}$$

where $\chi_0 \ge \alpha > 0$, as per hypothesis (H3). Hence,

$$\lim_{p\to\infty} \liminf_{\varepsilon\to 0} \|\widetilde{f}_\varepsilon\|_{L^p(\Omega)} \ge \lim_{p\to\infty} \left\|\frac{f_0}{\chi_0}\right\|_{L^p(\Omega)} = \left\|\frac{f_0}{\chi_0}\right\|_{L^\infty(\Omega)}. \tag{7.15}$$

As a result, (7.13) follows from (7.14) and (7.15). □

In the case when $\widetilde{f}_\varepsilon = \chi_{\Omega_\varepsilon} f$, we also have the following result, which is more precise than (7.13).

Proposition 7.15. *Let f be a fixed element of $L^\infty(\Omega)$. Then for the sequence $\{\widetilde{f}_\varepsilon = \chi_{\Omega_\varepsilon} f\}_{\varepsilon>0}$, we have*

$$\lim_{\varepsilon\to 0} \|\widetilde{f}_\varepsilon\|_{L^\infty(\Omega)} = \|f\|_{L^\infty(\Omega)}. \tag{7.16}$$

Proof. Since $\widetilde{f}_\varepsilon \overset{*}{\rightharpoonup} \chi_0 f$ in $L^\infty(\Omega)$, by Proposition 7.14, we have

$$\liminf_{\varepsilon\to 0} \left\|\widetilde{f}_\varepsilon\right\|_{L^\infty(\Omega)} \ge \|f_0\|_{L^\infty(\Omega)}.$$

On the other hand, $\|\chi_{\Omega_\varepsilon} f\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)}$, and hence,

$$\limsup_{\varepsilon\to 0} \|\widetilde{f}_\varepsilon\|_{L^\infty(\Omega)} = \limsup_{\varepsilon\to 0} \|\chi_{\Omega_\varepsilon} f\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)}.$$

Combining this with the previous inequality, we obtain the required result. This completes the proof. □

Remark 7.16. Note that in Proposition 7.15, the limit of $\{\widetilde{f}_\varepsilon\}_{\varepsilon>0}$ is $\chi_0 f$, not f. Thus, (7.16) does not imply the strong convergence in $L^\infty(\Omega)$. In fact, if $\chi_0 \ne 1$, then for any $1 \le p \le \infty$, given a sequence $\{\widetilde{f}_\varepsilon = \chi_{\Omega_\varepsilon} f_\varepsilon\}_{\varepsilon>0}$ that converges weakly to f_0 in $L^p(\Omega)$ (weakly-$*$ in $L^\infty(\Omega)$ if $p = \infty$), we cannot, in general, have the strong convergence if $f_0 \ne 0$.

To see this, assume that $\widetilde{f}_\varepsilon \to f_0$ strongly in $L^p(\Omega)$. If we take any element $g \in L^q(\Omega)$, where $1/p + 1/q = 1$, then

$$\int_\Omega \widetilde{f}_\varepsilon g \,\mathrm{d}x = \int_\Omega \chi_{\Omega_\varepsilon} \widetilde{f}_\varepsilon \,\mathrm{d}x, \quad \forall\, \varepsilon > 0.$$

Passing to the limit (using the strong convergence of $\widetilde{f}_\varepsilon$ to f_0 and the weak convergence of $\chi_{\Omega_\varepsilon} g$ to $\chi_0 g$ for the term on the right-hand side), we get

$$\int_\Omega f_0 g \,\mathrm{d}x = \int_\Omega \chi_0 f_0 g \,\mathrm{d}x.$$

Since g was arbitrarily chosen in $L^q(\Omega)$, it follows that $f_0 = \chi_0 f_0$ in $L^p(\Omega)$, which is not generally true (except in the case when $f_0 = 0$) if $\chi_0 \neq 1$.

7.3 Convergence of sets associated with thin periodic structures

To illustrate some specific properties of set convergence in $\mathbb{R}^n$, we consider a particular case when each of the sets Ω_ε can be associated with a periodic thin structure. In general, this highly contrasting medium depends on two small related parameters ε and h which control the size of a periodicity cell and thickness of the reinforcement. To describe this structure in a general manner, we assume that there is a singular structure $\mathcal{F}$ in $\mathbb{R}^n$ with $\mathcal{L}_n(\mathcal{F}) = 0$ such that $\mathcal{F}$ is a generating skeleton of the thin structure. For instance, for thin grids, $\mathcal{F}$ can be associated with a periodic singular graph in the corresponding Euclidean space that consists of straight segments (see Figs. 1.1 and 1.3). For 3D box structures, the singular structure $\mathcal{F}$ can be represented as some family of planes (see Fig. 1.2). We refer to Chechkin and Zhikov [58, 257] for the details.

Let $\square = [0,1)^n$ be a periodicity cell of $\mathcal{F}$. We define a thin structure $\mathcal{F}^h$ applying the procedure of h-thickening to the skeleton $\mathcal{F}$. In the case $n = 2$, it means that $\mathcal{F}^h$ can be obtained if any component I of $\mathcal{F}$ is replaced by a strip I^h of width h whose median coincides with I. For the 3D case, each plane of the structure $\mathcal{F}$ turns out to be the middle of the corresponding plate from the structure $\mathcal{F}^h$ whose thickness is $h > 0$.

Having applied the procedure of homothetic contraction of $\mathbb{R}^n$ with the factor of ε^{-1}, we transform the structures $\mathcal{F}^h$ and $\mathcal{F}$ into $\mathcal{F}^h_\varepsilon$ and $\mathcal{F}_\varepsilon$, respectively. In particular, $\mathcal{F}^h_\varepsilon = \varepsilon \mathcal{F}^h = \{\varepsilon x \,:\, x \in \mathcal{F}^h\}$. Note that in this case, $\mathcal{F}^h_\varepsilon$ consists of components (rods, planes, slabs, etc.) whose thicknesses are εh.

Let Ω be an open bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. We define the thin periodic structure $\Omega_{\varepsilon,h}$ on Ω as $\Omega_{\varepsilon,h} = \Omega \cap \mathcal{F}^h_\varepsilon$. In spite of the fact that the sets $\Omega_{\varepsilon,h}$ fill up the whole of the domain Ω in the limit as $\varepsilon \to 0$, the sequence $\{\Omega_{\varepsilon,h}\}_{\varepsilon>0}$ does not strongly χ-converge to Ω in the sense of Definition 7.4. Indeed, let χ^h_ε and χ^h be the characteristic functions of the sets $\Omega_{\varepsilon,h}$ and $\mathcal{F}^h$, respectively. Then we have the following result.

Theorem 7.17. *Let Ω be an open bounded subdomain of $\mathbb{R}^n$ and let Ω be measurable in the Jordan sense. Let $\Omega_{\varepsilon,h}$ be a thin periodic structure in Ω, the geometry of which depends on two small parameters ε and $h(\varepsilon)$ related to each other by the relation $h = h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then*

$$\chi_{\Omega_{\varepsilon,h}} \rightharpoonup 0 = \chi_{\emptyset} \quad \text{in} \quad L^2(\Omega). \tag{7.17}$$

Proof. Since $C_0^\infty(\Omega)$ is a dense subset of $L^2(\Omega)$, it is sufficient to show that

$$\lim_{\varepsilon\to 0} \int_\Omega \psi(x)\chi_{\Omega_{\varepsilon,h}}(x)\,\mathrm{d}x = 0, \quad \forall\, \psi \in C_0^\infty(\Omega).$$

Let $\psi \in C_0^\infty(\Omega)$ be a fixed function and let $\square_{\mathbf{k}} = \square + \mathbf{k}$, where $\mathbf{k}$ is a vector in $\mathbb{R}^n$ with integer components. Then $\{\varepsilon\square_{\mathbf{k}}\}$ is a partition of $\mathbb{R}^n$. Hence,

$$\begin{aligned}\int_\Omega \psi(x)\chi_{\Omega_{\varepsilon,h}}(x)\,\mathrm{d}x = &\sum \int_{\varepsilon\square_{\mathbf{k}}} \psi(x)\chi_{\Omega_{\varepsilon,h}}(x)\,\mathrm{d}x \\ &+ \sum \int_{\varepsilon\square_{\mathbf{k}}\cap\Omega} \psi(x)\chi_{\Omega_{\varepsilon,h}}(x)\,\mathrm{d}x,\end{aligned} \tag{7.18}$$

where the first sum is taken over all $\mathbf{k}$ such that $\varepsilon\square_{\mathbf{k}}$ is inside and the second sum is over all $\mathbf{k}$ such that $\varepsilon\square_{\mathbf{k}}$ and $\partial\Omega$ have common points.

Let us first consider the second sum in (7.18). We have the following estimate:

$$\left|\sum \int_{\varepsilon\square_{\mathbf{k}}\cap\Omega} \psi(x)\chi_{\Omega_{\varepsilon,h}}(x)\,\mathrm{d}x\right| \le \max_{x\in\Omega}|\psi(x)| \int_{\varepsilon\square} \chi_{\mathcal{F}_\varepsilon^h}(x)\,\mathrm{d}x M(\varepsilon), \tag{7.19}$$

where $M(\varepsilon)$ is the number of cubes $\varepsilon\square_{\mathbf{k}}$ containing the boundary of Ω. However,

$$\int_{\varepsilon\square} \chi_{\mathcal{F}_\varepsilon^h}(x)\,\mathrm{d}x = \varepsilon^n \int_\square \chi_{\mathcal{F}^h}\,\mathrm{d}x \le \varepsilon^n.$$

Combining this relation with (7.19) and using the fact that $\varepsilon^n M(\varepsilon) \to 0$, since the boundary of Ω has Lebesgue measure zero, we obtain

$$\lim_{\varepsilon\to 0} \sum \int_{\varepsilon\square_{\mathbf{k}}\cap\Omega} \psi(x)\chi_{\Omega_{\varepsilon,h}}(x)\,\mathrm{d}x = 0. \tag{7.20}$$

Let us now consider the first sum in (7.18). Since ψ is continuous and bounded, there exist points $x_{\mathbf{k}} \in \varepsilon\square_{\mathbf{k}}$ such that

$$\int_{\varepsilon\square_{\mathbf{k}}} \psi(x)\chi_{\Omega_{\varepsilon,h}}\,\mathrm{d}x = \psi(x_{\mathbf{k}}) \int_{\varepsilon\square} \chi_{\mathcal{F}_\varepsilon^h}(x)\,\mathrm{d}x = \psi(x_{\mathbf{k}})\varepsilon^n \int_\square \chi_{\mathcal{F}^h}(x)\,\mathrm{d}x.$$

This together with the fact

$$\sum \psi(x_{\mathbf{k}})\varepsilon^n \to \int_\Omega \psi(x)\,\mathrm{d}x \quad \text{as} \quad \varepsilon \to 0 \quad \text{and} \quad \int_\square \chi_{\mathcal{F}^h}(x)\,\mathrm{d}x \to 0 \quad \text{as} \quad h \to 0$$

implies that

$$\lim_{\varepsilon\to 0}\sum\int_{\varepsilon\square_{\mathbf{k}}}\psi(x)\chi_{\Omega_{\varepsilon,h}}\,\mathrm{d}x\;=\;\int_{\Omega}\psi(x)\,\mathrm{d}x\lim_{\varepsilon\to 0}\int_{\square}\chi_{\mathcal{F}^{h(\varepsilon)}}(x)\,\mathrm{d}x\;=\;0. \quad (7.21)$$

As a result, the desired conclusion (7.17) follows by taking (7.18)–(7.21) into account. □

As an evident consequence of this result and Theorem 7.8, we admit the following conclusion.

Corollary 7.18. *The empty set is the strong χ-limit of the thin periodic structures $\{\Omega_{\varepsilon,h}\}_{\varepsilon>0}$ as $\varepsilon\to 0$ in the sense of Definition 7.4.*

At the same time, it is interesting to observe that for any point $x\in\Omega$ and its arbitrary neighborhood $U(x)$, there exists $\varepsilon_0>0$ such that $U(x)\cap\Omega_{\varepsilon,h}\neq\emptyset$ for every $\varepsilon<\varepsilon_0$. This provides good reasons to take a new view of Ω as a limit set for the sequence $\{\Omega_{\varepsilon,h}\}_{\varepsilon>0}$ in some appropriate sense. To do so, we make use of the geometrical approach. Denote by λ a measure concentrated on $\mathcal{F}$ and proportional to the corresponding Lebesgue measure. Denote by λ^h a measure concentrated on $\mathcal{F}^h$ and proportional to the Lebesgue measure $\mathcal{L}_n$. It is assumed that the measures λ and λ^h are normed by the following condition: $\int_{\square}\mathrm{d}\lambda=\int_{\square}\mathrm{d}\lambda^h=1$. Clearly, $\lambda^h\rightharpoonup\lambda$ as $h\to 0$, that is,

$$\lim_{h\to 0}\int_{\square}\varphi\,\mathrm{d}\lambda^h=\int_{\square}\varphi\,\mathrm{d}\lambda\quad\text{for any function}\quad\varphi\in C^{\infty}_{per}(\square). \quad (7.22)$$

Moreover, each of the structures $\mathcal{F}^h_\varepsilon$ and $\mathcal{F}_\varepsilon$ can be associated with the so-called ε-periodic scaling measures λ^h_ε and λ_ε, respectively. Here,

$$\lambda^h_\varepsilon(B)=\varepsilon^n\lambda^h(B)\;\;\text{and}\;\;\lambda_\varepsilon(B)=\varepsilon^n\lambda(B)\quad\text{for any Borel set}\;\;B\subset\mathbb{R}^n.$$

As usual, we relate parameters h and ε assuming that $h=h(\varepsilon)\to 0$ as $\varepsilon\to 0$. Since $\lambda^h_\varepsilon(\varepsilon\square)=\lambda_\varepsilon(\square)=\varepsilon^n$, it follows that (see Lemma 6.50) the measures λ^h_ε and λ_ε converge weakly to the Lebesgue measure $\mathcal{L}_n$ as $\varepsilon\to 0$, that is,

$$\int_{\mathbb{R}^n}\varphi\,\mathrm{d}\lambda^h_\varepsilon\to\int_{\mathbb{R}^n}\varphi\,\mathrm{d}x,\quad\int_{\mathbb{R}^n}\varphi\,\mathrm{d}\lambda_\varepsilon\to\int_{\mathbb{R}^n}\varphi\,\mathrm{d}x,\quad\forall\,\varphi\in C^{\infty}_0(\mathbb{R}^n).$$

We are now in a position to state the main result of this section.

Theorem 7.19. *Let $\chi_{\Omega_{\varepsilon,h}}$ and $\chi_{\mathcal{F}^h}$ be the characteristic functions of the sets $\Omega_{\varepsilon,h}$ and $\mathcal{F}^h$, respectively. Then*

$$\chi_{\mathcal{F}^h}\;\longrightarrow\;1\quad\textit{in}\quad L^2(\square,\mathrm{d}\lambda^h), \quad (7.23)$$

$$\chi_{\Omega_{\varepsilon,h}}\;\overset{2}{\longrightarrow}\;\chi_{\Omega}\quad\textit{in}\quad L^2(\Omega,\mathrm{d}\lambda^h_\varepsilon), \quad (7.24)$$

$$\chi_{\Omega_{\varepsilon,h}}\longrightarrow\chi_{\Omega}\quad\textit{in}\quad L^2(\Omega,\mathrm{d}\lambda^h_\varepsilon). \quad (7.25)$$

Proof. Since $\lambda^h \rightharpoonup \lambda$ as $h \to 0$ and

$$\int_{\square} \varphi \, \mathrm{d}\lambda^h = \int_{\square} \varphi \chi_{\mathcal{F}^h} \, \mathrm{d}\lambda^h, \quad \varphi \in C^{\infty}_{per}(\square),$$

it follows that (7.22) implies $\chi_{\mathcal{F}^h} \rightharpoonup 1 \ni L^2(\square, \mathrm{d}\lambda)$ in the variable space $L^2(\square, \mathrm{d}\lambda^h)$ (see Definition 6.14). Therefore, to prove assertion (7.23), it remains only to take into account Lemma 6.19 and the fact that

$$\int_{\square} \mathrm{d}\lambda^h = \int_{\square} |\chi_{\mathcal{F}^h}|^2 \, \mathrm{d}\lambda^h, \quad \forall\, h > 0.$$

Due to this result and Theorem 6.51, we have

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\Omega} \varphi(x) \chi_{\Omega_{\varepsilon,h}} \, \mathrm{d}\lambda_{\varepsilon}^{h(\varepsilon)} &= \lim_{\varepsilon \to 0} \int_{\Omega} \varphi(x) \chi_{\mathcal{F}^{h(\varepsilon)}} \left(\frac{x}{\varepsilon} \right) \mathrm{d}\lambda_{\varepsilon}^{h(\varepsilon)} \\ &= \int_{\Omega} \int_{\square} \varphi(x) \, 1 \, \mathrm{d}\lambda(y) \, \mathrm{d}x = \int_{\Omega} \varphi(x) \chi_{\Omega} \, \mathrm{d}x \end{aligned}$$

for any continuous bounded function φ. Hence,

$$\chi_{\Omega_{\varepsilon,h}} \overset{2}{\rightharpoonup} \chi_{\Omega} \ \text{ in } \ L^2(\Omega, \mathrm{d}\lambda_{\varepsilon}^h), \quad \chi_{\Omega_{\varepsilon,h}} \rightharpoonup \chi_{\Omega} \ \text{ in } \ L^2(\Omega, \mathrm{d}\lambda_{\varepsilon}^h).$$

Further, we note that

$$\int_{\Omega} \varphi(x) \, \mathrm{d}\lambda_{\varepsilon}^{h(\varepsilon)} = \int_{\Omega} \varphi(x) \chi_{\Omega_{\varepsilon,h}} \, \mathrm{d}\lambda_{\varepsilon}^{h(\varepsilon)} = \int_{\Omega} \varphi(x) \left| \chi_{\Omega_{\varepsilon,h}} \right|^2 \mathrm{d}\lambda_{\varepsilon}^{h(\varepsilon)}.$$

Since the weak convergence of measures $\mathrm{d}\lambda_{\varepsilon}^h \rightharpoonup \mathrm{d}x$ possesses the property

$$\lim_{\varepsilon \to 0} \int_{\Omega} \varphi \, \mathrm{d}\lambda_{\varepsilon}^h = \int_{\Omega} \varphi \, \mathrm{d}x, \quad \forall\, \varphi \in C(\overline{\Omega})$$

(see Lemma 1.1 in [258]), it follows that

$$\lim_{\varepsilon \to 0} \int_{\Omega} \left| \chi_{\Omega_{\varepsilon,h}} \right|^2 \mathrm{d}\lambda_{\varepsilon}^{h(\varepsilon)} = \int_{\Omega} |\chi_{\Omega}|^2 \, \mathrm{d}x = \int_{\Omega} \int_{\square} |\chi_{\Omega}|^2 \, \mathrm{d}\lambda(y) \, \mathrm{d}x.$$

It remains only to apply the criterium of the strong convergence in variable spaces (see (6.107) and Lemma 6.19). □

We can, therefore, draw the following conclusion: The fact that the thin periodic structure $\Omega_{\varepsilon,h}$ fills up the domain Ω in the limit as $\varepsilon \to 0$ (in symbols, $\Omega_{\varepsilon,h} \rightsquigarrow \Omega$) means that its characteristic function $\chi_{\Omega_{\varepsilon,h}}$ converges to χ_{Ω} in the variable space $L^2(\Omega, \mathrm{d}\lambda_{\varepsilon}^h)$.

To end of this section, we observe that the weak convergence $\mathrm{d}\lambda_{\varepsilon}^h \rightharpoonup \mathrm{d}x$ as $\varepsilon \to 0$ implies

$$\int_\Omega \varphi \frac{\chi_{\Omega_{\varepsilon,h}}}{\int_\square \chi_{\mathcal{F}^{h(\varepsilon)}}\,\mathrm{d}x}\,\mathrm{d}x \equiv \int_\Omega \varphi\,\mathrm{d}\lambda_\varepsilon^h \longrightarrow \int_\Omega \varphi\,\mathrm{d}x \text{ as } \varepsilon \to 0, \quad \forall\,\varphi \in C_0(\Omega),$$

where $\int_\square \chi_{\mathcal{F}^{h(\varepsilon)}}\,\mathrm{d}x \to 0$ by supposition that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

However, this fact can be always interpreted as weak-$*$ convergence in $L^\infty(\Omega)$ of $\left(\int_\square \chi_{\mathcal{F}^{h(\varepsilon)}}\,\mathrm{d}x\right)^{-1} \chi_{\Omega_{\varepsilon,h}}$ to χ_Ω. Thus, for comparison with the strong χ-convergence, we have the following implication:

$$\Omega_{\varepsilon,h} \rightsquigarrow \Omega \implies \left(\int_\square \chi_{\mathcal{F}^{h(\varepsilon)}}\,\mathrm{d}x\right)^{-1} \chi_{\Omega_{\varepsilon,h}} \overset{*}{\rightharpoonup} \chi_\Omega \text{ in } L^\infty(\Omega).$$

7.4 Set convergence in the sense of Kuratowski and in the Hausdorff metric

To begin, we recall a general concept of the set convergence, named Kuratowski convergence [161]. Let $(\mathbb{X}, \tau)$ be an arbitrary topological space and let $\{E_k\}_{k=1}^\infty$ be a sequence of subsets of $\mathbb{X}$.

Definition 7.20. *The K-lower limit of the sequence $\{E_k\}_{k=1}^\infty$, denoted by $K-\liminf_{k\to\infty} E_k$, is the set of all points $x \in \mathbb{X}$ with the following property: For every neighborhood U of x, there exists $k_0 \in \mathbb{N}$ such that $U \cap E_k \neq \emptyset$ if $k \geq k_0$. The K-upper limit, denoted by $K-\limsup_{k\to\infty} E_k$, is the set of all points $x \in \mathbb{X}$ with the following property: For every neighborhood U of x and for every $k_0 \in \mathbb{N}$ there exists $k \geq k_0$ such that $U \cap E_k \neq \emptyset$. If there exists a subset E of $\mathbb{X}$ such that $E = K-\liminf_{k\to\infty} E_k = K-\limsup_{k\to\infty} E_k$, then we write $E = K-\lim_{k\to\infty} E_k$ and say that $\{E_k\}_{k=1}^\infty$ converges to the set E in the sense of Kuratowski, or K-converges to E.*

We also need the sequential version of such convergence.

Definition 7.21. *We define the sequential K-lower and K-upper limits of $\{E_k\}_{k=1}^\infty$ by*

$$K_s-\liminf_{k\to\infty} E_k = \{x \in \mathbb{X} \,:\, \exists\, x_k \to x,\ \exists\, k_0 \in \mathbb{N},\ \forall\, k \geq k_0 \,:\, x_k \in E_k\},$$

$$K_s-\limsup_{k\to\infty} E_k = \left\{x \in \mathbb{X} : \exists\, \sigma(k) \to +\infty, \exists\, x_k \to x, \forall\, k \in \mathbb{N} : x_k \in E_{\sigma(k)}\right\},$$

respectively. Now, we say that the sequence $\{E_k\}_{k=1}^\infty$ K_s-converges to E if

$$K_s-\liminf_{k\to\infty} E_k = K_s-\limsup_{k\to\infty} E_k = E.$$

Sometimes we will write $K_s(\tau)-\lim$, $K(\tau)-\lim$, and so forth, to indicate the topology τ explicitly.

Remark 7.22. We note that K-convergence coincides with K_s-convergence if the space $\mathbb{X}$ satisfies the first axiom of countability [161]. It is also clear that $K_s-\liminf_{k\to\infty} E_k \subseteq K_s-\limsup_{k\to\infty} E_k$; hence, $\{E_k\}_{k=1}^{\infty}$ K_s-converges to E if and only if

$$K_s-\limsup_{k\to\infty} E_k \subseteq E \subseteq K_s-\liminf_{k\to\infty} E_k.$$

Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{X}$. If $E_k = \{x_k\}$ for every $k \in \mathbb{N}$, then the K-upper limit of $\{E_k\}_{k=1}^{\infty}$ is the set of all cluster points of $\{x_k\}_{k=1}^{\infty}$ and the K-lower limit of $\{E_k\}_{k=1}^{\infty}$ is the (possibly empty) set of all limits of $\{x_k\}_{k=1}^{\infty}$ (recall that we do not assume that $\mathbb{X}$ satisfies the Hausdorff separation axiom, so $\{x_k\}_{k=1}^{\infty}$ may have more that one limit). If $E_k = \{x_i : i \geq k\}$, then $\{E_k\}_{k=1}^{\infty}$ K-converges to the set of all cluster points of $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{X}$.

In the following proposition, we list the main properties of K-convergence (see, for instance, [161]).

Proposition 7.23. *(i) If $E \subset \mathbb{X}$ and $E_k = E$ for every $k \in \mathbb{N}$, then $K-\lim E_k = \overline{E}$ is the closure of E in $\mathbb{X}$.*

(ii) For any subsequence $\{\sigma(k)\}_{k=1}^{\infty}$, we have

$$K_s-\liminf_{k\to\infty} E_k \subseteq K_s-\liminf_{k\to\infty} E_{\sigma(k)},$$
$$K_s-\limsup_{k\to\infty} E_k \supseteq K_s-\limsup_{k\to\infty} E_{\sigma(k)}.$$

(iii) If $E_k \subseteq F_k$ for every $k \in \mathbb{N}$, then

$$K_s-\liminf_{k\to\infty} E_k \subseteq K_s-\liminf_{k\to\infty} F_k,\ K_s-\limsup_{k\to\infty} E_k \subseteq K_s-\limsup_{k\to\infty} F_k.$$

(iv) The sets $K-\liminf_{k\to\infty} E_k$ and $K-\limsup_{k\to\infty} E_k$ are closed. We also have

$$K-\liminf_{k\to\infty} E_k = K-\liminf_{k\to\infty} \overline{E}_k,\ K-\limsup_{k\to\infty} E_k = K-\limsup_{k\to\infty} \overline{E}_k.$$

(v) A sequence $\{E_k\}_{k=1}^{\infty}$ K_s-converges to E if and only if any subsequence of $\{E_k\}_{k=1}^{\infty}$ has a further subsequence K_s-converging to E.

The main result on K-convergence is the following Kuratowski compactness theorem.

Theorem 7.24. *Assume that $\mathbb{X}$ satisfies the second axiom of countability (i.e., $\mathbb{X}$ has a countable base). Then for every sequence $\{E_k\}_{k=1}^{\infty}$, there exists a subsequence $\left\{E_{\sigma(k)}\right\}_{k=1}^{\infty}$ such that it K-converges to a subset $E \subset \mathbb{X}$ (possibly, empty).*

For the proof and more details, we refer to [161].

Example 7.25. Let $\mathbb{X} = \mathbb{R}$ and let $E_k = [0, 1/k] \cup [k, +\infty)$. Then the sequence of sets $\{E_k\}_{k=1}^{\infty}$ K-converges to $\{0\}$.

Example 7.26. Let $\mathbb{X} = \mathbb{R}^2$ and let $E_k = \{(1/k, y) \ : \ 0 < y < 1\}$. Then

$$K\text{-}\lim_{k\to\infty} E_k = \{(0, y) \ : \ 0 \le y \le 1\}\,.$$

Example 7.27. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{X}$. If $E_k = \{x_k\}$ for every $k \in \mathbb{N}$, then $K\text{-}\limsup_{k\to\infty} E_k$ is the set of all cluster points of $\{x_k\}_{k=1}^{\infty}$ and $K\text{-}\liminf_{k\to\infty} E_k$ is the (possibly empty) set of all limits of $\{x_k\}_{k=1}^{\infty}$ (we do not assume $\mathbb{X}$ to be a Hausdorff space). If $E_k = \{x_j \ : \ j \ge k\}$, then $K\text{-}\lim_{k\to\infty}$ is the set of all cluster points of $\{x_k\}_{k=1}^{\infty}$.

In many cases, the natural topology on the class of closed sets is given by the so-called Hausdorff distance.

Definition 7.28. *The Hausdorff distance between two nonempty closed subsets E_1 and E_2 of $\mathbb{R}^n$ (taking possibly the value $+\infty$) is defined by*

$$d_H(E_1, E_2) = \max\left\{\sup_{x\in E_1} \operatorname{dist}(x, E_2), \sup_{x\in E_2} \operatorname{dist}(x, E_1)\right\}, \tag{7.26}$$

where $\operatorname{dist}(x, E_1) = \inf\{|x - y| \ : \ y \in E_1\}$.

Definition 7.29. *We say that a sequence $\{E_k\}_{k=1}^{\infty}$ of nonempty closed subsets of $\mathbb{R}^n$ converges to a nonempty closed subset E in the Hausdorff metric (in symbols, $E_k \overset{H}{\to} E$) if $d_H(E_k, E)$ converges to 0 as $k \to \infty$.*

Remark 7.30. The convergence in the Hausdorff metric implies the convergence in the sense of Kuratowski and, in general, the converse is false. However, if $\{E_k\}_{k=1}^{\infty}$ is a uniformly bounded sequence of nonempty closed sets (i.e., there exists a bounded subset of $\mathbb{R}^n$ which contains all sets of the sequence), then $\{E_k\}_{k=1}^{\infty}$ converges to a closed set E in the Hausdorff metric if and only if $\{E_k\}_{k=1}^{\infty}$ converges to E in the sense of Kuratowski.

Remark 7.31. It is well known (see [107]) that the class of all closed subsets of a given compact set is compact with respect to the Hausdorff distance. Moreover, the convergence $E_k \overset{H}{\to} E$ (induced by the Hausdorff distance) is equivalent to the so-called uniform convergence, which occurs if for every $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that

$$E_k \subset E + B_{0,\varepsilon} \quad \text{and} \quad E \subset E_k + B_{0,\varepsilon} \quad \forall\, k \ge k_\varepsilon.$$

Here $B_{0,\varepsilon}$ is the ball in $\mathbb{R}^n$ centered at the origin and of radius ε.

Further, we consider the concept of set convergence adapted to the case of open subsets of $\mathbb{R}^n$. Let D be a bounded open set in $\mathbb{R}^n$. Let A and B be two open sets containing in the bounded domain D of $\mathbb{R}^n$.

Definition 7.32. *We say that $\delta_\infty(A,B)$ is the distance between open sets A and B if it is the Pompeiu–Hausdorff distance between the closed sets $\overline{D}\setminus A$ and $\overline{D}\setminus A$, that is,*

$$\delta_\infty(A,B)=\max\{\rho(A,B),\rho(B,A)\}, \tag{7.27}$$

where

$$\rho(A,B)=\sup_{x\in\overline{D}\setminus A}\sup_{y\in\overline{D}\setminus B}\|x-y\|_{\mathbb{R}^n}.$$

Definition 7.33. *A sequence of open sets $\{A_k\subset D\}_{k=1}^{\infty}$ is said to be δ_∞-convergent to an open set $A\subset D$ if $\delta_\infty(A_k,A)\to 0$ as $k\to\infty$.*

As a result, we have: $A_k \xrightarrow{\delta_\infty} A$ if and only if $\overline{D}\setminus A_k$ converges to $\overline{D}\setminus A$ in the sense of Pompeiu–Hausdorff (in symbols, $\overline{D}\setminus A = PH-\lim_{k\to\infty}\overline{D}\setminus A_k$).

Remark 7.34. Note that without supplementary regularity assumptions on the boundaries of the sets, there is no connection between the distance δ_∞ and distance μ defined in (7.2). For instance, let $\overline{B(0,1)}$ be the open unit ball in $\mathbb{R}^n$. Add k (closed) rays of length 2, starting from the origin, into the ball such that the union of the rays is dense in $\overline{B(0,2)}$ as $k\to\infty$ and denote the resulting (closed) sets by A_k. Then

$$PH-\lim_{k\to\infty}A_k=\overline{B(0,2)}, \tag{7.28}$$

$$\mu(A_k,\overline{B(0,1)})\to 0 \text{ as } k\to\infty. \tag{7.29}$$

In Chenais [60], it was proved that for uniformly Lipschitz domains, convergence in the metric (7.27) yields convergence in the metric (7.2) with the same limit up to a set of zero measure.

Note that the convergence with respect to the Pompeiu–Hausdorff distance is closely related to the convergence in the Kuratowski sense, but in some important respects, it is distinct from it. It particular, if a sequence $\{\overline{A}_k\}_{k=1}^{\infty}$ is uniformly bounded in $\mathbb{R}^n$, then its convergence in the sense of Kuratowski is equivalent to the convergence with respect to Pompeiu–Hausdorff distance. However, the HP-convergence is not equivalent to K-convergence without this boundedness restriction. Note also that, in general, the PH-limits of $\{\overline{\Omega}_k\}_{k=1}^{\infty}$, where Ω_k are open subsets of D, and of $\{\overline{D}\setminus\Omega_k\}_{k=1}^{\infty}$ may be not complementary to each other (see the above example with the sets $\Omega_k:=A_k$).

As for the topology on the class of open sets, we introduce several useful notions. We set

$$\mathcal{A}=\{\Omega\ :\ \Omega\subseteq D,\ \Omega\ \text{ is open}\}.$$

Definition 7.35. *We say that τ is the Hausdorff complementary topology on $\mathcal{A}$ if it given by the metric*

$$d_{H^c}(\Omega_1,\Omega_2)=d_H(\Omega_1^c,\Omega_2^c), \tag{7.30}$$

where Ω_k^c are the complements of Ω_k in $\mathbb{R}^n$.

Definition 7.36. *We say that a sequence* $\{\Omega_k\}_{k=1}^{\infty}$ *of open subsets of* $\mathbb{R}^n$ *converges to an open* Ω *in the Hausdorff complementary topology (in symbols,* $\Omega_k \overset{H^c}{\to} \Omega$*) if* $d_{H^c}(\Omega_k, \Omega)$ *converges to* 0 *as* $k \to \infty$.

Remark 7.37. Since every uniformly bounded sequence of nonempty closed sets is compact with respect to the Hausdorff convergence, it follows that every uniformly bounded sequence of open sets is compact with respect to the Hausdorff complementary topology. Moreover, a uniformly bounded sequence of open sets $\{\Omega_k\}_{k=1}^{\infty}$ converges to an open set Ω in the Hausdorff complementary topology if and only if the sequence $\{\Omega_k^c\}_{k=1}^{\infty}$ converges to Ω^c in the sense of Kuratowski.

Proposition 7.38. *The following properties of the Hausdorff convergence hold:*

1. $(\mathcal{A}, d_{H^c})$ *is a compact metric space.*
2. *If* $\Omega_k \overset{H^c}{\to} \Omega$*, then for every compact set* $K \subseteq \Omega$*, there exists* $N_K \in \mathbb{N}$ *such that for every* $k \geq N_K$ *we have* $K \subset \Omega_k$.
3. *The Lebesgue measure is lower semicontinuous in the* H^c *topology.*
4. *The number of connected components of the complement of an open set is lower semicontinuous in the* H^c *topology, that is, if* $\{\Omega_k\}_{k=1}^{\infty}$ *and* Ω *are open subsets of* $\mathbb{R}^n$ *and*

$$\Omega_k \overset{H^c}{\to} \Omega, \quad \text{then} \quad \liminf_{k\to\infty} \sharp\Omega_k^c \geq \sharp\Omega^c,$$

   where by $\sharp$ *we denote the number of connected components.*
5. *A sequence of open sets* $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset D$ H^c*-converges to an open set* Ω *if and only if the sequence of complements* $\{\Omega_\varepsilon^c\}_{\varepsilon>0}$ *converges to* Ω^c *in the sense of Kuratowski.*

Proof. The proof of this proposition is quite simple; we refer to [121] for further details. For the convenience of the reader, we only recall that property 1 is a consequence of the Ascoli–Arzelâ theorem. □

For these and other properties on H^c-topology, we refer to [107]. We recall here the well known fact (see [38]) that in the case when $p > n$, the H^c-convergence of open sets $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset D$ is equivalent to the convergence in the sense of Mosco of the associated Sobolev spaces $W_0^{1,p}(\Omega_\varepsilon)$.

Definition 7.39. *We say that a sequence of Sobolev spaces* $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ *converges in the sense of Mosco to* $W_0^{1,p}(\Omega)$ *if the following conditions are satisfied (see [191]):*

(M_1) *For every* $y \in W_0^{1,p}(\Omega)$*, there exists a sequence* $\left\{y_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ *such that* $\widetilde{y}_\varepsilon \to \widetilde{y}$ *strongly in* $W^{1,p}(\mathbb{R}^n)$.

(M_2) *fI* $\{\varepsilon_k\}_{k\in\mathbb{N}}$ *is a sequence converging to* 0, $\{y_k\}_{k\in\mathbb{N}}$ *is a sequence such that* $y_k \in W_0^{1,p}(\Omega_{\varepsilon_k})$ *for every* $k \in \mathbb{N}$ *and* $\widetilde{y}_k \to \psi$ *weakly in* $W^{1,p}(\mathbb{R}^n)$, *then there exists a function* $y \in W_0^{1,p}(\Omega)$ *such that* $y = \psi|_\Omega$.

Here, we denote by $\widetilde{y}_\varepsilon$ *(respectively,* $\widetilde{y}$*) the zero-extension to* $\mathbb{R}^n$ *of a function defined on* Ω_ε *(respectively, on* Ω*), that is,* $\widetilde{y}_\varepsilon = \widetilde{y}_\varepsilon \chi_{\Omega_\varepsilon}$ *and* $\widetilde{y} = \widetilde{y}\chi_\Omega$.

Remark 7.40. As follows from Definition 7.39, the sequence of the spaces $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ converges in the sense of Mosco to $W_0^{1,p}(\Omega)$ if the following two inclusions hold:

$$W_0^{1,p}(\Omega) \subseteq K_s(s)\text{-}\liminf_{\varepsilon\to 0} W_0^{1,p}(\Omega_\varepsilon), \tag{7.31}$$

$$W_0^{1,p}(\Omega) \supseteq K_s(w)\text{-}\limsup_{\varepsilon\to 0} W_0^{1,p}(\Omega_\varepsilon). \tag{7.32}$$

Here, by $K_s(s)\text{-}\liminf_{\varepsilon\to 0} W_0^{1,p}(\Omega_\varepsilon)$ and $K_s(w)\text{-}\limsup_{\varepsilon\to 0} W_0^{1,p}(\Omega_\varepsilon)$ we respectively denote the strong lower limit and the weak upper limit in the sense of Kuratowski of the sequence of Sobolev spaces $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$, which are defined by (see Definition 7.20)

$$K_s(s)\text{-}\liminf_{\varepsilon\to 0} W_0^{1,p}(\Omega_\varepsilon) = \left\{ u \in W_0^{1,p}(D) \,\middle|\, \exists\, u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon), \quad \text{and} \quad u_\varepsilon \overset{W_0^{1,p}(D)}{\to} u \right\}, \tag{7.33}$$

$$K_s(w)\text{-}\limsup_{\varepsilon\to 0} W_0^{1,p}(\Omega_\varepsilon) = \left\{ u \in W_0^{1,p}(D) \,\middle|\, \exists\, u_{\varepsilon_i} \in W_0^{1,p}(\Omega_{\varepsilon_i}), \quad \text{and} \quad u_{\varepsilon_i} \overset{W_0^{1,p}(D)}{\rightharpoonup} u \right\}. \tag{7.34}$$

Note that, in general,

$$K_s(s)\text{-}\liminf_{\varepsilon\to 0} W_0^{1,p}(\Omega_\varepsilon) \subseteq K_s(w)\text{-}\limsup_{\varepsilon\to 0} W_0^{1,p}(\Omega_\varepsilon).$$

Therefore, if $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ converges in the sense of Mosco to $W_0^{1,p}(\Omega)$, then

$$K_s(s)\text{-}\liminf_{\varepsilon\to 0} W_0^{1,p}(\Omega_\varepsilon) = W_0^{1,p}(\Omega) = K_s(w)\text{-}\limsup_{\varepsilon\to 0} W_0^{1,p}(\Omega_\varepsilon). \tag{7.35}$$

As for the general case ($p \le n$), we have the following (see [39]):

Theorem 7.41. *Let* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *be a sequence of open subsets of* D *such that* $\Omega_\varepsilon \overset{H^c}{\longrightarrow} \Omega$ *and* $\Omega_\varepsilon \in \mathcal{W}_w(D)$ *for every* $\varepsilon > 0$, *where the class* $\mathcal{W}_w(D)$ *is defined as*

$$\mathcal{W}_w(D) = \{\Omega \subseteq D \ : \ \forall\, x \in \partial\Omega, \forall\, 0 < r < R < 1;$$
$$\int_r^R \left(\frac{\mathrm{cap}_p(\Omega^c \cap \overline{B(x,t)}; B(x,2t))}{\mathrm{cap}_p(\overline{B(x,t)}; B(x,2t))} \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \geq w(r,R,x) \right\}, \quad (7.36)$$

where $B(x,t)$ is the ball with center at x and radius t, $\mathrm{cap}_p(K;D)$ is the capacity of K in D, and

$$w : (0,1) \times (0,1) \times D \to \mathbb{R}^+$$

is such that

1. $\lim_{r \to 0} w(r,R,x) = +\infty$*, locally uniformly on* $x \in D$*;*
2. w *is lower semicontinuous function in the third argument.*

Then $\Omega \in \mathcal{W}_w(D)$ *and the sequence of Sobolev spaces* $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ *converges in the sense of Mosco to* $W_0^{1,p}(\Omega)$*.*

Theorem 7.42. *Let* $n \geq p > n-1$ *and let* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *be a sequence of open subsets of* D *such that* $\Omega_\varepsilon \xrightarrow{H^c} \Omega$ *and* $\Omega_\varepsilon \in \mathcal{O}_l(D)$ *for every* $\varepsilon > 0$*, where the class* $\mathcal{O}_l(D)$ *is defined as*

$$\mathcal{O}_l(D) = \{\Omega \subseteq D \ : \ \sharp\Omega^c \leq l\} \quad (7.37)$$

(here, as usual, by $\sharp$ *one denotes the number of connected components). Then* $\Omega \in \mathcal{O}_l(D)$ *and the sequence of Sobolev spaces* $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ *converges in the sense of Mosco to* $W_0^{1,p}(\Omega)$*.*

However, the perturbation in H^c topology (without additional assumptions) may be very irregular (see, for instance, [38, 80]). In view of this, we make use once more of the concept of the set convergence. Following Dancer [83] (see also [84]), we have the following:

Definition 7.43. *A sequence* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *of open subsets of* D *topologically converges to an open set* $\Omega \subseteq D$ *(in symbols,* $\Omega_\varepsilon \xrightarrow{\mathrm{top}} \Omega$*) if there exists a compact set* $K_0 \subset \Omega$ *of* p*-capacity zero* $\left(\mathrm{cap}_p(K_0, D) = 0\right)$ *and a compact set* $K_1 \subset \mathbb{R}^n$ *of Lebesgue measure zero such that the following hold:*

(D_1) $\Omega' \subset\subset \Omega \setminus K_0$ *implies that* $\Omega' \subset\subset \Omega_\varepsilon$ *for* ε *small enough.*
(D_2) *For any open set* U *with* $\overline{\Omega} \cup K_1 \subset U$*, we have* $\Omega_\varepsilon \subset U$ *for* ε *small enough.*

Note that without supplementary regularity assumptions on the sets, there is no connection between this type of set convergence and the set convergence in the Hausdorff complementary topology. Moreover, the topological set convergence allows that certain parts of the subsets Ω_ε degenerate and are deleted in the limit. For instance, assume that Ω consists of two disjoint

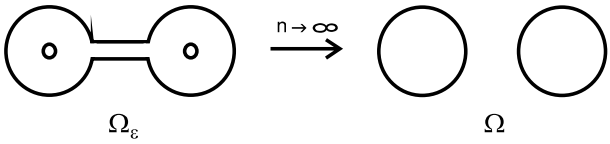


Fig. 7.4. Example of the set convergence in the sense of Definition 7.43

balls and that Ω_ε is a dumbbell with a small hole on each side. By shrinking the holes and the handle, we can approximate the set Ω by sets Ω_ε in the sense of Definition 7.43 as shown in Fig. 7.4. It is obvious that in this case, $d_{H^c}(\Omega_\varepsilon, \Omega)$ does not converge to 0 as $\varepsilon \to 0$. However, as an estimate of an "approximation" of Ω by elements of the above sequence $\Omega_\varepsilon \xrightarrow{\text{top}} \Omega$, we can take the Lebesgue measure of the symmetric set difference $\Omega_\varepsilon \triangle \Omega$, that is, $\mu(\Omega, \Omega_\varepsilon) = \mathcal{L}_n(\Omega \setminus \Omega_\varepsilon \cup \Omega_\varepsilon \setminus \Omega)$. It should be noted that in this case, the distance μ coincides with the well-known Ekeland metric in $L^\infty(D)$ applied to characteristic functions

$$d_E(\chi_\Omega, \chi_{\Omega_\varepsilon}) = \mathcal{L}_n\{x \in D : \chi_\Omega(x) \neq \chi_{\Omega_\varepsilon}(x)\} = \mu(\Omega, \Omega_\varepsilon).$$

Another example of subsets which are H^c-convergent but have no limit in the sense of Definition 7.43 is the sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ containing an oscillating crack with vanishing amplitude ε (see Fig. 7.5).

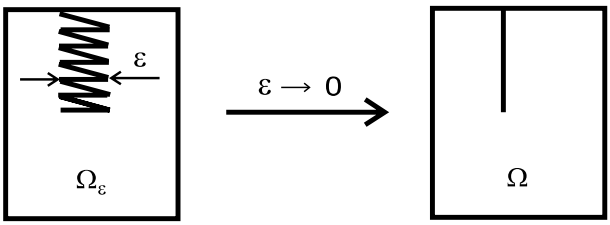


Fig. 7.5. The p-unstable sets which are compact with respect to the H^c topology

We refer to [37] for a detailed description of this topic.

7.5 Parametrical convergence of open sets and of associated mappings

Our main interest in this section is to introduce the so-called distance-type mappings and discuss their application to the convergence of associated sets. Let Ω be a given open subset of D. For an open subset $\Omega \subset D$, we introduce

the so-called distance-type mapping $d_\Omega : \overline{D} \to \mathbb{R}$ based on the Euclidean distance functions associated with the domain Ω and its complement

$$d_\Omega(x) = \begin{cases} \operatorname{dist}(x, \overline{D} \setminus \Omega) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \partial\Omega, \\ -\operatorname{dist}(x, \overline{\Omega}) & \text{if } x \in \overline{D} \setminus \overline{\Omega}. \end{cases} \tag{7.38}$$

It is well known that the mapping d_Ω is uniformly Lipschitzian for any open subset $\Omega \subset D$ (see Clarke [68]). Let $\{\Omega_k \subset D\}_{k=1}^{\infty}$ be a sequence of open sets, not necessarily connected. Let $d_k = d_{\Omega_k}$ be the associated mapping via (7.38). By the Ascoli theorem, on a subsequence again indexed by k, we have $d_k \to \widehat{d}$ uniformly in $\overline{D}$. However, $\widehat{d}$ is not necessary a function of the same type since, in general, the PH-limits of $\{\overline{\Omega}_k\}_{k=1}^{\infty}$ and of $\{\overline{D} \setminus \Omega_k\}_{k=1}^{\infty}$ may not be complementary to each other (see Definition 7.32). The following result indicates that the well known compactness property of the Pompeiu–Hausdorff distance is a direct consequence of the Ascoli compactness criterion (see [178]).

Proposition 7.44. *Let* $\widehat{\Omega} = \left\{x \in D \mid \widehat{d}(x) > 0\right\}$ *(possibly void). Then*

$$\widehat{\Omega} = D \setminus \left[PH - \lim_{k\to\infty} \left(\overline{D} \setminus \Omega_k\right)\right].$$

Proof. Let us fix an element x of $D \setminus \left[PH - \lim_{k\to\infty} \left(\overline{D} \setminus \Omega_k\right)\right]$ such that $x \notin PH - \lim_{k\to\infty} \left(\overline{D} \setminus \Omega_k\right)$. Then, by definition of the Pompeiu–Hausdorff distance, we have $\lim_{k\to\infty} \operatorname{dist}(x, \overline{D} \setminus \Omega_k) > 0$. Hence, $\lim_{k\to\infty} d_k(x) > 0$ (i.e., $x \in \widehat{\Omega}$). Thus, we come to the inclusion

$$D \setminus \left[PH - \lim_{k\to\infty} \left(\overline{D} \setminus \Omega_k\right)\right] \subseteq \widehat{\Omega}.$$

To obtain the reverse one, we assume that

$$x \in \widehat{\Omega} \quad \text{and} \quad x \notin D \setminus \left[PH - \lim_{k\to\infty} \left(\overline{D} \setminus \Omega_k\right)\right].$$

Then $\widehat{d}(x) > 0$ and $x \in PH - \lim_{k\to\infty} \left(\overline{D} \setminus \Omega_k\right)$, that is, $\widehat{d}(x) > 0$ and there are elements $x_k \in \overline{D} \setminus \Omega_k$ such that $x_k \to x$. This means that $\widehat{d}(x) > 0$, $d_k(x_k) \le 0$, and $x_k \to x$. By the uniform convergence, we have $\widehat{d}(x) > 0$ and $\widehat{d}(x) \le 0$, which leads to a contradiction. It follows that

$$\widehat{\Omega} \subseteq D \setminus \left[PH - \lim_{k\to\infty} \left(\overline{D} \setminus \Omega_k\right)\right],$$

and this concludes the proof. □

As a consequence of this proposition, we have the following result.

Proposition 7.45. *If* $PH-\lim_{k\to\infty}\left(\overline{D}\setminus\Omega_k\right)=\overline{D}\setminus\widehat{\Omega}$, *then for any compact set* $K\subset\overline{\Omega}$ *there is an integer* $k_K=k(K)\in\mathbb{N}$ *such that*

$$K\subset\Omega_k \quad \textit{for} \quad k\geq k_K.$$

Proof. Using the same notation as in Proposition 7.44, we note that since the mapping $\widehat{d}$ is continuous on $\overline{D}$ and strictly positive on K, it follows that there exists a constant $c_K>0$ satisfying the inequality

$$\widehat{d}(x)\geq c_K>0, \quad \forall\, x\in K.$$

By the uniform convergence, for any $k\geq k_K$, we obtain $d_k(x)\geq c_K/2>0$ for all $x\in K$, that is, $K\subset\Omega_k$ for all $k\geq k_K$, as required. □

Remark 7.46. This property is called the Γ-property by Liu [177], and as we will see later, it plays an essential role in the local convergence theory for the solutions of PDEs defined in a sequence of bounded domains. The same property is also proved in Pironneau [214], by different methods, together with other domain convergence results.

We are now in a position to introduce the following concept.

Definition 7.47. *A sequence of open sets* $\{\Omega_k\subset D\}_{k=1}^{\infty}$ *is called parametrically convergent to an open set* $\widetilde{\Omega}\subset D$ *if there is a sequence of continuous mappings* $\left\{p_k:\overline{D}\to\mathbb{R}\right\}_{k=1}^{\infty}$ *such that* $p_k\to\widetilde{p}$ *uniformly in* $\overline{D}$ *and*

$$\begin{aligned}
\Omega_k &= \{x\in D \mid p_k(x)>0\},\\
D\setminus\overline{\Omega}_k &= \{x\in D \mid p_k(x)<0\},\\
\widetilde{\Omega} &= \{x\in D \mid \widetilde{p}(x)>0\},\\
D\setminus\overline{\widetilde{\Omega}} &= \{x\in D \mid \widetilde{p}(x)<0\}.
\end{aligned}$$

We denote this limit by $\widetilde{\Omega}=p-\lim\limits_{k\to\infty}\Omega_k$.

Remark 7.48. It is possible to weaken the conditions of this definition by replacing the uniform convergence with other types of functional convergence for the mappings $\left\{p_k:\overline{D}\to\mathbb{R}\right\}_{k=1}^{\infty}$.

Note that the "parameterization" p_k associated with the domain Ω_k is not unique and the distance mapping d_k is just one example of such a parameterization. It means that, in general, the p-limit and the convergence properties depend on the parameterization. If it is different from the function d_Ω, then the convergence may differ from the Pompeiu–Hausdorff convergence.

Example 7.49. Let us choose the mapping $\widetilde{p}:\mathbb{R}\to\mathbb{R}$ as follows:

$$\widetilde{p}(x)=\begin{cases}-(x-1)^2+1/2 & \text{if } x\geq 1/2,\\ x^2 & \text{if } |x|\leq 1/2,\\ -(x+1)^2+1/2 & \text{if } x\leq -1/2.\end{cases} \tag{7.39}$$

We rotate its graph to define a continuous mapping $p : \mathbb{R}^2 \to \mathbb{R}$. Let $p_k : \mathbb{R}^2 \to \mathbb{R}$ be defined as $p_k(x) = p(x) + 1/k$. Then the corresponding domains are

$$\Omega_k = \left\{ x \in \mathbb{R}^2 \mid \|x\|_{\mathbb{R}^2} < 1 + \sqrt{(n+2)/2n} \right\},$$
$$\Omega = \left\{ x \in \mathbb{R}^2 \mid 0 < \|x\|_{\mathbb{R}^2} < 1 + 1/\sqrt{2} \right\}.$$

Notice that in this case, Ω is a nonsmooth domain, $\Omega = p - \lim \Omega_k$, and $\overline{\Omega} \neq PH - \lim_{k\to\infty} \overline{\Omega}_k$. Moreover, if $\widetilde{p}$ is 0 around $x = 0$ on some interval, then $p - \lim \Omega_k$ will be a circular crown, and so forth. By taking $\sup(p_k, p_m)$ or $\inf(p_k, p_m)$, one can easily parameterize the domains $\Omega_k \cup \Omega_m$ or $\Omega_k \cap \Omega_m$, respectively.

Following [178], we list the fine properties of the set convergence in the sense of Definition 7.47 and its applications.

Proposition 7.50. *The parametric convergence has the Γ-property for any parameterization.*

Proof. This is similar to the proof of Proposition 7.45. □

Proposition 7.51. *If $\Omega = p - \lim_{k\to\infty} \Omega_k$ and the closed set*

$$C = \left\{ x \in \overline{D} \mid \widetilde{p}(x) = 0 \right\}$$

has zero Lebesgue measure, then $\chi_{\Omega_k} \to \chi_\Omega$ a.e. in D.

Proof. If $x \in \Omega$, then, by definition, $\widetilde{p}(x) > 0$. Thus, there is an integer $k_x = k(x) \in \mathbb{N}$ such that $p_k(x) > 0$ for all $k \geq k_x$. Hence, $\chi_{\Omega_k}(x) = \chi_\Omega(x) = 1$, $\forall\, k \geq k_x$. If $x \in D \backslash \overline{\Omega}$, then $\widetilde{p}(x) < 0$ and there exists an integer $k_x = k(x) \in \mathbb{N}$ such that $p_k(x) < 0$ for all $k \geq k_x$. Consequently, $\chi_{\Omega_k}(x) = \chi_\Omega(x) = 0$ for $k \geq k_x$. Since the set C has zero measure, we get $\chi_{\Omega_k}(x) \to \chi_\Omega(x)$ almost everywhere in D. □

Proposition 7.50 motivates us to introduce the following concept.

Definition 7.52. *Let $\Omega = p - \lim_{k\to\infty} \Omega_k$ and let $\left\{ y_k \in H^1(\Omega_k) \right\}_{k=1}^{\infty}$ be a uniformly bounded sequence (i.e., $\sup_{k\in\mathbb{N}} \|y_k\|_{H^1(\Omega_k)} < +\infty$). We say that the sequence $\{y_k\}_{k=1}^{\infty}$ is locally convergent to $y \in H^1(\Omega)$, and we write $y = L - \lim y_k$ if for any open set $G \subset\subset \Omega$ (compactly embedded in Ω), we have*

$$y_k|_G \to y|_G \quad \text{weakly in} \quad H^1(G). \tag{7.40}$$

Notice that the limit mapping $y = L - \lim y_k$, given by Definition 7.52, is uniquely determined (see [178]). Moreover, in this case, we have the following compactness result.

Theorem 7.53. *Let $\Omega = p-\lim_{k\to\infty}\Omega_k$. Suppose that $\{y_k \in H^1(\Omega_k)\}_{k=1}^{\infty}$ is a uniformly bounded sequence in the scale of the spaces $\{H^1(\Omega_k)\}$. Then there are a function $y \in H^1(\Omega)$ and a subsequence of $\{y_k \in H^1(\Omega_k)\}_{k=1}^{\infty}$ still indexed by k such that $y = L-\lim y_k$.*

Proof. Let $\{G_j \subset\subset \Omega\}_{j=1}^{\infty}$ be a sequence of open subsets such that $G_j \subset G_{j+1}$ and $\cup_{j\in\mathbb{N}} G_j = \Omega$. Following the Eberlein–Šmuljan theorem, for each $j \in \mathbb{N}$ we can choose a subsequence $\{y_k \in H^1(\Omega_k)\}_{k=1}^{\infty}$ (one after another and still indexed by k) such that $y_k|_{G_j} \to y^j\big|_{G_j}$ weakly in $H^1(G_j)$. Using the properties of the family of subsets $\{G_j \subset\subset \Omega\}_{j=1}^{\infty}$, we define the function y on Ω by $y(x) = y^j(x)$ a.e. $x \in G_j$. Then it is clear that $y \in L^2(\Omega)$ since the sequence $\{\|y^j\|_{L^2(G_j)}\}_{j=1}^{\infty}$ is uniformly bounded. Let $\varphi \in C_0^{\infty}(\Omega)$ be any fixed test function. Then, by the initial supposition, there is an index j_0 such that $\varphi \in C_0^{\infty}(G_j)$ for all $j \geq j_0$. Therefore,

$$\begin{aligned}\int_{G_j} \nabla y\,\varphi\,\mathrm{d}x &= \int_{\Omega} \nabla y\,\varphi\,\mathrm{d}x = -\int_{\Omega} y\,\nabla\varphi\,\mathrm{d}x \\ &= -\int_{G_j} y^j\,\nabla\varphi\,\mathrm{d}x = \int_{G_j} \nabla y^j\,\varphi\,\mathrm{d}x.\end{aligned}$$

Hence, $\nabla y = \nabla y^j$ in G_j for all $j \geq j_0$. Since $\{y_k \in H^1(\Omega_k)\}_{k=1}^{\infty}$ is a uniformly bounded sequence in the scale of the spaces $\{H^1(\Omega_k)\}$, it follows that $\nabla y \in [L^2(\Omega)]^n$ (i.e., $y \in H^1(\Omega)$). As easy to see in this case, (7.40) holds true. This concludes the proof. □

Theorem 7.54. *Let $J : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a non-negative measurable mapping such that $J(x,\cdot,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ and $J(x,s,\cdot)$ is convex on $\mathbb{R}^n$. Let $\Omega = p-\lim_{k\to\infty}\Omega_k$. Then*

$$\liminf_{k\to\infty} \int_{\Omega_k} J(x, y_k, \nabla y_k)\,\mathrm{d}x \geq \int_{\Omega} J(x, y, \nabla y)\,\mathrm{d}x \tag{7.41}$$

provided that $y = L-\lim y_k$.

Proof. Let $\{G_j \subset\subset \Omega\}_{j=1}^{\infty}$ be a sequence of open sets selected as in the previous proof. Then we have $\chi_{G_j} \to \chi_{\Omega}$ a.e. in D. By the initial assumptions, we have that $y_k \to y$ weakly in $H^1(G_j)$ for any fixed G_j. Then, due to the property of the weak lower semicontinuity for the convex integrals in fixed domains, we obtain

$$\liminf_{k\to\infty} \int_{G_j} J(x, y_k, \nabla y_k)\,\mathrm{d}x \geq \int_{G_j} J(x, y, \nabla y)\,\mathrm{d}x.$$

As a result, using the positivity of J, Fatou's lemma, and the properties of the family of subsets $\{G_j \subset\subset \Omega\}_{j=1}^{\infty}$, we get

$$\begin{aligned}\int_{\Omega_j} J(x,y,\nabla y)\,\mathrm{d}x &= \int_{\Omega} \lim_{j\to\infty} \chi_{G_j} J(x,y,\nabla y)\,\mathrm{d}x \\ &\le \liminf_{j\to\infty} \int_{G_j} J(x,y,\nabla y)\,\mathrm{d}x \\ &\le \liminf_{j\to\infty} \liminf_{k\to\infty} \int_{\Omega_k} J(x,y_k,\nabla y_k)\,\mathrm{d}x \\ &= \liminf_{k\to\infty} \int_{\Omega_k} J(x,y_k,\nabla y_k)\,\mathrm{d}x.\end{aligned}$$

The proof is complete. □

Let $\mathcal{G}$ be a family of equibounded and equiuniformly continuous functions $g : B(0,l) \to \mathbb{R}$ with $l > 0$ fixed and $B(0,l) \subset \mathbb{R}^{n-1}$ an open ball centered at the origin with radius l. To indicate a class of open sets which is compact with respect to the parameterical convergence, we adopt the following uniform segment property (see [178]).

Definition 7.55. *We say that an open set $\Omega \subset D$ belongs to the class $\mathcal{O}$ (or equivalently, has the segment property) if the following conditions are fulfilled:*

(G1) There is a subset $\mathcal{G}_\Omega \subset \mathcal{G}$ and, for any $g \in \mathcal{G}_\Omega$, it can be associated to an orthogonal system of axes of center $o_g \in \partial\Omega$, "vertical" vector $b_g \in \mathbb{R}^n$ of unit length, and a rotation R_g in $\mathbb{R}^n$ such that $b_g = R_g(0,\dots,0,1)$ and

$$\bigcup_{g\in\mathcal{G}_\Omega} \{R_g(s,0) + o_g + g(s)b_g \mid \forall\, s \in B(0,l)\} = \partial\Omega.$$

(G2) There is an $a > 0$ such that for any $g \in \mathcal{G}_\Omega$, the uniform segment property is valid:

$$\begin{aligned} &R_g(s,0) + o_g + (g(s)+t)\,b_g \in R^n \setminus \overline{\Omega}, && \forall\, s \in B(0,l),\ \forall\, t \in (0,a), \\ &R_g(s,0) + o_g + (g(s)-t)\,b_g \in \Omega, && \forall\, s \in B(0,l),\ \forall\, t \in (0,a). \end{aligned}$$

(G3) There is a constant $r \in (0,l)$ such that

$$\bigcup_{g\in\mathcal{G}_\Omega} \left\{R_g(s,0) + o_g + g(s)b_g \mid \forall\, s \in \overline{B(0,r)}\right\} = \partial\Omega.$$

Remark 7.56. Let us note that conditions (G1) and (G2) represent the usual definition of boundaries of class C with added uniformity assumptions. Moreover, due to the compactness of the boundary $\partial\Omega$, it can be covered by a finite number of local charts; therefore, both conditions are automatically satisfied, and the only real requirement is the uniformity with respect to the whole family $\mathcal{G}$, which does not allow the local charts to shrink. As for condition (G3), this is a uniform restriction property for the whole family $\mathcal{G}$ of the local charts. It avoids, for instance, clustering of singularities (like cusps) near the

boundary of any local chart. Notice also that due to the finite number of the local charts, a positive number $0 < r_\Omega < l$ can be found in each Ω with the above property. However, condition (G3) ensures that this number can be chosen independent of Ω.

Theorem 7.57. *Let $\{\Omega_k\}_{k=1}^{\infty} \subset \mathcal{O}$ be a sequence of open subsets of D. Let $\{d_k : \overline{D} \to \mathbb{R}\}_{k=1}^{\infty}$ be the corresponding sequence of the distance-type functions introduced in (7.38). Then there are an open set $\widehat{\Omega} \subset D$ of the class $\mathcal{O}$, a function $\widehat{d} \in C(\overline{D})$, and a subsequence of $\{\Omega_k\}_{k=1}^{\infty} \subset \mathcal{O}$ still indexed by k such that the following hold:*

(i) $\widehat{\Omega} = p - \lim_{k\to\infty} \Omega_k$ *and* $\widehat{\Omega} = D \setminus \left[PH - \lim_{k\to\infty} \left(\overline{D} \setminus \Omega_k\right)\right]$.

(ii) $d_k \to \widehat{d}$ *in* $C(\overline{D})$.

(iii) $\chi_{\Omega_k} \to \chi_{\widehat{\Omega}}$ *a.e. in* D, *and* $\chi_{\Omega_k} \to \chi_{\widehat{\Omega}}$ *strongly in any* $L^q(D)$ *for all* $q \geq 1$.

(iv) $\partial\widehat{\Omega} = \left\{x \in D \mid \widehat{d}(x) = 0\right\}$ *has zero Lebesgue measure and it can be represented as a finite union of graphs of continuous functions.*

Proof. We refer to Propositions 7.44 and 7.51, and to [178] for the details of this proof. □

7.6 Kuratowski set convergence in variable spaces

Under suitable additional assumptions, the convergence of sets in the sense of Kuratowski can be extended to the case of variable Banach spaces $\mathbb{X}_\varepsilon$. To do so, we make use the initial suppositions of Sect. 6.1. Let the small parameter $\varepsilon > 0$ vary in a strictly decreasing sequence of positive numbers which converges to 0. For any $\varepsilon > 0$, let $\mathbb{X}_\varepsilon$ be a real reflexive separable Banach space, equipped with the norm $\|\cdot\|_{\mathbb{X}_\varepsilon}$. Let $\|\cdot\|_{0,\mathbb{X}_\varepsilon}$ be an auxiliary norm in $\mathbb{X}_\varepsilon$. Further, let $\mathbb{X}$ be a real reflexive separable Banach space equipped with the norm $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{0,\mathbb{X}}$ be an auxiliary norm in $\mathbb{X}$ relative to which the ball $B_{\mathbb{X}}(0,1) = \{u \in \mathbb{X} \ : \ \|u\|_{\mathbb{X}} \leq 1\}$ is precompact. For any $\varepsilon > 0$, we suppose that there exist linear mappings $Q_\varepsilon : \mathbb{X} \to \mathbb{X}_\varepsilon$ and $P_\varepsilon : \mathbb{X}_\varepsilon \to \mathbb{X}$ with properties (6.1)–(6.5). Moreover, by the weak convergence in the scale of variable spaces $\{\mathbb{X}_\varepsilon\}_{\varepsilon>0}$, we mean the convergence in the sense of Definition 6.2.

Definition 7.58. *We say that a sequence of sets $\{E_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ weakly K_s-converges in the scale of variable spaces $\{\mathbb{X}_\varepsilon\}_{\varepsilon>0}$ to a set $E \subset \mathbb{X}$ if the following conditions are satisfied:*

(a) For every $x \in E$, there exist a constant $\varepsilon_0 > 0$ and a sequence $\{x_\varepsilon\}_{\varepsilon>0}$ weakly converging to x in the scale of variable spaces $\{\mathbb{X}_\varepsilon\}$ such that $x_\varepsilon \in E_\varepsilon$ for every $\varepsilon < \varepsilon_0$.

(b) If $\{E_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{E_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ and $\{x_k\}_{k\in\mathbb{N}}$ is a sequence converging to x in the scale of variable spaces $\{\mathbb{X}_\varepsilon\}$ such that $x_k \in E_{\varepsilon_k}$ for every $k \in \mathbb{N}$, then $x \in E$.

We are now in a position to prove the compactness property of weak K_s-convergence in variable spaces provided the suppositions of Sect. 6.1 hold true.

Theorem 7.59. *Let $\{E_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ be an equibounded sequence of sets, that is,*

$$\sup_{\varepsilon>0} \sup_{x\in E_\varepsilon} \|x\|_{\mathbb{X}_\varepsilon} < +\infty.$$

Assume that for any $\varepsilon > 0$ there exist linear mappings $Q_\varepsilon : \mathbb{X} \to \mathbb{X}_\varepsilon$ and $P_\varepsilon : \mathbb{X}_\varepsilon \to \mathbb{X}$ with properties (6.1)–(6.5). Then $\{E_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ has a subsequence which weakly K_s-converges a subset $E \subset \mathbb{X}$ in the sense of Definition 7.58.

Proof. To begin, for every $\varepsilon > 0$ we define the sets

$$\widehat{E}_\varepsilon = P_\varepsilon(E_\varepsilon) = \{\widetilde{x} = P_\varepsilon x,\ \forall\, x \in E_\varepsilon\} \subset \mathbb{X}. \tag{7.42}$$

Since the sequence $\{E_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ is equibounded and operators $\{P_\varepsilon\}_{\varepsilon>0}$ satisfy property (6.3), it follows that there exists a norm bounded closed set $M \subset \mathbb{X}$ such that $\bigcup_{\varepsilon>0} \widehat{E}_\varepsilon \subseteq M$.

By definition, the weak topology on M is the topology induced by the weak topology of $\mathbb{X}$. Since the space $\mathbb{X}$ is separable and reflexive, there exists a metric d on $\mathbb{X}$ such that the weak topology on a norm bounded subset $M \subset \mathbb{X}$ coincides with the topology induced on M by the metric d. In other words, it means that for any sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\mathbb{X}$, the following statements are equivalent:

(j) $x_k \rightharpoonup x$ in $\mathbb{X}$.
(jj) $\{x_k\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{X}$ and $d(x_k, x) \to 0$ as $k \to \infty$

(see, e.g. [99]). We denote by τ the topology associated to the metric d on $\mathbb{X}$.

Since the topology τ has a countable base, by the Kuratowski compactness theorem (Theorem 7.24), there exists a subsequence of $\{\widetilde{E}_\varepsilon\}_{\varepsilon>0} \subset \mathbb{X}$, still denoted by $\{\widetilde{E}_\varepsilon\}_{\varepsilon>0}$, which $K_s(\tau)$-converges to a set $E \subset \mathbb{X}$. In view of (7.42) and the initial assumptions, it is clear that the set E is nonempty.

Now, we prove that E is the K_s-limit of $\{E_\varepsilon\}_{\varepsilon>0}$ in the scale of variable spaces $\{\mathbb{X}_\varepsilon\}_{\varepsilon>0}$.

First, let us verify condition (a) of Definition 7.58. Let x be any element of E. Since $E \subset M$ and (jj) implies (j), it follows that there exist a constant $\varepsilon_0 > 0$ and a sequence $\{\widetilde{x}_\varepsilon = P_\varepsilon(x_\varepsilon)\}_{\varepsilon>0}$ weakly converging to x in $\mathbb{X}$ such that $\widetilde{x}_\varepsilon \in \widetilde{E}_\varepsilon$ for every $\varepsilon < \varepsilon_0$. Using the assumption regarding the norm $\|\cdot\|_{0,\mathbb{X}}$, we find that $\|P_{\varepsilon_k} u_{\varepsilon_k} - u\|_{0,\mathbb{X}} \to 0$. Hence, from (6.4) and (7.42), we conclude that

$$x_\varepsilon \in E_\varepsilon \quad \text{for every} \quad \varepsilon < \varepsilon_0,$$
$$\|P_\varepsilon\,(x_\varepsilon - Q_\varepsilon x)\,\|_{0,\mathbb{X}} = \|x_\varepsilon - Q_\varepsilon x\|_{0,\mathbb{X}_\varepsilon} \to 0,$$

that is, by Definition 6.2, the sequence $\{x_\varepsilon \in \mathbb{X}_\varepsilon\}_{k=1}^{\infty}$ converges weakly to x (in the scale of variable spaces $\{\mathbb{X}_\varepsilon\}$). Thus, condition (a) is valid.

The verification of condition (b) can be done in a similar manner. This concludes the proof. □

Remark 7.60. As an evident consequence of Theorem 7.59, we have the following result: Under suppositions (6.15)–(6.17) of Example 6.10, any equibounded sequence of sets $\left\{E_\varepsilon \subset W^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ is relatively compact with respect to K_s-convergence. Moreover, it is easy to see that in this case, the sequence of periodically perforated domains $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset \Omega$ converges to Ω in the sense of Kuratowski. Hence, suppositions (6.15)–(6.17) imply the following limit property for Sobolev spaces:

$$\Omega_\varepsilon \overset{K}{\to} \Omega \implies W^{1,p}(\Omega_\varepsilon) \supset E_\varepsilon \overset{K_s}{\to} E \subset W^{1,p}(\Omega). \tag{7.43}$$

However, in general, a sequence of the entire Sobolev spaces $\left\{W^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ can fail to satisfy the stability property (7.43) (see, for instance, [36, 249]).

Our next step is to study the K_s-convergence property of sets in the variable spaces $\mathbb{X}_\varepsilon = L^2(\Omega, \mathrm{d}\mu_\varepsilon)$, where $\{\mu_\varepsilon\}_{\varepsilon>0}$ are some Radon measures on $\Omega \subset \mathbb{R}^n$ and $\mathbb{X} = L^2(\Omega)$. In contrast to the suppositions of Theorem 7.59, we make no assumptions on the existence of linear mappings $Q_\varepsilon : \mathbb{X} \to \mathbb{X}_\varepsilon$ and $P_\varepsilon : \mathbb{X}_\varepsilon \to \mathbb{X}$ and their properties. Moreover, we assume that each of the Radon measures μ_ε can be singular with respect to the Lebesgue measure $\mathcal{L}_n$ (see Definition 2.6). It means that $Q_\varepsilon : \mathbb{X} \to \mathbb{X}_\varepsilon$ are linear mappings like "trace operators". So, in this case, there are no well-defined extension operators $P_\varepsilon : \mathbb{X}_\varepsilon \to \mathbb{X}$ with properties (6.3)–(6.5).

For simplicity, we consider the case when each of the Radon measures μ_ε is associated with some ε-periodic graphlike structure on $\Omega \subset \mathbb{R}^2$. To be more specific, we introduce the graphlike structure on the domain Ω as follows: We say that the set $\square = [0,1)^2 = [0,1) \times [0,1)$ is the cell of periodicity for some graph $\mathcal{F}$ on $\mathbb{R}^2$ if $\square$ contains a "star"-structure such that (see Fig. 7.6) the following hold:

(i) All edges of this structure have a common point $M \in \operatorname{int} \square$, each edge is a line segment and all end points of these edges belong to the boundary of $\square$.

(ii) In the set of end points (vertices), there exist pairs $\underline{(M_i; M_k)}$ such that $x_1^{M_i} = x_1^{M_k}$ or $x_2^{M_i} = x_2^{M_k}$.

As follows from condition (ii), we admit the existence of isolated vertices in the $\square$-periodic graph $\mathcal{F}$ on $\mathbb{R}^2$. Let $\varepsilon \in E = (0, \varepsilon^0]$ be a small parameter.

Definition 7.61. *We say that $\mathcal{F}_\varepsilon$ is an ε-periodic graph on $\mathbb{R}^2$ if*

$$\mathcal{F}_\varepsilon = \varepsilon\mathcal{F} = \{\varepsilon x \,:\, x \in \mathcal{F}\}.$$

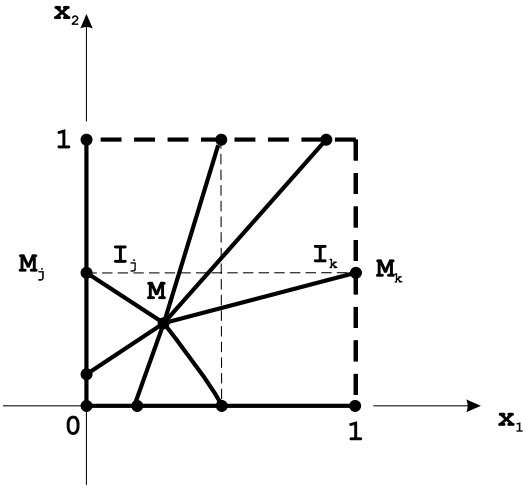


Fig. 7.6. The cell of periodicity

It is clear that the cell of periodicity for Ω_ε is $\varepsilon\square$. Let

$$I^{ed} = \{I_j \ , \ j = 1, 2, \ldots, K\} \tag{7.44}$$

be the set of all edges on $\square$. Let Ω be an open bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary such that

$$\Omega = \{(x_1, x_2) \ : \ x_1 \in \Gamma_1, 0 < x_2 < \gamma(x_1)\}\,, \tag{7.45}$$

where $\Gamma_1 = (0, a)$, $\gamma \in C^1([0, a])$, and $0 < \gamma_0 = \inf_{x_1 \in [0,a]} \gamma(x_1)$. Then $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_2 = \partial\Omega \setminus \Gamma_1$.

Definition 7.62. *We say that Ω_ε has an ε-periodic graphlike structure if $\Omega_\varepsilon = \Omega \cap \mathcal{F}_\varepsilon$ (see Fig. 7.7).*

For every segment $I_i \in I^{ed}, i = 1, 2, \ldots, K$, we denote by μ_i its corresponding Lebesgue measure. Now, we define the $\square$-periodic Borel measure μ in $\mathbb{R}^2$ as follows:

$$\mu = \sum_{i=1}^{K} g_i \, \mu_i \qquad \text{on } \square, \tag{7.46}$$

where $g_1, g_2, \ldots, g_K$ are non-negative weights such that $\displaystyle\int_\square \mathrm{d}\mu = 1$.

Thus, the support of the measure μ is the union of all edges $I_i \in I^{ed}$, each of which is a 1D manifold in $\mathbb{R}^2$. Since the homothetic contraction of the plane with a factor of ε^{-1} takes the grid $\mathcal{F}$ to $\mathcal{F}_\varepsilon = \varepsilon\mathcal{F}$, we introduce a "scaling" ε-periodic measure μ_ε as follows:

$$\mu_\varepsilon(B) = \varepsilon^2 \mu(\varepsilon^{-1} B) \text{ for every Borel set } B \subset \mathbb{R}^2. \tag{7.47}$$

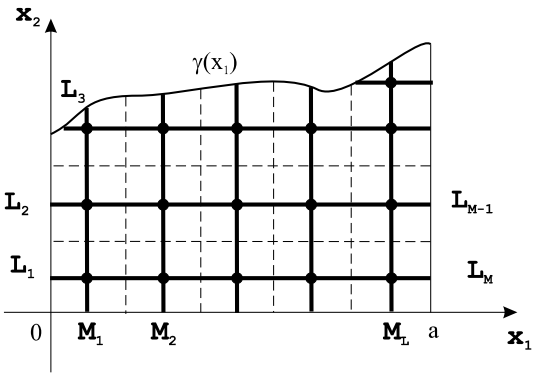


Fig. 7.7. Periodic grid on Ω

Then

$$\int_{\varepsilon\square} \mathrm{d}\mu_\varepsilon = \varepsilon^2 \int_{\square} \mathrm{d}\mu = \varepsilon^2.$$

Hence, the measure μ_ε is weakly convergent to the Lebesgue measure $\mathcal{L}_2$, that is,

$$d\mu_\varepsilon \rightharpoonup \mathrm{d}x \Leftrightarrow \lim_{\varepsilon\to 0} \int_{\mathbb{R}^2} \varphi \,\mathrm{d}\mu_\varepsilon = \int_{\mathbb{R}^2} \varphi \,\mathrm{d}x, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^2). \tag{7.48}$$

In the scale of spaces $\left\{L^2(\Omega, \mathrm{d}\mu_\varepsilon)\right\}$, let us consider the following sequence of sets:

$$E_\varepsilon = \left\{h \in L^2(\Omega, \mathrm{d}\mu_\varepsilon) \ : \ |h| \le C \quad \mu_\varepsilon\text{-a.e. in } \ \Omega\right\}. \tag{7.49}$$

It is clear that the spaces $\mathbb{X}_\varepsilon = L^2(\Omega, \mathrm{d}\mu_\varepsilon)$ and the "limit" space $\mathbb{X} = L^2(\Omega)$ do not satisfy the key initial suppositions of Theorem 7.59. Nevertheless, the following result shows that the sequence (7.49) is K_s-convergent as $\varepsilon \to 0$ and its K_s-limit set can be recovered in an explicit form.

Theorem 7.63. *Let C be a positive constant independent of ε. Let Ω be an open bounded measurable set in the sense of Jordan. Then the sequence (7.49) K_s-converges in the sense of Definition 7.58 to the set E_0, where*

$$E_0 = \left\{h \in L^2(\Omega) \ : \ |h| \le C \quad \textit{a.e. in } \ \Omega\right\}. \tag{7.50}$$

Proof. We divide our proof into two steps.

Step 1. We show that condition (a) of Definition 7.58 is valid. Let u_0 be any element of E_0. Since the space of smooth functions $C^\infty(\Omega)$ is dense in $L^2(\Omega)$, it follows that there is a sequence $\{u_\varepsilon \in C^\infty(\Omega)\}_{\varepsilon>0}$ satisfying the conditions

$$|u_\varepsilon| \le C \ \text{ on } \ \Omega \ \text{ for every } \ \varepsilon > 0, \quad \| \, u_\varepsilon - u^0 \, \|_{L^2(\Omega)} \to 0 \ \text{as } \ \varepsilon \to 0. \tag{7.51}$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi u_\varepsilon \,\mathrm{d}x = \int_\Omega \varphi u^0 \,\mathrm{d}x \ \text{ for every } \ \varphi \in C^0(\overline{\Omega}). \tag{7.52}$$

Further, we note that $u_\varepsilon \in L^2(\Omega, \mathrm{d}\mu_\varepsilon)$ (as a smooth function) and hence $|u_\varepsilon| \le C$ μ_ε-almost everywhere in Ω. We have to show that $u_\varepsilon \rightharpoonup u^0$ weakly in $L^2(\Omega, \mathrm{d}\mu_\varepsilon)$, that is,

$$\int_\Omega \varphi u_\varepsilon \,\mathrm{d}\mu_\varepsilon \to \int_\Omega \varphi u^0 \,\mathrm{d}x \quad \text{ for every } \quad \varphi \in C_0^\infty(\mathbb{R}^2). \tag{7.53}$$

We partition the domain Ω into the sets $\varepsilon\square_j$, where $\{\square_j\}$ is a periodic covering of $\mathbb{R}^2$ by the cell $\square$. Then

$$\int_\Omega \varphi u_\varepsilon \,\mathrm{d}\mu_\varepsilon = \sum_j \int_{\varepsilon\square_j} \varphi u_\varepsilon \,\mathrm{d}\mu_\varepsilon + \sum \int_{\Omega\cap\varepsilon\square_j} \varphi u_\varepsilon \,\mathrm{d}\mu_\varepsilon, \tag{7.54}$$

where the second sum is calculated over the set of the "boundary" squares such that $\varepsilon\square_j \cap \partial\Omega \ne \emptyset$. By the mean value theorem, for each index j there exist points x_j in the cells $\varepsilon\square_j$ such that

$$\begin{aligned}\int_{\varepsilon\square_j} \varphi u_\varepsilon \,\mathrm{d}\mu_\varepsilon &= \varphi(x_j)\, u_\varepsilon(x_j) \int_{\varepsilon\square_j} \mathrm{d}\mu_\varepsilon \\ &= \varphi(x_j)\, u_\varepsilon(x_j)\, \varepsilon^2 \int_\square \mathrm{d}\mu = \varphi(x_j)\ u_\varepsilon(x_j)\, \varepsilon^2, \quad \forall\, j. \end{aligned} \tag{7.55}$$

Then, in view of (7.54), we get

$$\begin{aligned}\int_\Omega \varphi u_\varepsilon \,\mathrm{d}\mu_\varepsilon &= \left(\sum_j \varphi(x_j)\ u_\varepsilon(x_j)\, \varepsilon^2 - \int_\Omega \varphi u_\varepsilon \,\mathrm{d}x\right) \\ &\quad + \sum \int_{\Omega\cap\varepsilon\square_j} \varphi u_\varepsilon \,\mathrm{d}\mu_\varepsilon + \int_\Omega \varphi u_\varepsilon \,\mathrm{d}x \\ &= I_1 + I_2 + \int_\Omega \varphi u_\varepsilon \,\mathrm{d}x. \end{aligned} \tag{7.56}$$

Note that

$$\begin{aligned}|I_2| &= \left|\sum \int_{\Omega\cap\varepsilon\square_j} \varphi u_\varepsilon \,\mathrm{d}\mu_\varepsilon\right| \\ &\le \sup_{j\in\, D(\varepsilon)} \left(\sup_{x\in\, \Omega\cap\varepsilon\square_j} |\varphi|\,|u_\varepsilon|\right) \varepsilon^2\, D(\varepsilon) \le C\|\varphi\|_{C(\Omega)}\varepsilon^2\, D(\varepsilon),\end{aligned}$$

where $D(\varepsilon)$ is the quantity of the "boundary" squares and $\varepsilon^2\, D(\varepsilon) \to 0$ by Jordan's measurability property of the set $\partial\Omega$. Hence, $I_2 \to 0$ as ε tends to 0.

Now, we show that $I_1 \to 0$. For this, we note that

$$
\begin{aligned}
|I_1| &= \left| \sum_j \varphi(x_j)\ u_\varepsilon(x_j)\, \varepsilon^2 - \int_\Omega \varphi u_\varepsilon \, \mathrm{d}x \right| \\
&\le \left| \sum_j \left(\varphi(x_j)\ u_\varepsilon(x_j) - \frac{1}{\varepsilon^2} \int_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \right) \varepsilon^2 \right| + \left| \sum \int_{\Omega \cap \varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \right| \\
&\le \sum_j \left| \varphi(x_j) u_\varepsilon(x_j) - \frac{1}{\varepsilon^2} \int_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \right| \varepsilon^2 + C \|\varphi\|_{C(\Omega)} \varepsilon^2 \, D(\varepsilon).
\end{aligned}
$$

Let us suppose the converse, that is,

$$
\lim_{\varepsilon \to 0} \sum_j \left| \varphi(x_j) u_\varepsilon(x_j) - \varepsilon^{-2} \int_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \right| \varepsilon^2 > 0.
$$

Since Ω is bounded, it is contained in a number of squares $\varepsilon\square_j$ smaller than $\widehat{C}/\varepsilon^2$, where $\widehat{C}$ does not depend on ε. So, there exist a constant $C^* > 0$ and a value $\varepsilon > 0$ such that

$$
\left| \varphi(x_j) u_\varepsilon(x_j) - \varepsilon^{-2} \int_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \right| \ge C^* \tag{7.57}
$$

(for an infinite number of indices j for every fixed ε). Hence, wild oscillations are present in the sequence $\{\varphi u_\varepsilon\}$. However (see [105, 245]), if we have rapid fluctuations in the functions $\{\varphi u_\varepsilon\}$, then the convergence $\varphi u_\varepsilon \to \varphi u^0$ almost everywhere in Ω is excluded. This fact immediately reflects the lack of strong convergence $\varphi u_\varepsilon \to \varphi u^0$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Indeed, by the initial assumptions, we have $|u_\varepsilon| \le C$ for every $\varepsilon \in E$, $\varphi u_\varepsilon \rightharpoonup \varphi u^0$ in $L^1(\Omega)$, and $\| \varphi u_\varepsilon - \varphi u^0 \|_{L^1(\Omega)} \to 0$ as $\varepsilon \to 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^2)$. Let A be any subset of Ω with $|A| \ne 0$. Then, by Valadier's theorem [245] (see also Theorem 2.28), $\varphi u_\varepsilon \to \varphi u^0$ strongly if and only if the following criterion is satisfied: $\forall\, \delta > 0\ \exists\, \varepsilon^0 > 0$, $\exists\, B \subset A$ with $|B| \ne 0$ such that $\forall\, \varepsilon < \varepsilon^0$,

$$
|B|^{-1} \int_B \left| \varphi u_\varepsilon - |B|^{-1} \int_B \varphi u_\varepsilon \, \mathrm{d}x \right| \mathrm{d}x < \delta.
$$

Hence, for any $\varepsilon < \varepsilon^0$, there is a square $\varepsilon\square_j \subset B$ such that

$$
\varepsilon^{-2} \int_{\varepsilon\square_j} \left| \varphi u_\varepsilon - \varepsilon^{-2} \int_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \right| \mathrm{d}x < \delta.
$$

Since the functions φu_ε are continuous and uniformly bounded, it follows that, for any point x_j of $\varepsilon\square_j$ satisfying the condition

$$
\varphi(x_j) u_\varepsilon(x_j) - \varepsilon^{-2} \int_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \ne 0,
$$

there is a constant $A_* > 0$ satisfying

$$\varepsilon^{-2} \int_{\varepsilon\square_j} \Big| \varphi u_\varepsilon - \varepsilon^{-2} \int_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \Big| \, \mathrm{d}x$$
$$= A_* \Big| \varphi(x_j) u_\varepsilon(x_j) - \varepsilon^{-2} \int_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \Big|.$$

Hence,

$$\left| \varphi(x_j) u_\varepsilon(x_j) - \varepsilon^{-2} \int\limits_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \right| < A_*^{-1} \delta,$$

and we come into conflict with (7.57). Hence, our assumption was wrong, and we get

$$\lim_{\varepsilon \to 0} \sum_j \Big| \varphi(x_j) u_\varepsilon(x_j) - \varepsilon^{-2} \int_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}x \Big| \varepsilon^2 = 0.$$

As a result, we have $I_1 \to 0$. Thus, summing up the results obtained above and relations (7.56) and (7.52), we come to the desired identity (7.53). Thereby the following inclusion $K_s\text{–}\lim_{\varepsilon \to 0}(E_\varepsilon) \supseteq E_0$ is established (see Definition 7.58).

Step 2. We show that the reverse inclusion to the above one holds true. To do so, we have to verify condition (b) of Definition 7.58. With this aim in view, we consider a bounded sequence $\{u_k\}_{k \in \mathbb{N}}$ with the following properties:

(a) $u_k \in E_{\varepsilon_k}$ for every $k \in \mathbb{N}$, where $\{\varepsilon_k\}$ is some subsequence of indices converging to 0 as k tends to ∞.

(aa) $u_k \rightharpoonup u^*$ in the variable space $L^2(\Omega, \mathrm{d}\mu_{\varepsilon_k})$.

It remains to show that the limit function $u^* \in L^2(\Omega)$ satisfies the following constraint:

$$|u^*| \le C \quad \text{a.e. in} \quad \Omega. \tag{7.58}$$

Indeed, for every $k \in \mathbb{N}$ and every positive function $\varphi \in C^\infty(\Omega)$, we have

$$\int_\Omega \varphi(C - u_k) \, \mathrm{d}\mu_{\varepsilon_k} \ge 0 \quad \text{and} \quad \int_\Omega \varphi(u_k + C) \, \mathrm{d}\mu_{\varepsilon_k} \ge 0. \tag{7.59}$$

Using the fact that $\mathrm{d}\mu_\varepsilon \rightharpoonup \mathrm{d}x$, property (6.121), and passing to the limit in (7.59) as $k \to \infty$, one gets

$$\int_\Omega \varphi(C - u^*) \, \mathrm{d}x \ge 0 \quad \text{and} \quad \int_\Omega \varphi(u^* + C) \, \mathrm{d}x \ge 0. \tag{7.60}$$

Since φ in (7.60) is an arbitrary positive function, it follows that inequalities (7.58) hold true. As a result, we deduce that $u^* \in E_0$. Thus, we have obtained the required inclusion $K_s\text{–}\lim_{\varepsilon \to 0} E_\varepsilon \subseteq E_0$, which concludes the proof. □

Remark 7.64. We provide a few comments on the scheme of the proof of Theorem 7.63. First, we note that the choice of points x_j in (7.55) is not arbitrary. These points $\{x_j\}$ must be chosen in such way that

$$\int_{\varepsilon\square_j} \varphi u_\varepsilon \, \mathrm{d}\mu_\varepsilon = \varphi(x_j) u_\varepsilon(x_j) \int_{\varepsilon\square_j} \mathrm{d}\mu_\varepsilon.$$

In particular, the position of x_j in $\varepsilon\square_j$ depends on φ and ε, in general. At the same time, the choice of the sequence $\{u_\varepsilon \in C^\infty(\Omega)\}_{\varepsilon>0}$ is only restricted by approximated condition (7.51). Neglecting these assumptions can lead us to inconsistent results. Indeed, if $\{u_\varepsilon\}_{\varepsilon>0}$ is a sequence in $C^\infty(\Omega)$ which converges to a function u^0 in $L^2(\Omega)$, $\varphi \in C^\infty(\Omega)$, and $\{x_j\}$ are arbitrary points in $\varepsilon\square_j$, then, by properties of Riemann integrals, we have

$$\lim_{\varepsilon\to 0} \sum_j \varphi(x_j) u_\varepsilon(x_j) \varepsilon^2 = \int_\Omega \varphi u^0 \, \mathrm{d}x. \tag{7.61}$$

However, this assertion is not true, in general. For example, let us consider the following counterexample. Assume that x_j is the center of the cube $\varepsilon\square_j$ for every j and consider a $C^\infty(\Omega)$ function u_ε such that for every j, $u_\varepsilon = 1$ in the cube of center x_j and side ε^2, $u_\varepsilon = 0$ in $\varepsilon\square_j$ outside the cube of center x_j and side $2\varepsilon^2$ and $0 \le u_\varepsilon \le 1$ in Ω. Then the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ clearly converges to 0 in $L^2(\Omega)$ strongly; however, for every $\varphi \in C^\infty(\Omega)$, we have

$$\lim_{\varepsilon\to 0} \sum_j \varphi(x_j) u_\varepsilon(x_j) \varepsilon^2 = \lim_{\varepsilon\to 0} \sum_j \varphi(x_j) \varepsilon^2 \to \int_\Omega \varphi \, \mathrm{d}x, \tag{7.62}$$

The strong convergence to 0 of u_ε in $L^2(\Omega)$ can be motivated as follows. Note that in each square $\varepsilon\square_j$, u_ε^2 is smaller than the characteristic function of the square of center x_j and side $2\varepsilon^2$. Thus, the integral of u_ε^2 in $\varepsilon\square_j$ is smaller than the measure of the square of center x_j and side $2\varepsilon^2$ (i.e., $(2\varepsilon^2)^2 = 4\varepsilon^4$). On the other hand, as Ω is bounded, it is contained in a number of squares smaller that $\widehat{C}/\varepsilon^2$ and $\widehat{C}$ does not depend on ε. So, we have

$$\int_\Omega u_\varepsilon^2 \, \mathrm{d}x \le \frac{\widehat{C}}{\varepsilon^2}(4\varepsilon^4) = 4\widehat{C}\varepsilon^2 \to 0.$$

Thus, $u_\varepsilon \to 0$ in $L^2(\Omega)$ and we come to relation (7.62) which contradicts (7.61).

To end of this section, we present one observation which is a direct consequence of Theorem 7.63 and which, we feel, will be interesting per se. It is well known that the space $C_0^\infty(\Omega)$ is not dense in $L^\infty(\Omega)$. However, as we will see later, any element of $y \in L^\infty(\Omega)$ can be approximated by a sequence of smooth functions $\{y_\varepsilon \in C_0^\infty(\Omega)\}_{\varepsilon>0}$ for which y is the weak limit in appropriate variable spaces.

For every $\varepsilon > 0$, let us define the space $L^\infty(\Omega,\, d\mu_\varepsilon)$ in the following way: $y_\varepsilon \in L^\infty(\Omega,\, \mathrm{d}\mu_\varepsilon)$ if and only if y_ε is a μ_ε-measurable function on Ω and there exists a constant $M > 0$ such that $|y_\varepsilon(x)| \le M$ μ_ε-everywhere in Ω. For a comparison, we refer to Sect. 6.2.

Definition 7.65. *We say that a sequence* $\{y_\varepsilon \in L^\infty(\Omega,\, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon\to 0}$ *is uniformly bounded if* $\sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega,\, \mathrm{d}\mu_\varepsilon)} < +\infty$.

Definition 7.66. *A uniformly bounded sequence* $\{y_\varepsilon \in L^\infty(\Omega,\, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon\to 0}$ *is said to be weakly-$*$ convergent in the variable space* $L^\infty(\Omega,\, \mathrm{d}\mu_\varepsilon)$ *to* $y \in L^\infty(\Omega)$ *if*

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi y_\varepsilon \,\mathrm{d}\mu_\varepsilon = \int_\Omega \varphi y \,\mathrm{d}x \qquad \text{for every } \varphi \in C_0^\infty(\Omega)$$

(in symbols, $y_\varepsilon \overset{*}{\rightharpoonup} y$*).*

We begin with the following result.

Proposition 7.67. *Let* $\{y_\varepsilon\}_{\varepsilon\to 0}$ *be any bounded sequence in the variable space* $L^\infty(\Omega,\, \mathrm{d}\mu_\varepsilon)$*. Then this sequence is relatively compact with respect to the weak-$*$ convergence in* $L^\infty(\Omega,\, \mathrm{d}\mu_\varepsilon)$*.*

Proof. Let us set

$$l_\varepsilon(\varphi) = \int_\Omega y_\varepsilon \varphi \,\mathrm{d}\mu_\varepsilon \quad \varphi \in C_0^\infty(\Omega).$$

Then, by Hölder inequality, we have

$$|l_\varepsilon(\varphi)| \le \int_\Omega |y_\varepsilon||\varphi| \,\mathrm{d}\mu_\varepsilon \le \|y_\varepsilon\|_{L^\infty(\Omega,\, \mathrm{d}\mu_\varepsilon)} \int_\Omega |\varphi| \,\mathrm{d}\mu_\varepsilon. \tag{7.63}$$

Hence,

$$|l_\varepsilon(\varphi)| \le \|y_\varepsilon\|_{L^\infty(\Omega,\, \mathrm{d}\mu_\varepsilon)} \|\varphi\|_{C(\Omega)} \mu_\varepsilon(K),$$

where by K we denote the support of φ in Ω. Since $\mathrm{d}\mu_\varepsilon \rightharpoonup \mathrm{d}x = \mathrm{d}\mathcal{L}_2$ in the space of Radon measures and

$$\limsup_{\varepsilon\to 0} \mu_\varepsilon(K) \le \mathcal{L}_2(K) \quad \text{for every compact subset of } \Omega$$

(see Zhikov [259]), it follows that

$$|l_\varepsilon(\varphi)| \le 2\|\varphi\|_{C(\Omega)} \mu(K) \sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega,\, \mathrm{d}\mu_\varepsilon)}$$

for $\varepsilon > 0$ small enough. On the other hand, the set

$$T(K) = \{\varphi \in C_0^\infty(\Omega),\ \operatorname{supp}\varphi \subseteq K\}$$

is separable with respect to the norm $\|\varphi\|_{C(\Omega)}$. Then, due to the Cantor diagonal method, it can be easy proved that the sequence $\{l_\varepsilon(\cdot)\}_{\varepsilon\to 0}$ contains

a subsequence which is pointwise convergent on $T(K)$. As a result, there exists a subsequence of values $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty} l_{\varepsilon_j}(\varphi) = l(\varphi), \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{7.64}$$

Taking into account inequality (7.63), we conclude

$$|l(\varphi)| \le \sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega,\,\mathrm{d}\mu_\varepsilon)} \lim_{\varepsilon\to 0} \int_\Omega |\varphi|\,\mathrm{d}\mu_\varepsilon = \sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega,\,\mathrm{d}\mu_\varepsilon)} \int_\Omega |\varphi|\,\mathrm{d}x.$$

So, $l(\cdot)$ is the linear continuous functional on $L^1(\Omega)$. Hence, the following representation holds true:

$$l(\varphi) = \int_\Omega v\varphi\,\mathrm{d}x,$$

where v is some element of $L^\infty(\Omega)$. Thus, in view of (7.64), v is a weak-$*$ limit of the subsequence $\{y_{\varepsilon_j}\}_{j=1}^\infty$ in the variable space $L^\infty(\Omega, \mathrm{d}\mu_\varepsilon)$. □

Combining this result with Theorem 7.63, we have the following conclusion which can be interpreted as the density concept of the locally convex space $C_0^\infty(\Omega)$ in $L^\infty(\Omega)$:

Theorem 7.68. *For any element $y \in L^\infty(\Omega)$, there is a sequence of smooth functions $\{y_\varepsilon \in C_0^\infty(\Omega)\}_{\varepsilon>0}$ satisfying the conditions*

$$|y_\varepsilon| \le \|y\|_{L^\infty(\Omega)} \text{ for every } \varepsilon \in E,$$
$$y_\varepsilon \overset{*}{\rightharpoonup} y \text{ in } L^\infty(\Omega, \mathrm{d}\mu_\varepsilon) \text{ as } \varepsilon \to 0.$$

Note that, in general, the set $C_0^\infty(\Omega)$ is not dense in $L^\infty(\Omega)$, that is, the assertion

> ... for any $f \in L^\infty(\Omega)$ a sequence $\{u_k \in C_0^\infty(\Omega)\}_{k=1}^\infty$ can be found such that $u_k \to f$ strongly in $L^\infty(\Omega)$ as $k \to \infty$...

is not true. So, Theorem 7.68 shows that it can be done through the concept of the weak-$*$ convergence in the variable spaces.

7.7 γ_p–Convergence of open sets and Mosco convergence of the associated Sobolev spaces

We begin this section with the following concept (see [38]).

Definition 7.69. *Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of open subsets of a bounded smooth domain $D \subset \mathbb{R}^n$ and let $1 < p < \infty$. We say that Ω_k γ_p-converges to Ω if for every $f \in W^{-1,q}(D)$ and $g \in W_0^{1,p}(D)$, the solutions $u_{\Omega_k,f,g}$ of the Dirichlet problem*

$$\left.\begin{aligned} -\Delta_p u_{\Omega_k,f,g} &= f \quad in \quad \Omega_k, \\ u_{\Omega_k,f,g} &= g \quad on \quad \partial\Omega_k \end{aligned}\right\} \tag{7.65}$$

extended by g on $D \setminus \Omega_k$ converge weakly in $W_0^{1,p}(D)$ to the function $u_{\Omega,f,g}$ of the same equation with Ω_k replaced by Ω. Here, $1/p + 1/q = 1$ and by Δ_p we denote the p-Laplacian in $W_0^{1,p}(D)$ (i.e., $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$).

As follows from this definition, the γ_p-convergence is precisely the topology on the space of open sets making the mapping $\Omega \mapsto u_{\Omega,f,g}$ continuous. This type of convergence is especially important in shape optimization problems [38]. The difficulty is to relate this convergence of domains to a geometrical one. The continuity with respect to the geometric domain, for the solution of the boundary value problem (7.65) or other related problems, has been studied by many authors (see, e.g., [36, 38, 40, 76, 121, 133, 217, 234, 249].) A natural question is: Given a sequence of open sets $\{\Omega_k\}_{k=1}^{\infty}$, does there exist an open set Ω such that the solution of the Dirichlet problem (7.65) on Ω is the weak limit of a subsequence of solutions on Ω_k? In general, the answer is negative – the "limit" domain being of relaxed type.

Let us consider a sequence $\{\Omega_k\}_{k=1}^{\infty}$ of quasi-open sets contained in a bounded design region D. We want to find a domain $\Omega \subseteq D$ such that (for a subsequence labeled using the same indices) we have $u_{\Omega_k,f,g} \rightharpoonup u_{\Omega,f,g}$ in $W_0^{1,p}(D)$. To do so, we associate with each domain Ω_k the corresponding Sobolev space $W_0^{1,p}(\Omega_k)$.

As a result, we have (see Definition 7.39) that the sequence of spaces $\left\{W_0^{1,p}(\Omega_k)\right\}_{k=1}^{\infty}$ converges in the sense of Mosco to $W_0^{1,p}(\Omega)$ if the following conditions are satisfied:

(M1) For all $u \in W_0^{1,p}(\Omega)$, there exists a sequence $\left\{u_k \in W_0^{1,p}(\Omega_k)\right\}_{k=1}^{\infty}$ such that u_k converges strongly in $W_0^{1,p}(D)$ to u.

(M2) For every sequence $\left\{u_{k_i} \in W_0^{1,p}(\Omega_{k_i})\right\}_{i=1}^{\infty}$ which is weakly convergent in $W_0^{1,p}(D)$ to a function u, we have $u \in W_0^{1,p}(\Omega)$.

It is easy to see (see [36]) that the γ_p-convergence of the sets Ω_k to Ω is equivalent to the Mosco convergence of the Sobolev spaces $W_0^{1,p}(\Omega_k)$ to $W_0^{1,p}(\Omega)$ (i.e., to both relations (7.31) and (7.32)). Even more, it is still equivalent with the continuity of the solutions $u_{\Omega_k,1,0} \overset{W_0^{1,p}(D)}{\rightharpoonup} u_{\Omega,1,0}$ (see [82]). In view of this, we cite the main properties of the γ_p-convergence following the brilliant book [38]:

1. The γ_p-convergence does not depend on $f \in W^{-1,q}(D)$ and $g \in W_0^{1,p}(D)$ in (7.65).
2. The γ_p-convergence is local, that is, $\Omega_k \xrightarrow{\gamma_p} \Omega$ if and only if there exists $\delta > 0$ such that for every $x \in D$ and, for every $r \in (0, \delta)$, we have that $\Omega_k \cap B_{x,r} \xrightarrow{\gamma_p} \Omega \cap B_{x,r}$, where by $B_{x,r}$ we denote the ball in $\mathbb{R}^n$ centered at x and of radius r.

3. For any two sequences of quasi-open sets $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ such that $A_k \xrightarrow{\gamma_p} A$ and $B_k \xrightarrow{\gamma_p} B$, we have $A_k \cap B_k \xrightarrow{\gamma_p} A \cap B$.
4. The first Mosco condition (7.31) is fulfilled for every sequence $\Omega_k \xrightarrow{H^c} \Omega$ with respect to the Hausdorff complementary topology on the set $\mathcal{A} = \{\Omega : \Omega \subseteq D, \Omega \text{ open}\}$ (see Definition 7.36). However, the second Mosco condition (7.32) does not hold, in general, for sequences converging in H^c. So, the geometrical constraints play a crucial role for this case. Indeed, the following counterexample shows that, in general, H^c-convergent sequences are not γ_p-convergent.

   Let $\{x_1, x_2, \dots\}$ be an enumeration of points of rational coordinates of the square $D = (0,1) \times (0,1)$ in $\mathbb{R}^2$. Defining $\Omega_k = D \setminus \{x_1, x_2, \dots, x_n\}$, we get that $\Omega_k \xrightarrow{H^c} \emptyset$ and $\Omega_k \xrightarrow{\gamma_p} D$ since $\mathrm{cap}_p(D \setminus \Omega_k = 0$ (see (2.44)).
4. Let $\mathcal{O}_{c,r}(D)$ be the class of open subsets of D having the so-called (r,c) capacity density condition, that is, $\Omega \in \mathcal{O}_{c,r}(D)$ if

$$\forall\, x \in \partial\Omega,\ \forall\, t \in (0,r), \quad \frac{\mathrm{cap}_p(\Omega^c \cap B_{x,t}, B_{x,2t})}{\mathrm{cap}_p(B_{x,t}, B_{x,2t})} \geq c.$$

   Let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{O}_{c,r}(D)$ which converges in the H^c-topology to an open set Ω. Then Ω_k γ_p-converges to Ω as well.

Note that each of the Mosco conditions (7.31) and (7.32) gives separately interesting information equation (7.65). For $p = 2$, relation (7.31) (respectively, (7.32)), gives the upper (respectively, lower) semicontinuity of the jth eigenvalue (counted with its multiplicity) of the Laplace operator with homogeneous Dirichlet boundary conditions on the moving domain, for any j.

If Ω_k "converges" in some "uniform" sense to Ω, then relations (7.31) and (7.32) can be easily established. A natural question is to give the minimal conditions on Ω_k and on Ω such that the Mosco convergence holds. Using the capacities and the framework of the Γ-convergence theory, Dal Maso [76, 77] and Dal Maso and Defranceschi [79] gave a characterization of the Mosco convergence. Essentially, $W_0^{1,p}(\Omega_k)$ converges in the sense of Mosco to $W_0^{1,p}(\Omega)$ if and only if there exists a family of sets $\mathcal{A} \subseteq \mathcal{P}(D)$ which is rich or dense in $\mathcal{P}(D)$ such that

$$\mathrm{cap}_p(\Omega^c \cap X, D) = \lim_{k\to\infty} \mathrm{cap}_p(\Omega_k^c \cap X, D), \quad \forall\, X \in \mathcal{A}.$$

As a result, it was shown in [79] that the convergence in the sense of Mosco is still equivalent with the following two relations, which have to be satisfied for all p-quasi open sets $A \subseteq D$ and p-quasi compact sets $F \subseteq A \subseteq D$:

$$\mathrm{cap}_p(\Omega^c \cap A, D) \geq \limsup_{k\to\infty} \mathrm{cap}_p(\Omega_k^c \cap F, D), \tag{7.66}$$

$$\mathrm{cap}_p(\Omega^c \cap A, D) \leq \liminf_{k\to\infty} \mathrm{cap}_p(\Omega_k^c \cap A, D). \tag{7.67}$$

However, the direct verification of these inequalities in the most practical situation is rather difficult. So, the question is to reduce the family of "test" sets to more simple ones. In [36, 38], the following results were proved:

1. Instead of all p-quasi open (respectively, p-quasi compact) sets, one can consider only open (respectively, closed) balls – namely, it is sufficient to consider only $A = F = \overline{B}_{x,r}$ in (7.66) and $A = B_{x,r}$ in (7.67), $\forall\, x \in \mathbb{R}^n$, $\forall\, r > 0$.
2. Relation (7.66) (respectively, (7.67)), on open (respectively, closed balls), represents a characterization of the strong lower (respectively, weak upper) limit in the sense of Kuratowski of the associated Sobolev spaces, being equivalent to (7.31) (respectively, (7.32)).

We also recall in this section some results of [45, 79, 82], where relaxed versions of the γ_p-convergence are studied. Let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of open sets containing in D. For every $f \in W^{-1,q}(D)$ and $g \in W_0^{1,p}(D)$, we consider the solutions $u_{\Omega_k,f,g}$ of the Dirichlet problem (7.65). We extend $u_{\Omega_k,f,g}$ to the whole of Ω by setting $u_{\Omega_k,f,g} = 0$ on $D \setminus \Omega_k$, and we regard $\{u_{\Omega_k,f,g}\}_{k=1}^{\infty}$ as a sequence in $W_0^{1,p}(D)$. Since the sequence $\{\Omega_k\}_{k=1}^{\infty}$ does not γ_p-converge in general, the problem is to describe the asymptotic behavior of $\{u_{\Omega_k,f,g}\}_{k=1}^{\infty}$ as k tends to infinity. The main idea is to rewrite the Dirichlet problem in a relaxed form. For this, each of the domains Ω_k can be associated with the non-negative Borel measure $\infty_{D\setminus\Omega_k}$ defined by

$$\infty_S(D) = \begin{cases} 0 & \text{if } \operatorname{cap}_p(D \cap S, \Omega) = 0, \\ +\infty & \text{otherwise,} \end{cases} \qquad \forall\, S \subseteq D. \tag{7.68}$$

If u and g are the functions in $W_0^{1,p}(D)$ such that $u - g = 0$ quasi-everywhere in $D \setminus \Omega_k$, then the restriction of $u - g$ to Ω_k belongs to $W_0^{1,p}(\Omega_k)$ (see [15]). Conversely, if $g \in W_0^{1,p}(D)$ and we extend a function $u \in W^{1,p}(\Omega_k)$ by setting $u = g$ in $D \setminus \Omega_k$, then u is quasi-continuous on D and belongs to $W_0^{1,p}(D)$. Therefore, if $\mu_k = \infty_{D\setminus\Omega_k}$, then a function u in $W_0^{1,p}(D)$ is the solution of the relaxed problem

$$\begin{cases} -\Delta_p u + |u - g|^{p-2}(u - g)\mu_k = f \ \text{ in } \ W_0^{1,p}(D) \cap L^2(D, \mathrm{d}\mu_k), \\ u - g \in W_0^{1,p}(D) \cap L^2(D, \mathrm{d}\mu_k) \end{cases} \tag{7.69}$$

if and only if $g \in W_0^{1,p}(D) \cap L^p(D, \mathrm{d}\mu_k)$ and the restriction of u to Ω_k is the solution of the classical Dirichlet problem (7.65) and, in addition, $u - g = 0$ quasi-everywhere on $D \setminus \Omega_k$.

In view of this, we have a good reason to introduce the following concept (see [38]).

Definition 7.70. *We say that a sequence of Borel measures $\{\mu_k\}_{k=1}^{\infty}$ in $\mathcal{M}_0^p(D)$ γ-converges to a measure $\mu \in \mathcal{M}_0^p(D)$ if and only if the corresponding solutions $u_{\mu_k,f,g}$ of relaxed Dirichlet problem (7.69) satisfy the condition*

$$u_{\mu_k,f,g} \rightharpoonup u_{\mu,f,g} \quad \text{in} \quad W_0^{1,p}(D), \quad \forall\, f \in W^{-1,q}(D),\ \forall\, g \in W_0^{1,p}(D).$$

The main result concerning an explicit identification of the γ-limit measure $\mu \in \mathcal{M}_0^p(D)$ and its formal computation can be stated as follows (see [45, 79, 82]).

Theorem 7.71. *Given a sequence of open sets $\{\Omega_k \subseteq D\}_{k=1}^{\infty}$ for which the solutions $u_{\Omega_k,1,0}$ of (7.65) with $f = 1$ and $g = 0$ converge weakly in $W_0^{1,p}(D)$ to a function w, there exists a Borel measure $\mu \in \mathcal{M}_0^p(D)$ which does not charge sets of p-capacity zero defined by*

$$\mu(A) = \begin{cases} \displaystyle\int_A \omega^{1-p} \, \mathrm{d}\nu & \text{if } \operatorname{cap}_p(A \cap \{w = 0\}) = 0, \\ +\infty & \text{if } \operatorname{cap}_p(A \cap \{w = 0\}) > 0, \end{cases}$$

where $\nu = 1 + \Delta_p w$, such that for any $f \in W^{-1,q}(D)$ and $g \in W_0^{1,p}(D)$, we have

$$u_{\Omega_k,f,g} \rightharpoonup u_{\mu,f,g}, \quad \text{weakly in } W_0^{1,p}(D),$$

where $u_{\mu,f,g} \in W_0^{1,p}(D) \cap L^p(D, \mathrm{d}\mu)$ and

$$\begin{cases} -\Delta_p u + |u - g|^{p-2}(u - g)\mu = f & \text{in } W_0^{1,p}(D) \cap L^2(D, \mathrm{d}\mu), \\ u - g \in W_0^{1,p}(D) \cap L^2(D, \mathrm{d}\mu). \end{cases}$$

8

Passing to the Limit in Constrained Minimization Problems

The main object of our consideration in this chapter is the following parameterized minimization problem

$$\left\langle \inf_{x\in\Xi_\varepsilon} I_\varepsilon(x) \right\rangle, \tag{8.1}$$

where $I_\varepsilon : (\Xi_\varepsilon \subseteq \mathbb{X}_\varepsilon) \to \overline{\mathbb{R}}$ is an objective functional, $\Xi_\varepsilon \subseteq \mathbb{X}_\varepsilon$ is a set of admissible solutions, and $\mathbb{X}_\varepsilon$ is some Banach space.

In what follows, we make a difference between

$$\inf_{x\in\Xi_\varepsilon} I_\varepsilon(x) \quad \text{and} \quad \left\langle \inf_{x\in\Xi_\varepsilon} I_\varepsilon(x) \right\rangle.$$

The first of one means the infimum $m_\varepsilon = \inf\{I_\varepsilon(x),\ x \in \Xi_\varepsilon\}$ of I_ε over the set Ξ_ε, whereas by the second one, we mean the constrained minimization problem as an object being defined by the triplet $(I_\varepsilon, \Xi_\varepsilon, \mathbb{X}_\varepsilon)$. Note that the basic space $\mathbb{X}_\varepsilon$, the set Ξ_ε, and the objective functional I_ε can be essentially dependent on ε. Hence, as the small parameter ε varies, we have a sequence of constrained minimization problems

$$\left\{ \left\langle \inf_{x\in\Xi_\varepsilon} I_\varepsilon(x) \right\rangle, \quad \varepsilon \to 0 \right\} \tag{8.2}$$

living in the corresponding scale of functional spaces.

In the sequel, each of the spaces $\mathbb{X}_\varepsilon$ will be associated with some thin or reticulated ε-periodic structure. In view of this, it is convenient in many practical situations to study the asymptotic behavior of the family of problems (8.2), not through the study of the "limit" properties of their solutions x_ε^0 or minimum values $m_\varepsilon = \inf\{I_\varepsilon(x),\ x \in \Xi_\varepsilon\}$, but by defining a limit functional I_0 and its domain Ξ_0 such that as $\varepsilon \to 0$ the problem

$$\left\langle \inf_{x\in\Xi_0} I_0(x) \right\rangle \quad \text{with } m_0 = \inf\{I_0(x),\ x \in \Xi_0\} \tag{8.3}$$

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_8,
© Springer Science+Business Media, LLC 2011

is a "good approximation" to the family (8.2), that is,

$$m_\varepsilon \to m_0 \quad \text{and} \quad x_\varepsilon^0 \to x^0 \quad \text{in variable space} \quad \mathbb{X}_\varepsilon,$$

where $x^0 \in \Xi_0$ is itself a solution of the "limit" problem (8.3). Our prime interest in this chapter is to discuss the formalism of variational convergence of the minimization problems (8.1) as ε tends to 0, which would possess the approximated property indicated above.

8.1 A short survey of Γ-convergence theory in metric spaces

This section is devoted to the definition of De Giorgi's Γ-convergence and its main properties. We do not want here to enter into the details of that theory, but only to use it in order to characterize the notion of limit minimization problem to the family (8.2). We refer the reader for all the details to [10, 29, 78], many of which are, nevertheless, simple applications of the definitions.

Let (X, ρ) be a separable metric space, endowed with a distance ρ. In applications, we will usually deal with metric spaces (as L^p-spaces) or metrizable spaces (as bounded subsets of Sobolev spaces or of spaces of measures, equipped with the weak topology) that in addition are also separable. In what follows in this section, we will consider a family of functionals $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}$ indexed by the positive parameter ε under the supposition that $\Xi_\varepsilon \subset \mathbb{X}$ for all $\varepsilon > 0$ and the sequential K-lower limit $K_s - \liminf_{\varepsilon\to 0} \Xi_\varepsilon$ is a nonempty set (see Definition 7.21). We also suppose that each of the functionals $I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}$ is defined on its domain Ξ_ε only and it may be undefined outside of this set. In this case, we identify such functionals by

$$\widetilde{I}_\varepsilon(x) = \begin{cases} I_\varepsilon(x) \text{ if } & x \in \Xi_\varepsilon, \\ +\infty \text{ if } & x \in \mathbb{X} \setminus \Xi_\varepsilon. \end{cases} \tag{8.4}$$

Definition 8.1. *Given a sequence of functionals $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$, we say that $\widetilde{I}_\varepsilon : \mathbb{X} \to \overline{\mathbb{R}}$ Γ-converges to a functional $I : \mathbb{X} \to \overline{\mathbb{R}}$ as $\varepsilon \to 0$ if for all $x \in \mathbb{X}$, we have the following:*

(i) (lim inf inequality) For every sequence $\{x_\varepsilon\}_{\varepsilon>0}$ converging to x,

$$I(x) \le \liminf_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x_\varepsilon). \tag{8.5}$$

(ii) (lim sup inequality) For all $\eta > 0$, there exists a sequence $\{x_\varepsilon\}_{\varepsilon>0}$ converging to x such that

$$I(x) \ge \limsup_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x_\varepsilon) - \eta. \tag{8.6}$$

If (i) and (ii) hold, we write $I(u) = \Gamma - \lim_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x)$ *and* I *is the* Γ*-limit of* $\widetilde{I}_\varepsilon$.

Sometimes we will write $\Gamma(\rho)-\lim_{\varepsilon\to 0}$ to indicate the topology associated with the metric ρ explicitly. We also introduce the notation

$$I'(x) = \Gamma - \liminf_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x) = \inf\left\{\liminf_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x_\varepsilon) \ : \ x_\varepsilon \to x\right\},$$

$$I''(x) = \Gamma - \limsup_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x) = \inf\left\{\limsup_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x_\varepsilon) \ : \ x_\varepsilon \to x\right\},$$

so that the equality $I' = I''$ is equivalent to the existence of the limit

$$\Gamma - \lim_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x).$$

We list the main properties of Γ-convergence.

- *Lower semicontinuity.* Every Γ-limit is lower semicontinuous on $\mathbb{X}$.
- *Convergence of minima.* If $\left\{\widetilde{I}_\varepsilon : \mathbb{X} \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ Γ-converges to I and is equicoercive on $\mathbb{X}$, that is, for every $t \in \mathbb{R}$ there exists a compact set $K_t \subset \mathbb{X}$ such that

$$\left\{x \in \mathbb{X} \ : \ \widetilde{I}_\varepsilon(x) \le t\right\} \equiv \{x \in \Xi_\varepsilon \ : \ I_\varepsilon(x) \le t\} \subset K_t, \quad \forall\, \varepsilon > 0,$$

  then I is coercive too, and so it attains its minimum on $\mathbb{X}$. Moreover, in this case, we have

$$\min_{x\in\mathbb{X}} I(x) = \lim_{\varepsilon\to 0}\left[\inf_{x\in\,\Xi_\varepsilon} I_\varepsilon(x)\right].$$

- *Convergence of minimizers.* Let $\left\{\widetilde{I}_\varepsilon : \mathbb{X} \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ be an equicoercive sequence of functionals which Γ-converges to a functional I. If

$$\left\{x_\varepsilon \in \arg\min_{x\in\Xi_\varepsilon} I_\varepsilon(x)\right\}_{\varepsilon>0}$$

  is a sequence with $x_\varepsilon \to x$ in $\mathbb{X}$, then we have $x \in \arg\min_{x\in\mathbb{X}} I(x)$. Moreover, if I is not identically $+\infty$ and if $x_\varepsilon \in \arg\min_{x\in\Xi_\varepsilon} I_\varepsilon(x)$, then there exists a subsequence of $\{x_\varepsilon\}_{\varepsilon>0}$ which converges to an element of $\arg\min_{x\in\mathbb{X}} I(x)$. In particular, if I has a unique minimum point x on $\mathbb{X}$, then every sequence $\left\{x_\varepsilon \in \arg\min_{x\in\Xi_\varepsilon} I_\varepsilon(x)\right\}_{\varepsilon>0}$ converges to x in $\mathbb{X}$.
- *Compactness.* From every sequence $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ of functionals, it is possible to extract a subsequence Γ-converging to a functional I on $\mathbb{X}$.
- *Metrizability.* The Γ-convergence, considered on the family $S(\mathbb{X})$ of all lower semicontinuous functions on $\mathbb{X}$, does not come from a topology unless the

space $\mathbb{X}$ is locally compact, which never occurs in the infinite-dimensional case. However, if instead of considering the whole family $S(\mathbb{X})$, we take the smaller classes

$$S_\Psi(\mathbb{X}) = \left\{ I : \Xi \to \overline{\mathbb{R}} \ : \ I \geq \Psi \text{ on } \ \Xi, \Xi \subset \mathbb{X}, \widetilde{I} \text{ l.s.c. on } \ \mathbb{X} \right\},$$

where $\Psi : \mathbb{X} \to \overline{\mathbb{R}}$ is lower semicontinuous and coercive (and non-negative, for simplicity), then the Γ-convergence on $S_\Psi(\mathbb{X})$ is metrizable. More precisely, it turns out to be equivalent to the convergence associated to the distance

$$d_\Gamma(I, J) = \sum_{i,j=1}^{\infty} 2^{-i-j} \left| \arctan(I_j(x_i) - \arctan(J_j(x_i) \right|,$$

where $\{x_i\}_{i=1}^{\infty}$ is a dense sequence in $\mathbb{X}$ and H_j denotes the Moreau–Yosida transform of a functional $\widetilde{H} : \mathbb{X} \to \overline{\mathbb{R}}$, defined by

$$H_j(x) = \inf \left\{ \widetilde{H}(x) + j\rho(x, y) \ : \ x \in \mathbb{X} \right\}.$$

According to the compactness property seen above, the family $S_\Psi(\mathbb{X})$ endowed with the distance d_Γ turns out to be a compact metric space.

- *Semiadditivity.* Let $\left\{ I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}} \right\}_{\varepsilon > 0}$ and $\left\{ J_\varepsilon : \mathbb{X} \to \overline{\mathbb{R}} \right\}_{\varepsilon > 0}$ be any two sequences of functionals. Then each of the inequalities

$$\Gamma\text{-}\liminf_{\varepsilon \to 0} \left(\widetilde{I}_\varepsilon + J_\varepsilon \right) \geq \Gamma\text{-}\liminf_{\varepsilon \to 0} \widetilde{I}_\varepsilon + \Gamma\text{-}\liminf_{\varepsilon \to 0} J_\varepsilon, \tag{8.7}$$

$$\Gamma\text{-}\limsup_{\varepsilon \to 0} \left(\widetilde{I}_\varepsilon + J_\varepsilon \right) \geq \Gamma\text{-}\liminf_{\varepsilon \to 0} \widetilde{I}_\varepsilon + \Gamma\text{-}\limsup_{\varepsilon \to 0} J_\varepsilon \tag{8.8}$$

is true, provided that the sums occurring in it are well defined on $\mathbb{X}$. In particular, if $\widetilde{I}_\varepsilon : \mathbb{X} \to \overline{\mathbb{R}}$ Γ-converges to I, $J_\varepsilon : \mathbb{X} \to \overline{\mathbb{R}}$ Γ-converges to J, and the sum $\left(\widetilde{I}_\varepsilon + J_\varepsilon \right)$ Γ-converges to H, then $I + J \leq H$, provided that the functionals $\widetilde{I}_\varepsilon + J_\varepsilon$ and $I + J$ are well defined on $\mathbb{X}$.
- *Stability property.* If $G : \mathbb{X} \to \overline{\mathbb{R}}$ is a continuous functional and $I = \Gamma\text{-}\lim_{\varepsilon \to 0} \widetilde{I}_\varepsilon$ then $I + G = \Gamma\text{-}\lim_{\varepsilon \to 0} \left(\widetilde{I}_\varepsilon + G \right)$.
- *Relaxation.* If $H(x) \leq I_\varepsilon(x)$, $\forall\, x \in \Xi_\varepsilon$, then

$$\mathrm{sc}^- H(x) \leq \Gamma\text{-}\lim_{\varepsilon \to 0} \widetilde{I}_\varepsilon(x) \leq \liminf_{\varepsilon \to 0} \widetilde{I}_\varepsilon(x), \quad \forall\, x \in \mathbb{X}, \tag{8.9}$$

where by $\mathrm{sc}^- H : \mathbb{X} \to \overline{\mathbb{R}}$ we denote the lower semicontinuous envelope (or relaxed functional) of $H : \mathbb{X} \to \overline{\mathbb{R}}$.
- *Convexity.* If $\left\{ I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}} \right\}_{\varepsilon > 0}$ is a sequence of convex functionals and if $I = \Gamma\text{-}\lim_{\varepsilon \to 0} \widetilde{I}_\varepsilon$, then $I : \mathbb{X} \to \overline{\mathbb{R}}$ is a convex closed functional.

Example 8.2. Let $\mathbb{X} = \mathbb{R}$ and let $I_\varepsilon : \mathbb{R} \to \mathbb{R}$ for all $\varepsilon > 0$ be defined as $I_\varepsilon(x) = \sin(x/\varepsilon)$. It is easy to calculate that in this case we have

$$\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon(x) = -1.$$

Example 8.3. Let $\varepsilon = 1/n$, where $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we define the functional $I_\varepsilon : \Xi_\varepsilon \to \mathbb{R}$ as follows:

$$I_\varepsilon(x) = \begin{cases} 1 & \text{if } x \le 0, \\ x^2 + 1 & \text{if } x > 0, \end{cases} \quad \text{if } \varepsilon = \frac{1}{2n}, \quad I_\varepsilon(x) = \frac{x^2}{2} + 1 \text{ if } \varepsilon = \frac{1}{2n+1};$$

$$\Xi_\varepsilon = \begin{cases} (-1, 1) & \text{if } \varepsilon = 1/(2n), \\ (-2, 2) & \text{if } \varepsilon = 1/(2n+1). \end{cases}$$

Then, following Definition 8.1, straightforward calculations show that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x) = \begin{cases} x^2/2 + 1 & \text{if } x \in [-2, -1), \\ 1 & \text{if } x \in [-1, 0), \\ x^2/2 + 1 & \text{if } x \in [0, 2], \\ +\infty & \text{if } x \notin [-2, 2], \end{cases}$$

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x) = \begin{cases} x^2/2 + 1 & \text{if } x \in [-1, 0), \\ x^2 + 1 & \text{if } x \in [0, 1], \\ +\infty & \text{if } x \notin [-1, 1]. \end{cases}$$

Disregarding the domains Ξ_ε in Definition 8.1 leads us to the following analytic representation of Γ-limits of $I_\varepsilon : \mathbb{R} \to \mathbb{R}$:

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} I_\varepsilon(x) = \begin{cases} 1 & \text{if } x \le 0, \\ x^2/2 + 1 & \text{if } x > 0, \end{cases}$$

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} I_\varepsilon(x) = \begin{cases} x^2/2 + 1 & \text{if } x \le 0, \\ x^2 + 1 & \text{if } x > 0. \end{cases}$$

Thus, in general, we have

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} \widetilde{I}_\varepsilon \ne \Gamma\text{-}\liminf_{\varepsilon\to 0} I_\varepsilon \quad \text{and} \quad \Gamma\text{-}\limsup_{\varepsilon\to 0} \widetilde{I}_\varepsilon \ne \Gamma\text{-}\limsup_{\varepsilon\to 0} I_\varepsilon.$$

Further, we focus on a topological aspect of the Γ-convergence. For this, we make use of the following result which shows that the K-convergence of a sequence of sets (see Definition 7.21) is equivalent to the Γ-convergence of the corresponding indicator functions (we refer to (5.68) for the definition of the indicator function).

Proposition 8.4 ([10]). *Let $\{\Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of subsets of $\mathbb{X}$ and let $\{\chi_{\Xi_\varepsilon}\}_{\varepsilon>0}$ be the corresponding indicator functions. Also let*

$$\Xi' = K\text{-}\liminf_{\varepsilon\to 0} \Xi_\varepsilon, \quad \Xi'' = K\text{-}\limsup_{\varepsilon\to 0} \Xi_\varepsilon.$$

Then

$$\mathbf{1}_{\Xi'} = \Gamma - \limsup_{\varepsilon\to 0} \mathbf{1}_{\Xi_\varepsilon}, \quad \mathbf{1}_{\Xi''} = \Gamma - \liminf_{\varepsilon\to 0} \mathbf{1}_{\Xi_\varepsilon}, \tag{8.10}$$

where $\mathbf{1}_{\Xi_\varepsilon}$ *denotes the indicator function of the set* Ξ_ε*:*

$$\mathbf{1}_{\Xi_\varepsilon} = \begin{cases} 0, & x \in \Xi_\varepsilon, \\ +\infty, & x \notin \Xi_\varepsilon. \end{cases}$$

In particular, $\{\Xi_\varepsilon\}_{\varepsilon>0}$ *K-converges to* Ξ *if and only if* $\{\mathbf{1}_{\Xi_\varepsilon}\}_{\varepsilon>0}$ Γ*-converges to* $\mathbf{1}_\Xi$ *in* $\mathbb{X}$*.*

Having observed that each of the functionals $I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}$ is well defined and has a sense only on the set of admissible solutions Ξ_ε yields the following object:

$$\operatorname{epi}(I_\varepsilon; \Xi_\varepsilon) = \{(x,\lambda) \in \mathbb{X}\times\mathbb{R} \mid x \in \Xi_\varepsilon, \quad I_\varepsilon(x) \le \lambda\}; \tag{8.11}$$

we call it the epigraph of $I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}$. Note that the set $\operatorname{epi}(I_\varepsilon; \Xi_\varepsilon)$ coincides with the epigraph of the extended functional $\widetilde{I}_\varepsilon : \mathbb{X} \to \overline{\mathbb{R}}$, namely

$$\operatorname{epi}(I_\varepsilon; \Xi_\varepsilon) \equiv \operatorname{epi}(\widetilde{I}_\varepsilon) := \left\{(x,\lambda) \in \mathbb{X}\times\mathbb{R} \ : \ \widetilde{I}_\varepsilon(x) \le \lambda\right\}, \quad \forall\, \varepsilon > 0. \tag{8.12}$$

Obviously, $\operatorname{epi}(\widetilde{I}_\varepsilon) = \mathbb{X}\times\mathbb{R}$ if $\widetilde{I}_\varepsilon \equiv -\infty$ and $\operatorname{epi}(\widetilde{I}_\varepsilon) = \emptyset$ if $\widetilde{I}_\varepsilon \equiv +\infty$. A functional $I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}$ is said to be closed, or lower semicontinuous, if $\operatorname{epi}(\widetilde{I}_\varepsilon)$ is a closed set.

A notable feature of the set $\operatorname{epi}(I_\varepsilon; \Xi_\varepsilon)$ is that for a fixed $x \in \Xi_\varepsilon$, the set of all $\lambda \in \mathbb{R}$ such that $(x,\lambda) \in \operatorname{epi}(I_\varepsilon; \Xi_\varepsilon)$ can either be empty, or coincide with a closed half-line $\lambda \ge \lambda_\varepsilon$, or coincide with the whole line of real numbers. A converse statement can also be made: Assume that a given subset $F_\varepsilon \subset \mathbb{X}\times\mathbb{R}$ satisfies the above property for every $x \in \Pi\mathrm{r}_{\mathbb{X}} F_\varepsilon = \{x \in \mathbb{X} \ : \ \exists \lambda \in \mathbb{R}, (x,\lambda) \in F_\varepsilon\}$. Then there exists a functional I_ε with domain $\Xi_\varepsilon = \Pi\mathrm{r}_{\mathbb{X}} F_\varepsilon$ for which $\operatorname{epi}(I_\varepsilon; \Xi_\varepsilon) = F_\varepsilon$.

Taking this into account, it is important to note that the K-lower and K-upper limits of a sequence of epigraphs $\left\{\operatorname{epi}\left(\widetilde{I}_\varepsilon\right)\right\}_{\varepsilon>0}$ are always closed epigraphs (see [10]). Moreover, the following theorem shows that Γ-convergence of functionals $\widetilde{I}_\varepsilon : \mathbb{X} \to \overline{\mathbb{R}}$ can be defined in terms of the K-convergence of their epigraphs. This is the reason why Γ-convergence is sometimes called epi-convergence.

Theorem 8.5 ([10, 78]). *Let* $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ *be a sequence of functionals and let*

$$J'(x) = \Gamma - \liminf_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x), \quad J''(x) = \Gamma - \limsup_{\varepsilon\to 0} \widetilde{I}_\varepsilon(x).$$

Then

$$\operatorname{epi}(J') = K - \limsup_{\varepsilon\to 0} \operatorname{epi}\left(\widetilde{I}_\varepsilon\right), \quad \operatorname{epi}(J'') = K - \liminf_{\varepsilon\to 0} \operatorname{epi}\left(\widetilde{I}_\varepsilon\right), \tag{8.13}$$

where K-limits are taken in the product metric of $\mathbb{X}\times\mathbb{R}$. In particular, $\{\widetilde{I}_\varepsilon : \mathbb{X}\to\overline{\mathbb{R}}\}_{\varepsilon>0}$ Γ-converges to J in $\mathbb{X}$ if and only if $\{\operatorname{epi}(\widetilde{I}_\varepsilon)\}_{\varepsilon>0}$ K-converges to $\operatorname{epi}(J)$ in $\mathbb{X}\times\mathbb{R}$.

According to (8.12), we have the obvious relations

$$K\text{-}\limsup_{\varepsilon\to 0}\operatorname{epi}\big(\widetilde{I}_\varepsilon\big) = K\text{-}\limsup_{\varepsilon\to 0}\operatorname{epi}\big(I_\varepsilon;\Xi_\varepsilon\big), \tag{8.14}$$

$$K\text{-}\liminf_{\varepsilon\to 0}\operatorname{epi}\big(\widetilde{I}_\varepsilon\big) = K\text{-}\liminf_{\varepsilon\to 0}\operatorname{epi}\big(I_\varepsilon;\Xi_\varepsilon\big). \tag{8.15}$$

Further, we make the observation: $\widetilde{I}_\varepsilon(x) = \widetilde{I}_\varepsilon(x) + \mathbf{1}_{\Xi_\varepsilon}(x)$ for every $\varepsilon>0$. From this, using (8.7) and (8.8), Proposition 8.4, and Theorem 8.5, we deduce that there exist functionals

$$I' : \Big(K\text{-}\limsup_{\varepsilon\to 0}\Xi_\varepsilon\Big)\to\overline{\mathbb{R}} \quad\text{and}\quad I'' : \Big(K\text{-}\liminf_{\varepsilon\to 0}\Xi_\varepsilon\Big)\to\overline{\mathbb{R}} \tag{8.16}$$

such that

$$\operatorname{epi}(J') = \operatorname{epi}\big(I';K\text{-}\limsup_{\varepsilon\to 0}\Xi_\varepsilon\big), \quad \operatorname{epi}(J'') = \operatorname{epi}\big(I'';K\text{-}\liminf_{\varepsilon\to 0}\Xi_\varepsilon\big).$$

Definition 8.6. *Given a sequence of functionals $\{I_\varepsilon : \Xi_\varepsilon\to\overline{\mathbb{R}}\}_{\varepsilon>0}$, we state the following:*

(i) A functional $I' : (K\text{-}\limsup_{\varepsilon\to 0}\Xi_\varepsilon)\to\overline{\mathbb{R}}$ is the constrained Γ-lower limit of the sequence $\{I_\varepsilon : \Xi_\varepsilon\to\overline{\mathbb{R}}\}_{\varepsilon>0}$ as $\varepsilon\to 0$ (we write $I' = \Gamma\text{-}\mathrm{li}_{\varepsilon\to 0}\, I_\varepsilon$) if

$$\operatorname{epi}\Big(I';K\text{-}\limsup_{\varepsilon\to 0}\Xi_\varepsilon\Big) = K\text{-}\limsup_{\varepsilon\to 0}\operatorname{epi}\big(I_\varepsilon;\Xi_\varepsilon\big). \tag{8.17}$$

(ii) A functional $I'' : (K\text{-}\liminf_{\varepsilon\to 0}\Xi_\varepsilon)\to\overline{\mathbb{R}}$ is the constrained Γ-upper limit of the sequence $\{I_\varepsilon : \Xi_\varepsilon\to\overline{\mathbb{R}}\}_{\varepsilon>0}$ as $\varepsilon\to 0$ (we write $I'' = \Gamma\text{-}\mathrm{ls}_{\varepsilon\to 0}\, I_\varepsilon$) if

$$\operatorname{epi}\Big(I'';K\text{-}\liminf_{\varepsilon\to 0}\Xi_\varepsilon\Big) = K\text{-}\liminf_{\varepsilon\to 0}\operatorname{epi}\big(I_\varepsilon;\Xi_\varepsilon\big). \tag{8.18}$$

If there exist a set $\Xi\subseteq\mathbb{X}$ and a functional $I : \Xi\to\overline{\mathbb{R}}$ such that

$$K\text{-}\liminf_{\varepsilon\to 0}\Xi_\varepsilon = K\text{-}\limsup_{\varepsilon\to 0}\Xi_\varepsilon = \Xi, \tag{8.19}$$

$$I'(x) = I''(x) = I(x) \quad \text{on} \quad \Xi, \tag{8.20}$$

then we write $I(x) = \Gamma\text{-}\mathrm{lm}_{\varepsilon\to 0}\, I_\varepsilon(x)$ and we say that $I : \Xi\to\overline{\mathbb{R}}$ is the constrained Γ-limit of the sequence $\{I_\varepsilon : \Xi_\varepsilon\to\overline{\mathbb{R}}\}_{\varepsilon>0}$.

Remark 8.7. Coming back to Example 8.3, it is easy to see that

$$\Gamma\text{-}\operatorname*{li}_{\varepsilon\to 0} I_\varepsilon(x) = \begin{cases} x^2/2+1 & \text{if } x\in[-2,-1),\\ 1 & \text{if } x\in[-1,0),\\ x^2/2+1 & \text{if } x\in[0,2] \end{cases} \quad \text{on} \quad K\text{-}\limsup_{\varepsilon\to 0} \Xi_\varepsilon = [-2,2],$$

$$\Gamma\text{-}\operatorname*{ls}_{\varepsilon\to 0} I_\varepsilon(x) = \begin{cases} x^2/2+1 & \text{if } x\in[-1,0),\\ x^2+1 & \text{if } x\in[0,1] \end{cases} \quad \text{on} \quad K\text{-}\liminf_{\varepsilon\to 0} \Xi_\varepsilon = [-1,1].$$

Remark 8.8. Taking into account (8.5) and (8.6) and the sequential version of Kuratowski convergence (see Definition 7.21), the definition of the constrained Γ-limit can be rewritten in a more elaborate form. We say that a functional $I : \Xi \to \overline{\mathbb{R}}$ is the constrained Γ-limit of a sequence $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ if for all $x \in \Xi$, the following conditions are satisfied:

(a) For all $\eta > 0$, there exist a constant $\varepsilon_0 > 0$ and a sequence $\{x_\varepsilon\}_{\varepsilon>0}$ converging to x in $\mathbb{X}$ such that

$$x_\varepsilon \in \Xi_\varepsilon \quad \text{for every} \quad \varepsilon \le \varepsilon_0, \qquad I(x) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(x_\varepsilon) - \eta. \tag{8.21}$$

(aa) For every sequence $\{x_k\}_{k=1}^\infty$ converging to x as $k \to \infty$ in $\mathbb{X}$, there exists a subsequence $\{\Xi_{\varepsilon_k}\}_{k=1}^\infty$ of the sequence $\{\Xi_\varepsilon\}_{\varepsilon>0}$ such that

$$x_k \in \Xi_{\varepsilon_k} \quad \text{for all} \quad k \in \mathbb{N}, \quad I(x) \le \liminf_{k\to\infty} I_{\varepsilon_k}(x_k). \tag{8.22}$$

In what follows, we will use the following notation for the constrained Γ-convergence of $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ to $I : \Xi \to \overline{\mathbb{R}}$:

$$\langle I_\varepsilon; \Xi_\varepsilon\rangle \xrightarrow{\Gamma} \langle I; \Xi\rangle .$$

8.2 Γ-Convergence of functionals defined on a Banach space

Now, let $\mathbb{X}$ be a Banach space, and let $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ be a given sequence of functionals, where $\Xi_\varepsilon \subset \mathbb{X}$ for every $\varepsilon > 0$. It is clear that the properties of the Γ-limit can be considerably affected by the choice of the metric ρ. The natural metric $\|x - y\|_{\mathbb{X}}$ in $\mathbb{X}$ happens to be not very effective in many situations. Moreover, the existence of the Γ-limit in one metric does not imply the existence of Γ-limit in another. Therefore, in each particular case, the application of general results is always a separate problem. So, it is natural to use the notion of weak convergence in $\mathbb{X}$. Thus we come to the following concept.

Definition 8.9. *Let $\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon>0}$ be a sequence of functionals defined on the subsets of a Banach space $\mathbb{X}$. A functional $I : \Xi \to \overline{\mathbb{R}}$ is said to be the constrained Γ-limit of $\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon>0}$ with respect to the weak topology of $\mathbb{X}$ (we write $I = \Gamma(w)\text{-}\lim_{\varepsilon\to 0} I_\varepsilon$ or $\langle I_\varepsilon; \Xi_\varepsilon\rangle \overset{\Gamma(w)}{\longrightarrow} \langle I; \Xi\rangle$) if for any $x \in \Xi$, the following two conditions are satisfied:*

(b) There exist a constant $\varepsilon_0 > 0$ and a sequence $x_\varepsilon \rightharpoonup x$ as $\varepsilon \to 0$ in $\mathbb{X}$ (called a Γ-realizing sequence) such that

$$x_\varepsilon \in \Xi_\varepsilon \quad \text{for every} \quad \varepsilon \le \varepsilon_0 \quad \text{and} \quad I(x) = \limsup_{\varepsilon\to 0} I_\varepsilon(x_\varepsilon). \tag{8.23}$$

(bb) For every sequence $x_k \rightharpoonup x$ as $k \to \infty$ in $\mathbb{X}$, there exists a subsequence $\{\Xi_{\varepsilon_k}\}_{k=1}^{\infty}$ of $\{\Xi_\varepsilon\}_{\varepsilon>0}$ such that

$$x_k \in \Xi_{\varepsilon_k} \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad I(x) \le \liminf_{k\to\infty} I_{\varepsilon_k}(x_k). \tag{8.24}$$

To illustrate Definition 8.9, consider the following example.

Example 8.10. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain and let $S_\varepsilon \subset \Omega$ be a closed subset for each $\varepsilon > 0$. We set $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$. Let $\chi_{\Omega_\varepsilon}$ be the characteristic function of the perforated domain Ω_ε.

Let $\mathbb{X} = L^2(\Omega)$. For every $\varepsilon > 0$, we define the functional $I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}$ as follows:

$$I_\varepsilon(y) = \int_\Omega a_\varepsilon(x) y^2(x)\,\mathrm{d}x, \quad \Xi_\varepsilon = \left\{\chi_{\Omega_\varepsilon} z \ : \ \forall\, z \in L^2(\Omega)\right\},$$

where $a_\varepsilon \in L^\infty(\Omega)$. Our aim is to calculate the constrained Γ-limit of

$$\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$$

with respect to the weak topology of $\mathbb{X}$ as $\varepsilon \to 0$.

Assume that the following conditions are satisfied:

(i) $\chi_{\Omega_\varepsilon} \overset{*}{\rightharpoonup} \chi_0$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$;
(ii) $a_\varepsilon^{-1} \in L^\infty(\Omega)$ $\forall \varepsilon > 0$ and $\chi_{\Omega_\varepsilon}/a_\varepsilon \overset{*}{\rightharpoonup} g_0$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$;
(iii) $g_0^{-1} \in L^\infty(\Omega)$.

Let us show that, in this case,

$$K\text{-}\lim_{\varepsilon\to 0} \operatorname{epi}\left(I_\varepsilon; \Xi_\varepsilon\right) = \operatorname{epi}\left(I_0; \Xi_0\right),$$

where

$$I(y) = \int_\Omega g_0^{-1} y^2\,\mathrm{d}x, \quad \Xi_0 = L^2(\Omega). \tag{8.25}$$

First, we verify condition (b) of Definition 8.9. For $\eta = 0$ and $y \in L^2(\Omega)$, we construct a sequence $\{y_\varepsilon\}_{\varepsilon>0}$ weakly converging to y in $L^2(\Omega)$ by setting

$y_\varepsilon = g_0^{-1}\chi_{\Omega_\varepsilon}a_\varepsilon^{-1}\,y$, $\forall\,\varepsilon > 0$. Then using the fact that $\chi_{\Omega_\varepsilon}\chi_{\Omega_\varepsilon} = \chi_{\Omega_\varepsilon}$ and $g_0^{-2}y^2 \in L^1(\Omega)$, we have

$$y_\varepsilon \in \Xi_\varepsilon \quad \text{for each} \quad \varepsilon > 0,$$
$$I_\varepsilon(y_\varepsilon) = \int_\Omega g_0^{-2}\frac{\chi_{\Omega_\varepsilon}}{a_\varepsilon}\,y^2\,\mathrm{d}x \longrightarrow \int_\Omega\int_\Omega g_0^{-1}y^2\,\mathrm{d}x = I(y).$$

Next, we show that condition (bb) is satisfied, too. Let y be an element of $L^2(\Omega)$ and let $\{z_k\}_{k=1}^\infty$ be an arbitrary sequence in $L^2(\Omega)$ such that $z_k \rightharpoonup y$ in $L^2(\Omega)$ as $k \to \infty$ and $z_k \in \Xi_{\varepsilon_k}$ for all $k \in \mathbb{N}$, where $\varepsilon_k \to 0$ as k tends to ∞. Then, we obviously have

$$y_{\varepsilon_k} = g_0^{-1}\frac{\chi_{\Omega_{\varepsilon_k}}}{a_{\varepsilon_k}}\,y \rightharpoonup y \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad k \to \infty,$$

$$\begin{aligned}\int_\Omega a_{\varepsilon_k}\left(z_k^2 - y_{\varepsilon_k}^2\right)\,\mathrm{d}x &\geq 2\int_\Omega a_{\varepsilon_k}\left(z_k - y_{\varepsilon_k}\right)y_{\varepsilon_k}\,\mathrm{d}x\\ &= 2\int_\Omega g_0^{-1}\left(z_k - y_{\varepsilon_k}\right)y\,\mathrm{d}x \longrightarrow 0 \quad \text{as} \quad k \to \infty.\end{aligned}$$

Hence,

$$\liminf_{k\to\infty} I_{\varepsilon_k}(z_k) \geq \liminf_{k\to\infty} I_{\varepsilon_k}(y_{\varepsilon_k}) = I(y),$$

and we obtain the required result. In particular, as follows from (8.25), if $a_\varepsilon = 1$ for all $\varepsilon > 0$, then

$$I(y) = \Gamma(w)\text{-}\lim_{\varepsilon\to 0} I_\varepsilon(y) = \int_\Omega \frac{y^2}{\chi_0}\,\mathrm{d}x, \quad \forall\, y \in L^2(\Omega)$$

(see Proposition 7.12 for comparison).

It should be noted that Definition 8.9 cannot be reduced to the general conditions (8.21) and (8.22) because the weak convergence in $\mathbb{X}$ cannot be specified in terms of a metric. However, there exists a wide class of functionals for which Definition 8.9 can be formulated in terms of a metric; moreover, within that class, the constrained Γ-convergence can be expressed in "variational terms".

Let us consider a sequence of functionals $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ for which the following conditions hold true:

$$K_s(w)\text{-}\liminf_{\varepsilon\to 0} \Xi_\varepsilon \neq \emptyset, \tag{8.26}$$

$$\{\Xi_\varepsilon\}_{\varepsilon>0} \quad \text{are uniformly bounded in} \quad \mathbb{X}, \ \text{i.e.,} \quad \sup_{\varepsilon>0}\sup_{x\in\Xi_\varepsilon}\|x\|_{\mathbb{X}} < \infty. \tag{8.27}$$

The following result establishes the compactness property of the constrained Γ-convergence in a Banach space.

Theorem 8.11. *Assume that $\mathbb{X}$ is a Banach space endowed with its weak topology and that the dual $\mathbb{X}^*$ is separable. Then a sequence of functionals $\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon>0}$ satisfying properties (8.26)–(8.27) is compact with respect to the constrained Γ-convergence in the sense of Definition 8.9.*

Proof. To prove the compactness of the class (8.26)–(8.27), let us choose a countable dense set in $\mathbb{X}^*$ and denote its points by x_j^*. Set

$$\rho(x,y) = \sum_j \frac{|\langle x_j^*, x-y\rangle_{\mathbb{X}^*,\mathbb{X}}|}{j^2 + |\langle x_j^*, x-y\rangle_{\mathbb{X}^*,\mathbb{X}}|}.$$

Then for any sequence $\{x_k\}_{k=1}^\infty$ in $\mathbb{X}$, the following conditions are equivalent (see [99]):

(i) $x_k \rightharpoonup x$ in $\mathbb{X}$;
(ii) $\{x_k\}_{k=1}^\infty$ is bounded in $\mathbb{X}$ and $\rho(x_k, x) \to 0$.

Since the space $\mathbb{X}$ is separable with respect to the above metric, it follows that the general compactness principle holds. Therefore, there exist a subsequence of $\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon>0}$, still denoted by $\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon>0}$, a set $\Xi \subset \mathbb{X}$, and a functional $I : \Xi \to \overline{\mathbb{R}}$ such that (see [78])

$$\Xi = K(\rho)\text{-}\lim_{\varepsilon\to 0} \Xi_\varepsilon, \quad \operatorname{epi}\big(I;\Xi\big) = K(\rho\times\tau)\text{-}\lim_{\varepsilon\to 0} \operatorname{epi}\big(I_\varepsilon;\Xi_\varepsilon\big),$$

where by τ we denote the topology of pointwise convergence in $\mathbb{R}$. Moreover, as follows from (8.26), the following condition is satisfied: $\Xi \neq \emptyset$.

Thus, $I = \Gamma(\rho)\text{-}\lim_{\varepsilon\to 0} I_\varepsilon$. Now, we prove that the functional $I : \Xi \to \overline{\mathbb{R}}$ coincide with the Γ-limit of $\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\}_{\varepsilon>0}$ in the sense of Definition 8.9. First, we verify condition (bb). Let $\{x_k\}_{k=1}^\infty$ be a sequence in $\mathbb{X}$ such that

$$x_k \rightharpoonup x \ \text{ in } \ \mathbb{X}, \ x_k \in \Xi_{\varepsilon_k} \ \text{ for all } \ k\in\mathbb{N}, \ \text{ where } \ \lim_{k\to\infty} \varepsilon_k = 0.$$

Then $x_k \overset{\rho}{\to} x$, and therefore inequality (8.24) is satisfied. Next, we show that condition (b) holds, too. For any $x \in \Xi$, denote by $\{x_\varepsilon\}_{\varepsilon>0}$ a $\Gamma(\rho)$-realizing sequence:

$$I_\varepsilon(x_\varepsilon) \to I(x), \quad x_\varepsilon \overset{\rho}{\to} x, \quad x_\varepsilon \in \Xi_\varepsilon, \ \forall\, \varepsilon \le \varepsilon_0.$$

Since the norms of x_ε are uniformly bounded with respect to ε, it follows that $x_\varepsilon \rightharpoonup x$, and therefore condition (b) is satisfied. Thus, the compactness of class (8.26)–(8.27), with respect to the constrained $\Gamma(w)$-convergence in $\mathbb{X}$, is established. This completes the proof. □

For every $\varepsilon > 0$, we introduce the conjugate functionals

$$-I_\varepsilon^*(x^*) = \inf_{x\in\Xi_\varepsilon} \Big\{ I_\varepsilon(x) - \langle x^*, x\rangle_{\mathbb{X}^*,\mathbb{X}} \Big\}. \tag{8.28}$$

According to the results of Sect. 3.1, each of the constrained minimization problems (8.28) is solvable, provided Ξ_ε is a bounded closed set and I_ε is sequentially lower semicontinuous with respect to the weak topology of $\mathbb{X}$. Then there exists a sequence of minimizers $\left\{x_\varepsilon^0 \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ such that

$$\inf_{x\in\Xi_\varepsilon}\left\{I_\varepsilon(x) - \langle x^*, x\rangle_{\mathbb{X}^*,\mathbb{X}}\right\} = I_\varepsilon(x_\varepsilon^0) - \left\langle x^*, x_\varepsilon^0\right\rangle_{\mathbb{X}^*,\mathbb{X}}, \quad \forall\,\varepsilon > 0.$$

The following result establishes a relation between constrained Γ-convergence and variational problems.

Theorem 8.12. *Assume that $\mathbb{X}$ is a separable reflexive Banach space. Let $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ be a sequence of functionals satisfying properties (8.26) and (8.27). Assume also that for every $\varepsilon > 0$, Ξ_ε is a closed set and I_ε is sequentially lower semicontinuous with respect to the weak topology of $\mathbb{X}$. Then the constrained Γ-convergence $\langle I_\varepsilon; \Xi_\varepsilon\rangle \xrightarrow{\Gamma} \langle I; \Xi\rangle$ implies the pointwise convergence of the conjugate functionals:*

$$I_\varepsilon^*(x^*) \to I^*(x^*) \quad \text{as} \quad \varepsilon \to 0, \quad \forall\, x^* \in \mathbb{X}^*. \tag{8.29}$$

Moreover, Γ-convergence $\langle I_\varepsilon; \Xi_\varepsilon\rangle \xrightarrow{\Gamma} \langle I; \Xi\rangle$ is accompanied by the weak convergence of minimizers: If x_ε^0 is a minimizer for problem (8.28), then the sequence $\left\{x_\varepsilon^0\right\}_{\varepsilon>0}$ is bounded and any weak cluster point of $\left\{x_\varepsilon^0\right\}_{\varepsilon>0}$ is a minimizer of the limit variational problem

$$-I^*(x^*) = \inf_{x\in\Xi}\left\{I(x) - \langle x^*, x\rangle_{\mathbb{X}^*,\mathbb{X}}\right\}. \tag{8.30}$$

Proof. Assume that $I : \Xi \to \overline{\mathbb{R}}$ is the constrained Γ-limit of $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ in the sense of Definition 8.9. Let x_ε^0 be a minimizer for problem (8.28). Then $\left\{x_\varepsilon^0\right\}_{\varepsilon>0}$ is the bounded sequence in $\mathbb{X}$. So, we can assume, without loss of generality, that $x_\varepsilon^0 \rightharpoonup x^0$ in $\mathbb{X}$, where $x^0 \in \Xi$ by the properties of $K(w)$-convergence. It then follows from inequality (8.24) that

$$\liminf_{\varepsilon\to 0} I_\varepsilon(x_\varepsilon^0) \geq I(x^0).$$

Therefore,

$$\begin{aligned}\limsup_{\varepsilon\to 0} I_\varepsilon^*(x^*) &= \limsup_{\varepsilon\to 0}\sup_{x\in\Xi_\varepsilon}\left\{\langle x^*, x\rangle_{\mathbb{X}^*,\mathbb{X}} - I_\varepsilon(x)\right\}\\ &= \limsup_{\varepsilon\to 0}\left\{\left\langle x^*, x_\varepsilon^0\right\rangle_{\mathbb{X}^*,\mathbb{X}} - I_\varepsilon(x_\varepsilon^0)\right\}\\ &\leq \left\langle x^*, x^0\right\rangle_{\mathbb{X}^*,\mathbb{X}} - I(x^0) \leq I^*(x^*).\end{aligned} \tag{8.31}$$

Let $\overline{x}$ be a minimizers of the limit problem (8.30) and consider a $\Gamma(w)$-realizing sequence $\overline{x}_\varepsilon \rightharpoonup \overline{x}$ in $\mathbb{X}$. Then

$$\begin{aligned}\liminf_{\varepsilon\to 0} I_\varepsilon^*(x^*) &\geq \liminf_{\varepsilon\to 0}\left\{\langle x^*, \overline{x}_\varepsilon\rangle_{\mathbb{X}^*,\mathbb{X}} - I_\varepsilon(\overline{x}_\varepsilon)\right\}\\ &= \langle x^*, \overline{x}\rangle_{\mathbb{X}^*,\mathbb{X}} - I(\overline{x}) = I^*(x^*).\end{aligned}$$

Thus, we establish the pointwise convergence (8.29) and we also see that (8.31) holds an equality, that is, x^0 is a minimizer for the limit problem (8.30). This concludes the proof. □

8.3 Variational convergence of constrained minimization problems in Banach spaces

As mentioned in Chap. 5, it is often necessary to have a reasonable definition of the "limit problem" as ε tends to zero for a given sequence of constrained minimization problems

$$\left\{\left\langle \inf_{x\in\Xi_\varepsilon} I_\varepsilon(x)\right\rangle, \quad \varepsilon\to 0\right\}. \tag{8.32}$$

For instance, the homogenization theory of OCPs deals with a special sequence of minimization problems

$$(\mathrm{OCP}_\varepsilon) \ : \quad \left\langle \inf_{(u,y)\in\,\Xi_\varepsilon} I_\varepsilon(u,y)\right\rangle, \tag{8.33}$$

where the set of admissible solutions Ξ_ε is defined as

$$(\mathrm{SAS}_\varepsilon) \ : \ \Xi_\varepsilon = \left\{(u,y)\in U_\varepsilon^\partial\times\mathbb{Y}, U_\varepsilon^\partial\subset\mathbb{U},\ L_\varepsilon(u,y)+F_\varepsilon(y)=0\right\}, \tag{8.34}$$

and dependence on a small parameter ε in (8.33) and (8.34) usually has the form $(\cdot/\varepsilon)$. As a result, its object consists in passing to the limit, as $\varepsilon\to 0$, in problem (8.33). The expression "passing to the limit" means that we have to find a kind of "limit cost functional" I and a "limit set of constraints" Ξ with clearly defined structure $\Xi = \{(u,y) : (\mathrm{SE})\cap(\mathrm{CSC})\}$ (SE=state equation) such that the limit object $\langle\inf_{(u,y)\in\Xi} I(u,y)\rangle$ could be interpreted as some OCP

$$(\mathrm{OCP}) \ : \ \left\{\begin{array}{rcl}(\mathrm{CF}) & : & I(u,y)\to\inf,\\ & \text{subject to} & \\ (\mathrm{CSC}) & : & (u,y)\in\mathbb{U}\times\mathbb{Y}, u\in U^\partial\subset\mathbb{U},\\ (\mathrm{SE}) & : & L(u,y)+F(y)=0\end{array}\right\}. \tag{8.35}$$

In addition, the definition, used for the convergence of constrained minimization problems, has to satisfy two natural requirements:

(V1) If ${x_\varepsilon}^0$ is a minimizer for $\langle\inf_{x\in\Xi_\varepsilon} I_\varepsilon(x)\rangle$, and if ${x_\varepsilon}^0$ tends (in some sense) to $x^{\,0}$ then $x^{\,0}$ is a minimizer for the limit problem

$$\left\langle \inf_{x\in\Xi} I(x)\right\rangle.$$

(V2) $\lim_{\varepsilon\to 0}\inf_{x\in\,\Xi_\varepsilon} I_\varepsilon(x)=\inf_{x\in\,\Xi} I(x)$.

In the present section, we discuss the definition of a "limit minimization problem" for sequence (8.32) in the case when all elements of this sequence are defined on a fixed Banach space $\mathbb{X}$. In view of this, we begin with the following assumptions: Let $\mathbb{X}$ be a Banach space endowed with its weak topology. For simplicity, we assume that $\mathbb{X}$ is reflexive and separable. Let $\{\Xi_\varepsilon\}_{\varepsilon>0}$ be a given sequence of subsets of $\mathbb{X}$ satisfying the following property: There exist a sequence $\{x_\varepsilon\in\Xi_\varepsilon\}_{\varepsilon>0}$ and element $x\in\mathbb{X}$ such that $x_\varepsilon\rightharpoonup x$ in $\mathbb{X}$. Hence, by the properties of set convergence in the sense of Kuratowski, the sequential K-lower limit $K_s(w)\text{-}\liminf_{\varepsilon\to 0}\Xi_\varepsilon$ with respect to the weak topology of $\mathbb{X}$ is nonempty.

Taking into account the concept of constrained Γ-convergence, we adopt the following definition for the convergence of minimization problems.

Definition 8.13. *Let $\{I_\varepsilon:\Xi_\varepsilon\to\overline{\mathbb{R}}\}_{\varepsilon>0}$ be a sequence of functionals defined on the subsets of a Banach space $\mathbb{X}$. A problem $\langle\inf_{x\in\Xi} I(x)\rangle$ is said to be the variational limit of sequence (8.32) with respect to the weak topology of $\mathbb{X}$, we write*

$$\left\langle\inf_{x\in\Xi} I(x)\right\rangle=\Gamma(w)\text{-}\operatorname*{Lim}_{\varepsilon\to 0}\left\langle\inf_{x\in\Xi_\varepsilon} I_\varepsilon(x)\right\rangle, \tag{8.36}$$

if $\langle I_\varepsilon;\Xi_\varepsilon\rangle\overset{\Gamma(w)}{\longrightarrow}\langle I;\Xi\rangle$, or in other words, if the following conditions are satisfied:

(c) For any $x\in\Xi$, there exist a constant $\varepsilon_0>0$ and a sequence $x_\varepsilon\rightharpoonup x$ as $\varepsilon\to 0$ in $\mathbb{X}$ (called a Γ-realizing sequence) such that

$$x_\varepsilon\in\Xi_\varepsilon\ \text{ for every }\ \varepsilon\le\varepsilon_0\ \text{ and }\ I(x)=\limsup_{\varepsilon\to 0} I_\varepsilon(x_\varepsilon). \tag{8.37}$$

(cc) For any $x\in\Xi$ and for every sequence $x_k\rightharpoonup x$ as $k\to\infty$ in $\mathbb{X}$, there exists a subsequence $\{\Xi_{\varepsilon_k}\}_{k=1}^{\infty}$ of $\{\Xi_\varepsilon\}_{\varepsilon>0}$ such that

$$x_k\in\Xi_{\varepsilon_k}\ \text{ for all }\ k\in\mathbb{N}\ \text{ and }\ I(x)\le\liminf_{k\to\infty} I_{\varepsilon_k}(x_k). \tag{8.38}$$

Remark 8.14. As immediately follows from the definition of K_s-convergence, conditions (c) and (cc) imply the equality $\Xi=K_s(w)-\lim_{\varepsilon\to 0}\Xi_\varepsilon$.

In the following theorem, we prove that the variational convergence of a sequence (8.32) to problem (8.36) implies the convergence of the minimum values of I_ε on Ξ_ε to the minimum value of I on Ξ and, moreover, in this case, every weakly cluster point of the sequence of the minimizers of I_ε is a minimizer of I.

Theorem 8.15. *Assume that the minimization problem $\langle\inf_{x\in\Xi} I(x)\rangle$ is the variational limit of the sequence (8.32) in the sense of Definition 8.13 and this problem has a nonempty set of solutions. For every $\varepsilon>0$, let $x_\varepsilon^0\in\Xi_\varepsilon$*

be a solution of the corresponding problem (8.32). If the sequence $\{x_\varepsilon^0\}_{\varepsilon>0}$ *is relatively weakly compact in* $\mathbb{X}$*, then every weakly cluster point* x^0 *of this sequence is a minimizer of* I *on* Ξ *and*

$$I(x^0) = \min_{x\in\Xi} I(x) = \lim_{\varepsilon\to 0} \min_{x\in\Xi_\varepsilon} I_\varepsilon(x). \tag{8.39}$$

Proof. Let $\{x_{\varepsilon_k}\}_{k\in\mathbb{N}}$ be a subsequence of the sequence of minimizers weakly converging to some element x^*. Then by properties of K-convergence, we have $x^0 \in \Xi$. Moreover, due to property (ccc) of Definition 8.13, one gets

$$\liminf_{k\to\infty} \min_{x\in\Xi_{\varepsilon_k}} I_{\varepsilon_k}(x) = \liminf_{k\to\infty} I_{\varepsilon_k}(x^0_{\varepsilon_k}) \geq I(x^*) \geq \min_{x\in\Xi} I(x). \tag{8.40}$$

Let x^0 be a minimizer for the limit problem (8.36). Since $x^0 \in \Xi$, it follows that there exist a constant $\varepsilon_0 > 0$ and a sequence $\{x_\varepsilon\}_{\varepsilon>0}$ weakly converging to x^0 such that

$$x_\varepsilon \in \Xi_\varepsilon \quad \text{for every} \quad \varepsilon < \varepsilon_0$$

and

$$I(x^0) \geq \limsup_{\varepsilon\to 0} I_\varepsilon(x_\varepsilon).$$

Using this fact, we have

$$\begin{aligned} \min_{x\in\Xi} I(x) = I(x^0) &\geq \limsup_{\varepsilon\to 0} I_\varepsilon(x_\varepsilon) \geq \limsup_{\varepsilon\to 0} \min_{x\in\Xi_\varepsilon} I_\varepsilon(x) \\ &\geq \limsup_{k\to\infty} \min_{x\in\Xi_{\varepsilon_k}} I_{\varepsilon_k}(x) = \limsup_{k\to\infty} I_{\varepsilon_k}(x^0_{\varepsilon_k}). \end{aligned} \tag{8.41}$$

Therefore, in view of (8.40), we come to the following inequality:

$$\liminf_{k\to\infty} I_{\varepsilon_k}(x^0_{\varepsilon_k}) \geq \limsup_{k\to\infty} I_{\varepsilon_k}(x^0_{\varepsilon_k}).$$

As a result, combining (8.40) and (8.41), we conclude

$$\begin{aligned} I(x^*) = I(x^0) &= \min_{x\in\Xi} I(x), \\ I(x^*) &= \lim_{k\to\infty} \min_{x\in\Xi_{\varepsilon_k}} I_{\varepsilon_k}(x). \end{aligned} \tag{8.42}$$

Hence, x^* is a minimizer for the limit problem $\langle \inf_{x\in\Xi} I(x)\rangle$. On the other hand, since (8.42) holds for the weak limits of all subsequences $\{x_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{x_\varepsilon^0\}_{\varepsilon>0}$, it follows that the limits $\lim_{k\to\infty} \min_{x\in\Xi_{\varepsilon_k}} I_{\varepsilon_k}(x)$ are coincident and, therefore,

$$I(x^*) = \lim_{\varepsilon\to 0} \min_{x\in\Xi_\varepsilon} I_\varepsilon(x).$$

This concludes the proof. □

As an obvious consequence of Definition 8.13 and Theorem 8.11, we have the following compactness result.

Theorem 8.16. *Let $\mathbb{X}$ be a Banach space with the separable dual $\mathbb{X}^*$. Let $\{\langle \inf_{x\in\Xi_\varepsilon} I_\varepsilon(x)\rangle,\ \varepsilon\to 0\}$ be a sequence of constrained minimization problems satisfying properties (8.26) and (8.27). Then this sequence is relatively compact with respect to the variational convergence in the sense of Definition 8.13.*

Our next intention is to study some aspects of the concept of variational convergence applied to different classes of constrained minimization problems. Let $\mathbb{X}$ be a Banach space endowed with its weak topology μ. We define the class $\mathcal{P}(\mathbb{X})$ of all constrained minimization problems as

$$\left\langle \inf_{x\in\Xi} I(x) \right\rangle$$

such that $I:\Xi\to\mathbb{R}\cup\{+\infty\}=\overline{\mathbb{R}}$ are lower μ-semicontinuous functionals and Ξ are μ-closed subsets of $\mathbb{X}$.

Let us introduce the following binary relation $\langle L;\mathcal{P}(\mathbb{X})\rangle$ on $\mathcal{P}(\mathbb{X})$:

$$\left\langle \inf_{x\in\Xi_1} I_1(x) \right\rangle L \left\langle \inf_{x\in\Xi_2} I_2(x) \right\rangle$$

if and only if there exists a set $Q\subset\mathbb{X}$ such that

$$\text{(R1)} \qquad \mathrm{Dom}(I_1)\cap\Xi_1 = Q = \mathrm{Dom}(I_2)\cap\Xi_2, \tag{8.43}$$

$$\text{(R2)} \qquad I_1(x) = I_2(x) \quad \text{for every} \quad x\in Q. \tag{8.44}$$

Here, $\mathrm{Dom}(I)=\{x\in\mathbb{X}: I(x)<+\infty\}$. It is easily seen that $\langle L;\mathcal{P}(\mathbb{X})\rangle$ is an equivalence relation.

Definition 8.17. *We say that two sequences of constrained minimization problems*

$$\left\{\left\langle \inf_{x\in\Xi_\varepsilon} I_\varepsilon(x),\ \varepsilon\to 0\right\rangle\right\} \quad \textit{and} \quad \left\{\left\langle \inf_{x\in\Lambda_\varepsilon} J_\varepsilon(x),\ \varepsilon\to 0\right\rangle\right\} \tag{8.45}$$

are jointly L-equivalent if

$$\left\langle \inf_{x\in\Xi_\varepsilon} I_\varepsilon(x) \right\rangle L \left\langle \inf_{x\in\Lambda_\varepsilon} J_\varepsilon(x) \right\rangle, \quad \forall\,\varepsilon>0.$$

For the sake of convenience, we make use of the following notations:

$$P_\Xi I(x) = \begin{cases} I(x),\ x\in\Xi, \\ +\infty,\ x\in\mathbb{X}\backslash\Xi, \end{cases} \qquad \mathbf{1}_\Xi(x) = \begin{cases} 0, & x\in\Xi, \\ +\infty, & x\in\mathbb{X}\backslash\Xi. \end{cases}$$

We are now in a position to state the following result.

Theorem 8.18. *Assume that the sequences (8.45) are jointly L-equivalent. Moreover, assume that for the sequence*

$$\left\{\left\langle \inf_{x\in\Xi_\varepsilon} I_\varepsilon(x)\right\rangle,\ \varepsilon\to 0\right\}, \tag{8.46}$$

there exists a variational limit $\{\langle \inf_{x\in\Xi_0} I_0(x)\rangle\}$ (with respect to the μ-topology) and the sequence of functionals $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{\mathbb{R}}\right\}_{\varepsilon>0}$ is uniformly bounded below, that is, there exists a constant $\gamma > -\infty$ such that

$$I_\varepsilon(x) \ge \gamma \quad \forall\, x\in\Xi_\varepsilon, \quad \forall\, \varepsilon>0.$$

Then the constrained minimization problems

$$\left\langle \inf_{x\in K_s(\mu)-\liminf\limits_{\varepsilon\to 0}\Lambda_\varepsilon} \Gamma(\mu)\text{-}\limsup[P_{\Lambda_\varepsilon}J_\varepsilon](x)\right\rangle,$$

$$\left\langle \inf_{x\in K_s(\mu)-\limsup\limits_{\varepsilon\to 0}\Lambda_\varepsilon} \Gamma(\mu)\text{-}\liminf[P_{\Lambda_\varepsilon}J_\varepsilon](x)\right\rangle,$$

$$\left\langle \inf_{x\in\Xi_0} I_0(x)\right\rangle$$

belong to the same class of L-equivalence. Here,

$$\Gamma(\mu)\text{-}\limsup_{\varepsilon\to 0} P_{\Lambda_\varepsilon}J_\varepsilon \quad \textit{and} \quad \Gamma(\mu)\text{-}\liminf_{\varepsilon\to 0} P_{\Lambda_\varepsilon}J_\varepsilon$$

denote the Γ-lower and Γ-upper limits of $\{P_{\Xi_\varepsilon}I_\varepsilon\}_{\varepsilon>0}$, respectively.

Proof. By our assumptions, for every $x\in\mathbb{X}$ we have

$$\Gamma(\mu)\text{-}\limsup_{\varepsilon\to 0}[P_{\Xi_\varepsilon}I_\varepsilon](x) = \Gamma(\mu)\text{-}\liminf_{\varepsilon\to 0}[P_{\Xi_\varepsilon}I_\varepsilon](x) = P_{\Xi_0}I_0(x).$$

Moreover, since the sequences (8.45) are jointly L-equivalent, it follows that

$$P_{\Xi_\varepsilon}I_\varepsilon(x) = P_{\Lambda_\varepsilon}J_\varepsilon(x), \quad \forall\, x\in\mathbb{X},\ \forall\,\varepsilon>0. \tag{8.47}$$

Therefore, the sequence $\left\{P_{\Lambda_\varepsilon}J_\varepsilon : \mathbb{X}\to\overline{\mathbb{R}}\right\}_{\varepsilon>0}$ is Γ-convergent as well. Moreover, in this case we have

$$\Gamma(\mu)\text{-}\lim_{\varepsilon\to 0} P_{\Xi_\varepsilon}I_\varepsilon = \Gamma(\mu)\text{-}\lim_{\varepsilon\to 0} P_{\Lambda_\varepsilon}J_\varepsilon \qquad \text{on } \mathbb{X}. \tag{8.48}$$

It remains to verify that there exists a set Q_0 satisfying the condition

$$Q_0 = \mathrm{Dom}(I_0)\cap\Xi_0 = \mathrm{Dom}\left(\Gamma(\mu)\text{-}\lim_{\varepsilon\to 0}[P_{\Lambda_\varepsilon}J_\varepsilon]\right)\cap K_s(\mu)-\liminf_{\varepsilon\to 0}\Lambda_\varepsilon. \tag{8.49}$$

Using the uniform boundedness of the family $\left\{I_\varepsilon : \Xi_\varepsilon\to\overline{\mathbb{R}}\right\}_{\varepsilon>0}$, (8.48), and the properties of Γ-limits (see [78]), we obtain

$$\Gamma(\mu)\text{-}\lim_{\varepsilon\to 0}\left[P_{\Lambda_\varepsilon}J_\varepsilon\right](x)\geq\gamma+\Gamma(\mu)\text{-}\limsup_{\varepsilon\to 0}\mathbf{1}_{Q_\varepsilon}(x),\ \forall x\in\mathbb{X},$$

where the sets Q_ε are defined as

$$Q_\varepsilon=\mathrm{Dom}(I_\varepsilon)\cap\Xi_\varepsilon=\mathrm{Dom}(J_\varepsilon)\cap\Lambda_\varepsilon,\ \forall\varepsilon>0. \tag{8.50}$$

As follows from the Γ-limit properties, we have

$$\Gamma(\mu)\text{-}\limsup_{\varepsilon\to 0}\mathbf{1}_{Q_\varepsilon}(x)=\mathbf{1}_{K_s(\mu)\text{-}\liminf_{\varepsilon\to 0}Q_\varepsilon}(x).$$

Hence,

$$\begin{aligned}\emptyset\neq\mathrm{Dom}\left(\Gamma(\mu)\text{-}\lim_{\varepsilon\to 0}[P_{\Lambda_\varepsilon}J_\varepsilon]\right)\\ =\mathrm{Dom}(P_{\Xi_0}I_0)\subset K_s(\mu)\text{-}\liminf_{\varepsilon\to 0}Q_\varepsilon.\end{aligned} \tag{8.51}$$

Using (8.50), we see that

$$\begin{aligned}K_s(\mu)\text{-}\liminf_{\varepsilon\to 0}Q_\varepsilon&\subseteq K_s(\mu)\text{-}\lim_{\varepsilon\to 0}\Xi_\varepsilon\cap K_s(\mu)\text{-}\limsup_{\varepsilon\to 0}\left(\mathrm{Dom}(F_\varepsilon)\right)\\&=\Lambda_0\cap\mu\text{-}\limsup_{\varepsilon\to 0}\left(\mathrm{Dom}(I_\varepsilon)\right),\\K_s(\mu)\text{-}\liminf_{\varepsilon\to 0}Q_\varepsilon&\subset K_s(\mu)\text{-}\liminf_{\varepsilon\to 0}\Lambda_\varepsilon\cap K_s(\mu)\text{-}\limsup_{\varepsilon\to 0}\left(\mathrm{Dom}(J_\varepsilon)\right).\end{aligned}$$

Then, from (8.51), we deduce that

$$\begin{aligned}\mathrm{Dom}\left(\Gamma(\mu)\text{-}\lim_{\varepsilon\to 0}[P_{\Lambda_\varepsilon}J_\varepsilon]\right)\subseteq K_s(\mu)\text{-}\liminf_{\varepsilon\to 0}\Lambda_\varepsilon,\\ \mathrm{Dom}\left(P_{\Xi_0}I_0\right)\subseteq Q_0.\end{aligned}$$

Thus, in view of (8.48), equality (8.49) follows immediately. □

Let us consider the following application of this theorem.

Let $\left\{I_\varepsilon:\mathbb{X}\to\overline{\mathbb{R}}\right\}_{\varepsilon>0}$ be a sequence of functionals uniformly bounded below and let $\{\Xi_\varepsilon\}_{\varepsilon>0}$ and $\{\Lambda_\varepsilon\}_{\varepsilon>0}$ be two sequences of subsets of $\mathbb{X}$ such that

$$\Xi_\varepsilon=\Lambda_\varepsilon\cap W\quad\text{ for every }\varepsilon>0,$$

where W is some fixed subset.

Then it is easy to see that the sequences of constrained minimization problems

$$\left\{\left\langle\inf_{x\in\Xi_\varepsilon}I_\varepsilon(x)\right\rangle,\ \varepsilon\to 0\right\}\quad\text{and}\quad\left\{\left\langle\inf_{x\in\Lambda_\varepsilon}\left(I_\varepsilon(x)+\mathbf{1}_W(x)\right)\right\rangle,\ \varepsilon\to 0\right\}$$

are jointly L-equivalent.

Assume that for the sequence $\{\langle\inf_{x\in\Lambda_\varepsilon}\left(I_\varepsilon(x)+\mathbf{1}_W(x)\right)\rangle,\ \varepsilon\to 0\}$, there exists a variational limit

$$\left\langle \inf_{x \in \Lambda_0} J_0(x) \right\rangle. \tag{8.52}$$

Then, by Theorem 8.18, there exists a variational limit for the sequence

$$\left\{ \left\langle \inf_{x \in \Xi_\varepsilon} I_\varepsilon(x) \right\rangle, \ \varepsilon \to 0 \right\}$$

which is L-equivalent to (8.52).

8.4 Variational convergence of minimization problems in variable spaces

In this section, we study the limit properties of a sequence of constrained minimization problems

$$\left\{ \left\langle \inf_{x \in \Xi_\varepsilon} I_\varepsilon(x) \right\rangle, \quad \varepsilon \to 0 \right\} \tag{8.53}$$

in the case when for every $\varepsilon > 0$, the sets of admissible solutions Ξ_ε belong to the corresponding Banach spaces $\mathbb{X}_\varepsilon$. Our aim is to give for (8.53) a reasonable definition of the "limit problem" as ε tends to 0. In view of this, we begin with the following suppositions.

For any small positive value $\varepsilon > 0$, let $\mathbb{X}_\varepsilon$ be a real Banach space equipped with the norm $\|\cdot\|_{\mathbb{X}_\varepsilon}$. Following Khruslov's classification (see [138]), we say that the spaces $\mathbb{X}_\varepsilon$ are strongly connected with a real Banach space $\mathbb{X}$ if there exists a sequence of continuous linear mappings $\{P_\varepsilon : \mathbb{X}_\varepsilon \to \mathbb{X}\}_{\varepsilon>0}$ satisfying the property $\sup_{\varepsilon>0} \|P_\varepsilon\| < +\infty$. In contrast to this concept, the weak connectedness of the above spaces means the absence of the corresponding extension operators $P_\varepsilon : \mathbb{X}_\varepsilon \to \mathbb{X}$ with properties prescribed above. In what follows in this section, we make no assumptions of strong connectedness of the variable spaces $\mathbb{X}_\varepsilon$ with $\mathbb{X}$. Moreover, in our next applications, each of the spaces $\mathbb{X}_\varepsilon$ is usually associated with some reticulated structure. So, in general, we admit that $\mathbb{X}_\varepsilon$ can be taken as Sobolev spaces with respect to some variable Borel measure.

Let τ be a "convergence concept" in the scale of variable spaces $\{\mathbb{X}_\varepsilon\}_{\varepsilon>0}$. As follows from our previous considerations, this convergence can be introduced in a variety of ways (see Chap. 6 for the details). It should be stressed here that when we speak about "choice of the convergence concept in the scale of variable spaces", we should look at this problem from two opposite points of view. On the one hand, this convergence has to be weak enough in order to have compactness result for bounded sequences $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$. Still, this type of convergence has to be strong enough in order to have the so-called strong approximation property for the "limit" space $\mathbb{X}$. This means that for every element $u \in \mathbb{X}$, one can find a sequence $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ such that $u_\varepsilon \xrightarrow{\tau} u$

in $\mathbb{X}_\varepsilon$. Note that, in general, this is not easily attained. However, the choice of the convergence in the scale of variable spaces $\{\mathbb{X}_\varepsilon\}_{\varepsilon>0}$ is completely free, provided the compactness and approximation occur. For further analysis, we suppose that the τ-convergence in $\mathbb{X}_\varepsilon$ is given and $\mathbb{X}$ is the τ-"limit" space for the scale $\{\mathbb{X}_\varepsilon\}_{\varepsilon>0}$.

For any fixed $u \in \mathbb{X}$, let us denote by $\mathfrak{W}_\tau(u)$ the set of all sequences $\{u_\varepsilon\}_{\varepsilon>0}$ such that

$$u_\varepsilon \in \mathbb{X}_\varepsilon,\ \forall\, \varepsilon > 0, \quad \sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathbb{X}_\varepsilon} < +\infty,$$
$$u_\varepsilon \xrightarrow{\tau} u \ \text{ in } \ \mathbb{X}_\varepsilon.$$

So, hereinafter the inclusion $\{u_\varepsilon\}_{\varepsilon>0} \in \mathfrak{W}_\tau(u)$ means that $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded τ-convergent sequence in $\mathbb{X}_\varepsilon$ and an element u is its τ-limit.

Let $\{\Xi_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ be a given sequence of subsets satisfying the following properties:

(A_i) $\{\Xi_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded, that is, $\sup_{\varepsilon>0} \sup_{u\in\Xi_\varepsilon} \|u\|_{\mathbb{X}_\varepsilon} < +\infty$.

(A_{ii}) There exist a sequence $\{u_\varepsilon \in \Xi_\varepsilon\}_{\varepsilon>0}$ and element $u \in \mathbb{X}$ such that $\{u_\varepsilon\}_{\varepsilon>0} \in \mathfrak{W}_\tau(u)$.

Hence, by properties of set convergence in the sense of Kuratowski, the sequential K-lower limit $K_s(\tau)\text{-}\liminf_{\varepsilon\to 0} \Xi_\varepsilon$ with respect to the τ-convergence in variable space $\mathbb{X}_\varepsilon$ is nonempty.

Definition 8.19. *We say that a sequence of sets $\{\Xi_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ K_s-converges with respect to the τ-convergence in the variable spaces $\{\mathbb{X}_\varepsilon\}_{\varepsilon>0}$ to a set $\Xi \subset \mathbb{X}$ if the following conditions are satisfied:*

(a) For every $u \in \Xi$, there exist a constant $\varepsilon_0 > 0$ and a sequence $\{u_\varepsilon\}_{\varepsilon>0} \in \mathfrak{W}_\tau(u)$ such that $u_\varepsilon \in \Xi_\varepsilon$ for every $\varepsilon < \varepsilon_0$.

(b) If $\{\Xi_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{\Xi_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ and $\{u_k\}_{k\in\mathbb{N}}$ is a sequence τ-converging to u in the variable spaces $\{\mathbb{X}_\varepsilon\}$ such that $u_k \in \Xi_{\varepsilon_k}$ for every $k \in \mathbb{N}$, then $u \in \Xi$.

For our next analysis, we make use the following observation.

Lemma 8.20. *Let $\{\Xi_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ be a sequence of subsets satisfying properties (A_i) and (A_{ii}). Let $\{I_\varepsilon : \Xi_\varepsilon \to \mathbb{R}\}_{\varepsilon>0}$ be a sequence of functionals such that*

$$\sup_{\varepsilon>0} \sup_{x\in\Xi_\varepsilon} |I_\varepsilon(x)| \le C \tag{8.54}$$

for some constant $C > 0$. Assume that there exists a nonempty set $\mathfrak{D} \subset \mathbb{X}\times\mathbb{R}$ such that

$$\begin{aligned}\mathfrak{D} &= K_s(\tau\times\sigma)\text{-}\lim_{e\to 0} \operatorname{epi}(I_\varepsilon; \Xi_\varepsilon)\\ &= K_s(\tau\times\sigma)\text{-}\lim_{e\to 0} \{(x,\lambda) \in \mathbb{X}\times\mathbb{R} \mid x \in \Xi_\varepsilon, \quad I_\varepsilon(x) \le \lambda\}\end{aligned} \tag{8.55}$$

in the sense of Definition 8.19, where by σ we denote the topology of pointwise convergence in $\mathbb{R}$. Then there exist a nonempty set $\Xi \subset \mathbb{X}$ and a functional $I : \Xi \to \mathbb{R}$ such that

$$\mathfrak{D} = \mathrm{epi}(I; \Xi), \quad \Xi = K_s(\tau) - \lim_{\varepsilon \to 0} \Xi_\varepsilon. \tag{8.56}$$

Remark 8.21. In context of Definition 8.6, this result can be rephrased as follows: Under assumptions given above, the functional $I : \Xi \to \mathbb{R}$ is the constrained Γ-limit of the sequence $\{I_\varepsilon : \Xi_\varepsilon \to \mathbb{R}\}_{\varepsilon>0}$ with respect to the τ-convergence in the variable space $\mathbb{X}_\varepsilon$.

Proof. We first prove that the set $\mathfrak{D}$ can be represented in the form $\mathfrak{D} = \mathrm{epi}(I; \Xi)$. To do so, we note that in view of supposition (8.54), a notable feature of the set $\mathrm{epi}(I_\varepsilon; \Xi_\varepsilon)$ is the fact that for a fixed $u_\varepsilon \in \Xi_\varepsilon$, the set of all $\lambda \in \mathbb{R}$ such that $(u_\varepsilon, \lambda) \in \mathrm{epi}(I_\varepsilon; \Xi_\varepsilon)$ coincides with a closed half-line $\lambda \geq I_\varepsilon(u_\varepsilon)$.

Let (u_0, λ_0) be an arbitrary pair belonging to the set $\mathfrak{D}$. To begin, we show that $(u_0, \lambda^*) \in \mathfrak{D}$ for any $\lambda^* \geq \lambda_0$. Due to the sequential properties of K_s-convergence (see Definition 8.19), for the given pair $(u_0, \lambda_0) \in \mathfrak{D}$ there exist a constant $\varepsilon_0 > 0$ and a sequence $\{(u_\varepsilon, \lambda_\varepsilon)\}_{\varepsilon>0}$ such that $\{u_\varepsilon\}_{\varepsilon>0} \in \mathfrak{W}_\tau(u_0)$, $\lambda_\varepsilon \to \lambda_0$ in $\mathbb{R}$, and $(u_\varepsilon, \lambda_\varepsilon) \in \mathrm{epi}(I_\varepsilon; \Xi_\varepsilon)$ for every $\varepsilon < \varepsilon_0$. Hence, $I_\varepsilon(u_\varepsilon) \leq \lambda_\varepsilon$ for all $\varepsilon < \varepsilon_0$. Let us define the sequence $\{\lambda^*_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}$ as follows:

$$\lambda^*_\varepsilon = \begin{cases} \lambda^* \text{ if } \lambda^* \geq \lambda_\varepsilon, \\ \lambda_\varepsilon \text{ otherwise.} \end{cases}$$

Since $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$ and $\lambda^* \geq \lambda_0$, it follows that λ^*_ε converge to λ^* as ε tends to 0. Moreover, for every $\varepsilon < \varepsilon_0$, we have $(u_\varepsilon, \lambda^*_\varepsilon) \in \mathrm{epi}(I_\varepsilon; \Xi_\varepsilon)$ and $(u_\varepsilon, \lambda^*_\varepsilon) \to (u_0, \lambda^*)$ with respect to the $\tau \times \sigma$-convergence in variable space $\mathbb{X}_\varepsilon \times \mathbb{R}$. Then, by property (b) of Definition 8.19, we conclude that $(u_0, \lambda^*) \in \mathfrak{D}$ for all $\lambda^* \geq \lambda_0$. Since this property is valid for any pair $(u_0, \lambda_0) \in \mathfrak{D}$, it follows that there exists a functional I with the domain

$$\Xi = \Pi \mathrm{r}_{\mathbb{X}} \mathfrak{D} = \{x \in \mathbb{X} \ : \ \exists \lambda \in \mathbb{R}, (x, \lambda) \in \mathfrak{D}\},$$

for which $\mathrm{epi}(I; \Xi) = \mathfrak{D}$.

To prove the second equality in (8.56), it remains to apply Definition 8.19 to the case when each of the $\tau \times \sigma$-convergent sequences $\{(u_\varepsilon, \lambda_\varepsilon)\}_{\varepsilon>0}$ has the form $\{(u_\varepsilon, \widetilde{\lambda})\}_{\varepsilon>0}$, where $\widetilde{\lambda} \geq C$. Here, C is defined in (8.54). This concludes the proof. □

It is reasonable now to introduce the following concept.

Definition 8.22. *Let (8.53) be a given sequence of constrained minimization problems in variable spaces $\mathbb{X}_\varepsilon$. Let $\mathbb{X}$ be a Banach space with respect to which the τ-convergence in the scale $\{\mathbb{X}_\varepsilon\}_{\varepsilon>0}$ is defined. A problem $\langle \inf_{x \in \Xi} I(x) \rangle$,*

where $\Xi \subseteq \mathbb{X}$ and $I : \Xi \to \overline{\mathbb{R}}$, is said to be the variational limit of the sequence (8.53) with respect to the τ-convergence in X_ε, we write

$$\left\langle \inf_{x\in\Xi} I(x) \right\rangle = \Gamma(\tau)\text{-}\operatorname*{Lim}_{\varepsilon\to 0} \left\langle \inf_{x\in\Xi_\varepsilon} I_\varepsilon(x) \right\rangle, \tag{8.57}$$

if

$$\operatorname{epi}(I_\varepsilon; \Xi_\varepsilon) \overset{K_s(\tau)}{\longrightarrow} \operatorname{epi}(I; \Xi).$$

Before we give a more elaborate form of this definition, we introduce the following concept.

Definition 8.23. *We say that the space $\mathbb{X}$ possesses the weak τ-approximation property with respect to the scale of spaces $\{X_\varepsilon\}_{\varepsilon>0}$ if for every $\delta > 0$ and every $u \in \mathbb{X}$, there exist an element $u^* \in \mathbb{X}$ and a sequence $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ such that*

$$\|u - u^*\|_{\mathbb{X}} \le \delta \quad \text{and} \quad u_\varepsilon \overset{\tau}{\longrightarrow} u^* \quad \text{in} \quad \mathbb{X}_\varepsilon. \tag{8.58}$$

In this case, the sequence $\{u_\varepsilon \in \mathbb{X}_\varepsilon\}_{\varepsilon\in E}$ is called δ-realizing.

Hereinafter in this framework we suppose that the τ-approximation property for the limit space $\mathbb{X}$ holds true. We are now in a position to introduce a "natural" notion of the "limiting" minimization problem to the family (8.53) as ε tends to 0.

Definition 8.24. *A problem $\langle \inf_{x\in\Xi} I(x) \rangle$ is the variational limit of the sequence (8.53) as $\varepsilon \to 0$ if and only if the following conditions are satisfied:*

(d) If sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and $\{u_k\}_{k\in\mathbb{N}}$ are such that $\varepsilon_k \to 0$ as $k \to \infty$, $u_k \in \Xi_{\varepsilon_k}$ $\forall\, k \in \mathbb{N}$, and $u_k \overset{\tau}{\longrightarrow} u$ in $\mathbb{X}_\varepsilon$, then

$$u \in \Xi, \quad I(u) \le \liminf_{k\to\infty} I_{\varepsilon_k}(u_k). \tag{8.59}$$

(dd) For every element $u \in \Xi \subset \mathbb{X}$ and any $\delta > 0$, there are a constant $\varepsilon^0 > 0$ and a δ-realizing sequence $\{u_\varepsilon\}_{\varepsilon>0}$ such that

$$u_\varepsilon \in \Xi_\varepsilon, \ \forall\, \varepsilon \le \varepsilon^0, \quad u_\varepsilon \overset{\tau}{\longrightarrow} \widehat{u} \quad \text{in} \quad \mathbb{X}_\varepsilon, \quad \|u - \widehat{u}\|_{\mathbb{X}} \le \delta, \tag{8.60}$$

$$I(u) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) - \widehat{C}\delta, \tag{8.61}$$

with some constant $\widehat{C} > 0$ independent of δ.

The following result can be regarded as a natural generalization of Theorem 8.15 to the case when the original sequence of constrained minimization problems is defined in the scale of variable spaces.

Theorem 8.25. *Assume that the constrained minimization problem*

$$\left\langle \inf_{u\in\Xi_0} I_0(u)\right\rangle \tag{8.62}$$

is the variational limit of the sequence (8.53) in the sense of Definition 8.24 and this problem has a unique solution $u^0 \in \Xi_0$. For every $\varepsilon > 0$, let $u_\varepsilon^0 \in \Xi_\varepsilon$ be a minimizer of I_ε on the corresponding sets Ξ_ε. If the sequence $\{u_\varepsilon^0\}_{\varepsilon>0}$ is relatively compact with respect to the τ-convergence in $\mathbb{X}_\varepsilon$, then

$$u_\varepsilon^0 \xrightarrow{\tau} u^0, \tag{8.63}$$

$$\inf_{u\in\Xi_0} I_0(u) = I_0\left(u^0\right) = \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^0) = \lim_{\varepsilon\to 0} \inf_{u_\varepsilon\in\Xi_\varepsilon} I_\varepsilon(u_\varepsilon). \tag{8.64}$$

Proof. Let $\{u_{\varepsilon_k}^0\}_{k\in\mathbb{N}}$ be a subsequence of the sequence of minimizers τ-converging to some element $u^* \in \mathbb{X}$. Then due to property (ii) of Definition 8.24, we have $u^* \in \Xi_0$ and

$$\liminf_{k\to\infty} \min_{u\in\Xi_{\varepsilon_k}} I_{\varepsilon_k}(u) = \liminf_{k\to\infty} I_{\varepsilon_k}(u_{\varepsilon_k}^0) \ge I_0(u^*) \ge \min_{u\in\Xi_0} I_0(u). \tag{8.65}$$

Let $u^0 \in \Xi_0$ be a minimizer of the limit problem (8.62). Let us fix a value $\delta > 0$. Then, by property (ii) of Definition 8.24, there exist a δ-realizing sequence $\{\widehat{u}_\varepsilon \in \Xi_\varepsilon\}_{\varepsilon>0}$ such that

$$\widehat{u}_\varepsilon \xrightarrow{\tau} \widehat{u}, \quad \|u^0 - \widehat{u}\|_{\mathbb{X}} \le \delta, \quad \text{and} \quad I_0(u^0) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(\widehat{u}_\varepsilon) - \widehat{C}\delta.$$

Using this fact, we have

$$\begin{aligned}\min_{u\in\Xi_0} I_0(u) + \widehat{C}\delta &= I_0(u^0) + \widehat{C}\delta \\ &\ge \limsup_{\varepsilon\to 0} I_\varepsilon(\widehat{u}_\varepsilon) \ge \limsup_{\varepsilon\to 0} \min_{u\in\Xi_\varepsilon} I_\varepsilon(u) \\ &\ge \limsup_{k\to\infty} \min_{u\in\Xi_{\varepsilon_k}} I_{\varepsilon_k}(u) = \limsup_{k\to\infty} I_{\varepsilon_k}(u_{\varepsilon_k}^0).\end{aligned} \tag{8.66}$$

From this and (8.65), we deduce that

$$\liminf_{k\to\infty} I_{\varepsilon_k}(u_{\varepsilon_k}^0) \ge \limsup_{k\to\infty} I_{\varepsilon_k}(u_{\varepsilon_k}^0) - \widehat{C}\delta.$$

Since this inequality holds true for any $\delta > 0$ small enough, combining the (8.65) and (8.66), we get

$$I_0(u^*) = I_0(u^0) = \min_{u\in\Xi_0} I_0(u), \quad I_0(u^0) = \lim_{k\to\infty} \min_{u\in\Xi_{\varepsilon_k}} I_{\varepsilon_k}(u).$$

Using these relations and the fact that a minimizer of problem (8.62) is unique, we obtain $u^* = u^0$. Since this equality holds for the τ-limits of all subsequences of $\{u_\varepsilon^0\}_{\varepsilon\in E}$, it follows that these limits coincide and, therefore,

u^0 is the τ-limit of the whole sequence $\{u_\varepsilon^0\}_{\varepsilon>0}$. Hence, accomplishing for the sequence of minimizers what we did earlier with a subsequence $\{u_{\varepsilon_k}^0\}_{k\in\mathbb{N}}$, we have

$$\begin{aligned}\liminf_{\varepsilon\to 0}\min_{u\in\Xi_\varepsilon} I_\varepsilon(u) &= \liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^0)\\ &\geq I_0(u^0) = \min_{u\in\Xi_0} I_0(u) \geq \limsup_{\varepsilon\to 0} I_\varepsilon(\widehat{u}_\varepsilon) - \delta\\ &\geq \limsup_{\varepsilon\to 0}\min_{u\in\Xi_\varepsilon} I_\varepsilon(u) - \widehat{C}\delta\\ &= \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^0) - \widehat{C}\delta, \quad \forall\,\delta>0.\end{aligned}$$

Thus, we have obtained the required conclusion. This proof is complete. □

8.5 Asymptotic analysis of a Dirichlet optimal control problem

The aim of this section is to study the asymptotic behavior of an OCP with partial exact controllability constraints. More precisely, for the sake of simplicity, the main control object is a Dirichlet boundary value problem for the linear Laplacian with bounded controls in the space of Radon measures $\mathcal{M}(\Omega)$. We suppose that for every ε, where ε takes its values in a sequence of positive numbers tending to 0, there are some closed "holes" $T_i^\varepsilon, 1\leq i\leq n(\varepsilon)$, such that on the subdomain $S_\varepsilon = \cup T_i^\varepsilon$ the state is supposed to exactly match a certain profile Ψ_ε^i on each T_i^ε. We do not make any assumption on the holes other than meas $T_i^\varepsilon > 0$, $\forall\varepsilon>0$. The problem is to describe the asymptotic behavior of the parameterized OCP as ε tends to 0, using the variational convergence approach given above.

We show that this problem can be viewed as a particular example covered by the theory developed in this chapter. In particular, we consider holes having a critical size depending on their number and distribution. We show that the state equation and the cost functional in the homogenized OCP include additional terms, the so-called "strange terms," that are associated with the holes. The analysis is very much in the spirit of Dal Maso and Murat [82], where, however, OCPs have not been considered. It is to be noted that the scheme of asymptotic analysis, used in this section, does not provide estimates of the convergence of solutions with respect to some distance measure between the domains. Convergence estimates have been investigated by Savaré and Schimperna [223], again for problems without control.

8.5.1 The statement of the optimal control problem and preliminary results

Let Ω be a bounded connected and open subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Let $E=(0;\varepsilon^0]$ be an index set and let $\{S_\varepsilon\}_{\varepsilon\in E}$ be a family of closed

subsets such that $S_\varepsilon \subset \Omega$ with an nonempty interior and Lipschitz boundary for every $\varepsilon \in E$.

Let $\{\Psi_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)\}_{\varepsilon\in E}$ and $\{f_\varepsilon \in L^2(\Omega)\}_{\varepsilon\in E}$ be given functions such that

$$\Psi_\varepsilon \to \Psi_0 \qquad \text{weakly in } H_0^1(\Omega), \quad f_\varepsilon \to f_0 \qquad \text{weakly in } L^2(\Omega).$$

As usual, by $\operatorname{cap}(B,\Omega)$ we will denote the capacity of a set B with respect to Ω which is defined as the infimum of $\displaystyle\int_\Omega |\nabla y|^2\,\mathrm{d}x$ over the set of all functions $y \in H_0^1(\Omega)$ such that $y \geq 1$ almost everywhere (a.e.) in a neighborhood of B. Using the terminology of Sect. 2.5 (see also [267]), we say that some property $\mathcal{P}(\alpha)$ holds quasi-everywhere (q.e.) in the set Ω, if it holds for all $x \in \Omega$ except for a subset B of Ω with $\operatorname{cap}(B,\Omega) = 0$. We recall that every function $y \in H_0^1(\Omega)$ has a quasi-continuous representative which is uniquely defined up to a set of capacity 0. Here, a function $y : \Omega \to \mathbb{R}$ is said to be quasi-continuous if for every $\delta > 0$, there exists a subset $B \subseteq \Omega$ with $\operatorname{cap}(B,\Omega) < \delta$ such that the restriction of y to $\Omega \setminus B$ is continuous. We will always identify elements of $H_0^1(\Omega)$ with their quasi-continuous representatives.

We will frequently use the notion of a filter on the set E.

Definition 8.26. *The Fréchet filter on E is the family $\mathcal{H}$ of all subsets of E which contain an interval of the form $(0,\delta)$ for some $\delta > 0$.*

The family of all subsets of E that meet all sets H in $\mathcal{H}$ we denote by $\mathcal{H}^\#$.

Now, for every fixed $\varepsilon \in E$, we define the control problem in the class of bounded Radon measures as follows:

$$-\triangle y = f_\varepsilon + \nu \quad \text{in } \mathcal{D}'(\Omega), \tag{8.67}$$

$$y \in H_0^1(\Omega) \qquad \nu \in \mathcal{M}(\Omega), \tag{8.68}$$

$$y = \Psi_\varepsilon \qquad \text{quasi-everywhere in } S_\varepsilon, \tag{8.69}$$

$$I(\nu, y) = \int_{\Omega\setminus S_\varepsilon} |y - z_d|^2\,\mathrm{d}x + \| \nu \|_{\mathcal{M}(\Omega)} \to \inf, \tag{8.70}$$

where z_d is some fixed function of $L^2(\Omega)$ and by $\| \nu \|_{\mathcal{M}(\Omega)}$ we denote the norm of ν in the space of Radon measures $\mathcal{M}(\Omega)$, that is,

$$\| \nu \|_{\mathcal{M}(\Omega)} = \sup_{\|\varphi\|_{C_0(\Omega)\leq 1}} |\nu(\varphi)|.$$

It should be stressed here that, in general, this problem has no optimal solutions. The reason for this is as follows. First, for a Radon measure $\nu \in \mathcal{M}(\Omega)$, in general, the Dirichlet problem for the equation $-\Delta y = f_\varepsilon + \nu$ has no solution in $H_0^1(\Omega)$. Second, even if the set of admissible pairs

$$\Xi_\varepsilon = \big\{(\nu, y) \in \mathcal{M}(\Omega) \times H_0^1(\Omega) \ : \\ -\Delta y = f_\varepsilon + \nu \text{ in } \mathcal{D}'(\Omega),\ y = \Psi_\varepsilon \text{ q.e. on } S_\varepsilon\big\}$$

is nonempty for a fixed value ε, it is not clear that this set is closed with respect to the product of the weak-$*$ topology for $\mathcal{M}(\Omega)$ and the topology of the weak convergence in $H_0^1(\Omega)$ (see [106, 169]). Therefore, in order to solve this problem, we have to use some regularization approaches. The simplest way for this is penalizing the controllability constraints on S_ε and choosing a narrower class of admissible controls compared with the class of all bounded Radon measures $\mathcal{M}(\Omega)$.

8.5.2 On modifications of the optimal control problem with controllability constraints

We recall that a subset U of Ω is said to be quasi-open if for every $\delta > 0$, there exists an open subset V of Ω with $\operatorname{cap}(V, \Omega) < \delta$ such that $U \cup V$ is open. If μ is a non-negative Borel measure on Ω, we will use $L^2_\mu(\Omega)$ to denote the Lebesgue space with respect to the measure μ.

We introduce the space $V_\mu = H_0^1(\Omega) \cap L^2_\mu(\Omega)$ which is well defined for every $\mu \in \mathcal{M}_0^2(\Omega)$ since all functions in $H_0^1(\Omega)$ are defined μ-almost everywhere in Ω (for the details, we refer to Sect. 2.6). In general, this space is not dense in $L^2(\Omega)$. Moreover, this space is a Hilbert space with respect to the scalar product

$$(y, g)_\mu = \int_\Omega \nabla y \cdot \nabla g \, \mathrm{d}x + \int_\Omega y\, g \, \mathrm{d}\mu \quad \text{(see [45])}.$$

Let V_μ' denote the dual space of V_μ and let $\langle \cdot, \cdot \rangle_\mu$ be the duality pairing. We note that even if the transposed mappings to the embeddings of V_μ into $H_0^1(\Omega)$ and into $L^2(\Omega)$ are not injective, the spaces $H^{-1}(\Omega)$ and $L^2(\Omega)$ can be considered as the linear subspaces of V_μ^* and we have

$$\langle f, y \rangle_\mu = \langle f, y \rangle_{H_0^1(\Omega)} \quad \text{for } f \in H^{-1}(\Omega),\ y \in V_\mu$$

and, in particular,

$$\langle f, y \rangle_\mu = \int_\Omega f y \, \mathrm{d}x \quad \text{for } f \in L^2(\Omega),\ y \in V_\mu,$$

where we denote by $\langle \cdot, \cdot \rangle_{H_0^1(\Omega)}$ the duality pairing between $H_0^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Let $f \in H^{-1}(\Omega), \Psi \in H_0^1(\Omega)$, and $\mu \in \mathcal{M}_0^2$ be fixed elements. We introduce the following Dirichlet problem: Find $y \in H_0^1(\Omega)$ such that $y - \Psi \in V_\mu$ and

$$\int_\Omega (\nabla y, \nabla \phi) \, \mathrm{d}x + \int_\Omega (y - \Psi)\varphi \, \mathrm{d}\mu = \big\langle f, \varphi \big\rangle \quad \forall \varphi \in V_\mu. \tag{8.71}$$

It is easy to see that this problem has a unique solution since the operator $B : V_\mu \to V_\mu^*$, which is defined by

$$\langle Bz, \varphi \rangle = \int_\Omega (\nabla z, \nabla \varphi)\, \mathrm{d}x + \int_\Omega z\varphi \, \mathrm{d}\mu, \qquad \forall\, \varphi \in V_\mu,$$

is monotone, continuous, and coercive. Moreover, if we take $\varphi = y - \Psi$ as the test function in (8.71), we obtain

$$\int_\Omega |\nabla(y - \Psi)|^2 \, \mathrm{d}x + \int_\Omega (y - \Psi)^2 \, \mathrm{d}\mu = \langle f + \triangle\Psi, y - \Psi \rangle.$$

Therefore, by Young's inequality, we have

$$\| y - \Psi \|_{V_\mu} \leq \| f + \triangle\Psi \|_{H^{-1}(\Omega)} \leq \| f \|_{H^{-1}(\Omega)} + \| \triangle\Psi \|_{H^{-1}(\Omega)}. \tag{8.72}$$

Now, we note that whenever $f \in L^2(\Omega)$ and $\Psi \in H_0^1(\Omega) \cap H^2(\Omega)$, the solution of (8.71) is actually the solution of a new equation involving a Radon measure λ.

Indeed, let y be the corresponding solution of problem (8.71). Since $(f + \triangle\Psi) \in L^2(\Omega)$, it follows that the positive part of $f + \triangle\Psi$ is well defined. So, we define the element λ_1 of $H^{-1}(\Omega)$ such that

$$-\triangle(y - \Psi)^+ + \lambda_1 = (f + \triangle\Psi)^+, \tag{8.73}$$

where $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$ and $f = f^+ - f^-$.

Let $v \in H_0^1(\Omega)$ be any function so that $v \geq 0$ q.e. in Ω. Following the line of the proof of Proposition 2.8 from [82], we put

$$v_n = \frac{1}{n} v \wedge (y - \Psi)^+,$$

where by $y \wedge v$ we denote the minimum of $\{y, v\}$. Then it is easy to see that

$$v_n \geq 0 \qquad \text{q.e. in } \Omega \text{ and } \; v_n \in V_\mu, \quad \forall\, n \in N.$$

Since

$$(y - \Psi) \cdot v_n \geq 0 \qquad \text{q.e. in } \Omega,$$
$$(f + \triangle\Psi) v_n \leq (f + \triangle\Psi)^+ v_n \qquad \text{a.e. in } \Omega,$$

it follows that by taking v_n as the test function in (8.71), we obtain

$$\int_\Omega (\nabla(y - \Psi), \nabla v_n)\, \mathrm{d}x \leq \int_\Omega (f + \triangle\Psi)^+ v_n \, \mathrm{d}x \leq \frac{1}{n} \int_\Omega (f + \triangle\Psi)^+ v \, \mathrm{d}x. \tag{8.74}$$

We note that

$$\nabla v_n = \nabla v / n \qquad \text{a.e. in the domain } \left\{ \frac{1}{n} v < (y - \Psi)^+ \right\},$$

$$\nabla v_n = \nabla (y - \Psi)^+ \qquad \text{a.e. in } \left\{ \frac{1}{n} v \geq (y - \Psi)^+ \right\}.$$

Therefore, from (8.74) we have

$$\frac{1}{n}\int_{\{v<n(y-\Psi)^+\}}(\nabla(y-\Psi),\nabla v)\,\mathrm{d}x+\int_{\{v\geq n(y-\Psi)^+\}}|\nabla(y-\Psi)^+|^2\,\mathrm{d}x \leq \frac{1}{n}\int_\Omega (f+\triangle\Psi)^+ v\,\mathrm{d}x. \tag{8.75}$$

As the second term in (8.75) is non-negative, one gets

$$\int_{\{\frac{1}{n}v<(y-\Psi)^+\}}(\nabla(y-\Psi),\nabla v)\,\mathrm{d}x \leq \int_\Omega (f+\triangle\Psi)^+ v\,\mathrm{d}x.$$

Then taking the limits as $n\to\infty$, we have

$$\int_{\{(y-\Psi)^+>0\}}(\nabla(y-\Psi),\nabla v)\,\mathrm{d}x \leq \int_\Omega (f+\triangle\Psi)^+ v\,\mathrm{d}x.$$

Since $\nabla(y-\Psi)=\nabla(y-\Psi)^+$ a.e. in $\{(y-\Psi)^+>0\}$ and $\nabla(y-\Psi)=0$ a.e. in $\{(y-\Psi)^+=0\}$, it follows that

$$\int_\Omega(\nabla(y-\Psi)^+,\nabla v)\,\mathrm{d}x \leq \int_\Omega (f+\triangle\Psi)^+ v\,\mathrm{d}x,\ \forall v\in H_0^1(\Omega) \text{ with } v\geq 0 \text{ q.e. in } \Omega.$$

Hence, this implies that the element λ_1, which was defined in (8.73), is non-negative, that is, λ_1 is a Radon measure.

By analogy, we deduce that if $\lambda_2\in H^{-1}(\Omega)$ is defined as

$$-\triangle\left[-(y-\Psi)^-\right]-\lambda_2=-(f+\triangle\Psi)^-, \tag{8.76}$$

then λ_2 is a non-negative Radon measure as well.

Since $-\triangle y+\triangle\Psi=-\triangle(y-\Psi)^+-\triangle[-(y-\Psi)^-]$, it follows that the element $\lambda=\lambda_1-\lambda_2$ of $H^{-1}(\Omega)$ satisfies the equality

$$-\triangle(y-\Psi)+\lambda=f+\triangle\Psi, \tag{8.77}$$

and it is a Radon measure with $|\lambda|\leq\lambda_1+\lambda_2$. Thus, we have obtained the following result.

Lemma 8.27. *Let $\mu\in\mathcal{M}_0^2(\Omega)$, $\Psi\in H_0^1(\Omega)\cap H^2(\Omega)$, and $f\in L^2(\Omega)$ and let $y\in H_0^1(\Omega)$ be the corresponding solution of problem (8.71). Let λ_1, λ_2, and λ be the elements of $H^{-1}(\Omega)$ defined by (8.73), (8.76), and (8.77), respectively. Then λ_1, λ_2, and λ are Radon measures such that $\lambda=\lambda_1-\lambda_2$, $|\lambda|\leq\lambda_1+\lambda_2$, $\lambda_1\geq 0$, and $\lambda_2\geq 0$ and for every compact set $K\subseteq\Omega$, the following inequality holds:*

$$|\lambda|(K)\leq 2\sqrt{\mathrm{cap}\,(K,\Omega)}\big(\,\|\,\nabla y-\nabla\Psi\,\|_{L^2(\Omega)} + \|\,f\,\|_{L^2(\Omega)}+\|\,\Delta\Psi\,\|_{L^2(\Omega)}\,\big), \tag{8.78}$$

where

$$\mathrm{cap}(K,\Omega)=\inf\left\{\int_\Omega|\nabla z|^2\,\mathrm{d}x,\ z\geq 1 \text{ a.e. in the neighborhood of } K\right\}.$$

In order to prove (8.78), we note that from (8.73) we have the following upper bound:

$$\| \lambda_1 \|_{H^{-1}(\Omega)} \leq \| \nabla y - \nabla \Psi \|_{L^2(\Omega)} + \| f \|_{L^2(\Omega)} + \| \Delta \Psi \|_{L^2(\Omega)} .$$

By analogy with (8.76), we have

$$\| \lambda_2 \|_{H^{-1}(\Omega)} \leq \| \nabla y - \nabla \Psi \|_{L^2(\Omega)} + \| f \|_{L^2(\Omega)} + \| \Delta \Psi \|_{L^2(\Omega)} .$$

For every $\delta > 0$, we fix a function $z \in H_0^1(\Omega)$ such that

$$\| z \|^2_{H_0^1(\Omega)} \leq \operatorname{cap}(K, \Omega) + \delta$$

and $z \geq 0$ q.e. in Ω and $z \geq 1$ q.e. in a neighborhood of K, where $K \subseteq \Omega$ is any compact set. Then

$$\begin{aligned} |\lambda|(K) &\leq \int_\Omega z \, \mathrm{d}\lambda_1 + \int_\Omega z \, \mathrm{d}\lambda_2 \\ &\leq \| z \|_{H_0^1(\Omega)} \left(\| \lambda_1 \|_{H^{-1}(\Omega)} + \| \lambda_2 \|_{H^{-1}(\Omega)} \right) \\ &\leq 2(\operatorname{cap}(K, \Omega) + \delta)^{1/2} (\| \nabla y - \nabla \Psi \|_{L^2(\Omega)} \\ &\quad + \| f \|_{L^2(\Omega)} + \| \Delta \Psi \|_{L^2(\Omega)}). \end{aligned}$$

Passing to the limit as $\delta \to 0$, we obtain (8.78).

Let us consider the following modification of the state constrained OCP (8.67)–(8.70):

$$-\triangle y + \mu_\varepsilon (y - \Psi_\varepsilon) = f_\varepsilon + u \quad \text{in } V^*_{\mu_\varepsilon}, \tag{8.79}$$

$$y - \Psi_\varepsilon \in V_{\mu_\varepsilon} \quad u \in L^2(\Omega), \tag{8.80}$$

$$\begin{aligned} I_\varepsilon(u, y) = &\int_{\Omega \setminus S_\varepsilon} |y - z_d|^2 \, \mathrm{d}x + \int_\Omega u^2 \, \mathrm{d}x + \int_\Omega |\nabla (y - \Psi_\varepsilon)|^2 \, \mathrm{d}x \\ &+ \int_\Omega (y - \Psi_\varepsilon)^2 \, \mathrm{d}\mu_\varepsilon \longrightarrow \inf, \end{aligned} \tag{8.81}$$

where $f_\varepsilon \in L^2(\Omega)$, $\Psi_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)$, $z_d \in L^2(\Omega)$, a Borel measure μ_ε is defined in such a form that $\mu_\varepsilon \in \mathcal{M}_0^2(\Omega)$, and for any $u \in L^2(\Omega)$, the corresponding solution of the problem (8.79)–(8.80) satisfies condition (8.69) (see [82]). For instance, this can be achieved by the rule

$$\mu_\varepsilon(B) = \begin{cases} 0 & \text{if } \operatorname{cap}(B \cap S_\varepsilon, \Omega) = 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{8.82}$$

As we will see later, this problem always has a solution and good stability properties. In particular, in view of the results mentioned above and the well-known Lax–Milgram lemma, we may conclude the following:

Proposition 8.28. *Assume that for a fixed $\varepsilon \in E$, $f_\varepsilon \in L^2(\Omega)$, $\Psi_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)$, the Borel measure $\mu_\varepsilon \in \mathcal{M}_0^2(\Omega)$ is defined in (8.82). Then for every $u \in L^2(\Omega)$, there exists a unique solution of the problem*

$$y - \Psi_\varepsilon \in V_{\mu_\varepsilon},$$

$$\int_\Omega (\nabla y, \nabla \varphi)\,\mathrm{d}x + \int_\Omega (y - \Psi_\varepsilon)\varphi\,\mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + u)\,\varphi\,\mathrm{d}x, \quad \forall \varphi \in V_{\mu_\varepsilon},$$

such that $y_\varepsilon = \Psi_\varepsilon$ quasi-everywhere in S_ε and $\lambda = \triangle y_\varepsilon + f_\varepsilon$ is a Radon measure satisfying (8.78) for every compact set $K \subseteq \Omega$.

8.5.3 Passing to the limit in the modified optimal control problem

The aim of this subsection is to study the limiting behavior of the modified problem (8.79)–(8.81) as $\varepsilon \to 0$. To do so, we represent the OCP (8.79)–(8.81) in the following form:

$$\left\{ \left\langle \inf_{(u,y)\in \Xi_\varepsilon} I_\varepsilon(u,y) \right\rangle, \qquad \varepsilon \in E \right\}, \tag{8.83}$$

where the cost functionals $I_\varepsilon : \Xi_\varepsilon \to \mathbb{R}$ and their domains Ξ_ε are

$$\Xi_\varepsilon = \left\{ (u,y) \,\middle|\, \begin{array}{ll} -\triangle y + \mu_\varepsilon(y - \Psi_\varepsilon) = f_\varepsilon + u & \text{in } V_{\mu_\varepsilon}^* \\ y - \Psi_\varepsilon \in V_{\mu_\varepsilon}, \quad u \in L^2(\Omega) & \end{array} \right\}, \tag{8.84}$$

$$\begin{aligned} I_\varepsilon(u,y) = &\int_{\Omega\setminus S_\varepsilon} (y - z_d)^2\,\mathrm{d}x + \int_\Omega u^2\,\mathrm{d}x \\ &+ \int_\Omega (y - \Psi_\varepsilon)\,\mathrm{d}\mu_\varepsilon + \int_\Omega |\nabla(y - \Psi_\varepsilon)|^2\,\mathrm{d}x. \end{aligned} \tag{8.85}$$

First, we emphasize the fact that for every $\varepsilon \in E$, each of the sets of admissible pairs Ξ_ε is nonempty, convex and closed in the following sense: We say that a sequence$\{(u_n, y_n) \in \Xi_\varepsilon\}_{n\in\mathbb{N}}$ τ_ε-converges to some pair (u,y) if

$$u_n \to u \qquad \text{weakly in } L^2(\Omega),$$

$$\begin{aligned} \int_\Omega (\nabla y_n, \nabla \varphi)\,\mathrm{d}x + \int_\Omega (y_n - \Psi_\varepsilon)\varphi\,\mathrm{d}\mu_\varepsilon \to &\int_\Omega (\nabla y, \nabla \varphi)\,\mathrm{d}x \\ &+ \int_\Omega (y - \Psi_\varepsilon)\varphi\,\mathrm{d}\mu_\varepsilon, \quad \forall\, \varphi \in V_{\mu_\varepsilon}. \end{aligned}$$

It should be remarked that for different values $\varepsilon \in E$, the sets Ξ_ε belong to different functional spaces. We proceed to define the concept of convergence in "varying functional spaces." For this, we use the notion of γ^Δ-convergence in $\mathcal{M}_0^2(\Omega)$ that was introduced in [81] (see also Sect. 7.7).

Definition 8.29. *Let $\{\mu_\varepsilon\}_{\varepsilon\in E}$ be a sequence of Borel measures of $\mathcal{M}_0^2(\Omega)$ and let $\mu \in \mathcal{M}_0^2(\Omega)$. We say that $\{\mu_\varepsilon\}_{\varepsilon\in E}$ $\gamma^\triangle$-converges to μ in Ω if for every $f \in H^{-1}(\Omega)$, the solution y_ε of the problem*

$$y_\varepsilon \in V_{\mu_\varepsilon}, \quad \big\langle -\triangle y_\varepsilon, \varphi\big\rangle + \int_\Omega y_\varepsilon \varphi \,\mathrm{d}\mu_\varepsilon = \big\langle f, \varphi\big\rangle, \qquad \forall\, \varphi \in V_{\mu_\varepsilon}, \tag{8.86}$$

converges weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0$ to the solution $y \in V_\mu$ of the problem

$$\big\langle -\triangle y, \varphi\big\rangle + \int_\Omega y\varphi \,\mathrm{d}\mu = \big\langle f, \varphi\big\rangle, \qquad \forall\, \varphi \in V_\mu. \tag{8.87}$$

We quote the following compactness result by Dal Maso and Murat [82].

Theorem 8.30. *Every sequence of measures of $\mathcal{M}_0^2(\Omega)$ contains a $\gamma^\triangle$-convergent subsequence, the $\gamma^\triangle$-limit of which is unique.*

Remark 8.31. Note also that since the solutions of (8.86) depend continuously on f uniformly with respect to the measure μ_ε (see [82]), it follows that if $\mu \in \mathcal{M}_0^2(\Omega)$ is the $\gamma^\triangle$-limit of $\{\mu_\varepsilon\}_{\varepsilon\in E}$ and $\{f_\varepsilon \in H^{-1}(\Omega)\}_{\varepsilon\in E}$ is strongly convergent to an $f \in H^{-1}(\Omega)$, then the functions $y_\varepsilon \in V_{\mu_\varepsilon}$ defined as the solutions of the problem

$$\big\langle -\triangle y_\varepsilon, \varphi\big\rangle + \int_\Omega y_\varepsilon \varphi \,\mathrm{d}\mu_\varepsilon = \big\langle f_\varepsilon, \varphi\big\rangle \qquad \text{for every } \varphi \in V_{\mu_\varepsilon}$$

converge weakly in $H_0^1(\Omega)$ to the solution $y \in H_0^1(\Omega)$ of the limit problem (8.87).

Due to this remark, we immediately have the following result.

Lemma 8.32. *Let $\{\mu_\varepsilon \in \mathcal{M}_0^2(\Omega)\}_{\varepsilon\in E}$, $\{f_\varepsilon \in L^2(\Omega)\}_{\varepsilon\in E}$, $\{u_\varepsilon \in L^2(\Omega)\}_{\varepsilon\in E}$, and $\{\Psi_\varepsilon \in H_0^1(\Omega)\cap H^2(\Omega)\}_{\varepsilon\in E}$ be sequences such that*

$$\left.\begin{array}{ll} \mu_\varepsilon \to \mu_0 & \text{in the sense of } \gamma^\triangle\text{-convergence,}\\ f_\varepsilon \to f_0 & \text{weakly in } L^2(\Omega),\\ u_\varepsilon \to u_0 & \text{weakly in } L^2(\Omega),\\ \Psi_\varepsilon \to \Psi_0 & \text{weakly in } H_0^1(\Omega),\\ \triangle\Psi_\varepsilon \to \triangle\Psi_0 & \text{strongly in } H^{-1}(\Omega). \end{array}\right\} \tag{8.88}$$

Then the solution y_ε of the problem

$$\left.\begin{array}{c} y_\varepsilon - \Psi_\varepsilon \in V_{\mu_\varepsilon},\\ \big\langle -\triangle y_\varepsilon, \varphi\big\rangle + \displaystyle\int_\Omega (y_\varepsilon - \Psi_\varepsilon)\varphi \,\mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + u_\varepsilon)\varphi, \mathrm{d}x, \quad \forall\, \varphi \in V_{\mu_\varepsilon}, \end{array}\right\} \tag{8.89}$$

converges weakly in $H_0^1(\Omega)$ to the solution y_0 of the problem

$$\left.\begin{array}{c} y_0 - \Psi_0 \in V_{\mu_0},\\ \big\langle -\triangle y_0, \varphi\big\rangle + \displaystyle\int_\Omega (y_0 - \Psi_0)\varphi \,\mathrm{d}\mu_0 = \int_\Omega (f_0 + u_0)\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in V_{\mu_0} \end{array}\right\}. \tag{8.90}$$

In view of this lemma, we give the definition of the limit object for the sequence of constrained minimization problems (8.83) (see Definition 8.24 for comparison).

Definition 8.33. *We say that a constrained minimization problem*

$$\left\langle \inf_{(u,y)\in\Xi_0} I_0(u,y) \right\rangle \tag{8.91}$$

is the variational limit of the sequence (8.83) with respect to the $\gamma^{\triangle}$-convergence if the following conditions are satisfied:

(V1)

$$\Xi_0 = \left\{ (u,y) \;\middle|\; \begin{array}{c} u \in L^2(\Omega),\ y - \Psi_0 \in V_{\mu_0}, \\ \langle -\triangle y, \varphi\rangle + \displaystyle\int_\Omega (y-\Psi_0)\varphi \,\mathrm{d}\mu_0 = \int_\Omega (f_0+u)\varphi \,\mathrm{d}x, \\ \forall\ \varphi \in V_{\mu_0} \end{array} \right\}, \tag{8.92}$$

where the elements f_0, Ψ_0, and μ_0 are defined by (8.88).

(V2) For every pair $(u,y) \in \Xi_0$, every index set $H \in \mathcal{H}^{\#}$, and every sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon\in H}$ such that $(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon$, $\forall \varepsilon \in H$, $u_\varepsilon \to u$ weakly in $L^2(\Omega)$ and $y_\varepsilon \to y$ weakly in $H^1_0(\Omega)$, we have

$$I_0(u,y) \le \liminf_{H\ni\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon).$$

(V3) For every pair $(u,y) \in \Xi_0$, one can find an index set $H \in \mathcal{H}$ and a sequence $\{(\bar u_\varepsilon, \bar y_\varepsilon)\}_{\varepsilon\in H}$ such that $(\bar u_\varepsilon, \bar y_\varepsilon) \in \Xi_\varepsilon$ for all $\varepsilon \in H$, $\bar u_\varepsilon \to u$ weakly in $L^2(\Omega)$, $\bar y_\varepsilon \to y$ weakly in $H^1_0(\Omega)$, and

$$I_0(u,y) \ge \limsup_{H\ni\varepsilon\to 0} I_\varepsilon(\bar u_\varepsilon, \bar y_\varepsilon).$$

Remark 8.34. As follows from condition (V1), the set Ξ_0 is the sequential version of the limit for $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ in Kuratowski's sense. Indeed, since $\mu_0 \in \mathcal{M}^2_0(\Omega)$ is a $\gamma^{\triangle}$-limit of $\{\mu_\varepsilon\}_{\varepsilon\in E}$, it follows that for any weakly in $L^2(\Omega)$ convergent subsequence of controls $\{u_\varepsilon\}_{\varepsilon\in H}$ to some u^0, $H \in \mathcal{H}^{\#}$, the corresponding sequence of solutions $\{y_\varepsilon = y(u_\varepsilon, \mu_\varepsilon)\}_{\varepsilon\in H}$, in accordance with Lemma 8.32, converges weakly in $H^1_0(\Omega)$ to the solution $y_0 = y(u_0, \mu_0)$ of the problem (8.90). Hence, the limit pair (u_0, y_0) of $\{(u_\varepsilon, y_\varepsilon)\}$ belongs to Ξ_0. Conversely, for any pair $(u_0, y_0) \in \Xi_0$, the sequence $\{u_0, y_\varepsilon = y(u_0, \mu_\varepsilon)\}_{\varepsilon\in H}$ is such that

$$(u_0, y_\varepsilon) \in \Xi_\varepsilon \ \text{ for every } \varepsilon \in H, \quad y_\varepsilon \to y_0 \ \text{ weakly in } H^1_0(\Omega) \text{ (Lemma 8.32).}$$

Thus, Ξ_0 satisfies all sequential properties of the Kuratowski's limit (see [10]).

Now, we are in a position to prove the main result of this subsection.

Theorem 8.35. *Let* $\{f_\varepsilon \in L^2(\Omega)\}_{\varepsilon\in E}$, $\{\mu_\varepsilon \in \mathcal{M}_0^2(\Omega)\}$, *and* $\{\Psi_\varepsilon \in H_0^1(\Omega)\cap H^2(\Omega)\}_{\varepsilon\in E}$ *be arbitrary sequences such that* $f_\varepsilon \to f_0$ *weakly in* $L^2(\Omega)$, $\Psi_\varepsilon \to \Psi_0$ *weakly in* $H_0^1(\Omega)$, $\triangle\Psi_\varepsilon \to \triangle\Psi_0$ *strongly in* $H^{-1}(\Omega)$, *and* $\mu_\varepsilon(B) = +\infty$ *for every* $B \subseteq \Omega$ *such that* $\operatorname{cap}(B\cap S_\varepsilon, \Omega) > 0$. *Then there exist a non-negative measure* $\mu_0 \in \mathcal{M}_0^2(\Omega)$ *and a subsequence of (8.83) for which the variational limit exists, and it can be recovered in the form of the following OCP*

$$-\triangle y + (y-\Psi_0)\mu_0 = f_0 + u \quad \text{in } V_{\mu_0}^*, \tag{8.93}$$

$$y - \Psi_0 \in V_{\mu_0}, \qquad u \in L^2(\Omega), \tag{8.94}$$

$$I_0(u,y) = \int_\Omega \chi_*(y-z_d)^2\,\mathrm{d}x + \int_\Omega u^2\,\mathrm{d}x + \int_\Omega |\nabla(y-\Psi_0)|^2\,\mathrm{d}x + \int_\Omega |y-\Psi_0|^2\,\mathrm{d}\mu_0 \to \inf. \tag{8.95}$$

Proof. Let $\{\chi_\varepsilon\}_{\varepsilon\in E}$ be the sequence of the characteristic functions of the sets $\Omega\setminus S_\varepsilon$. We denote by χ_* the weak-$*$ limit point in $L^\infty(\Omega)$ of this sequence. For the sake of simplicity, we always assume that $\mu_\varepsilon \to \mu_0$ in the sense of $\gamma^\triangle$-convergence and $\chi_\varepsilon \to \chi_*$ weakly-$*$ in $L^\infty(\Omega)$. For every $\{y_\varepsilon\}_{\varepsilon\in E}$ such that $y_\varepsilon \to y_0$ weakly in $H_0^1(\Omega)$, we have $y_\varepsilon^2 \to y_0^2$ strongly in $L^1(\Omega)$. Therefore,

$$\int_{\Omega\setminus S_\varepsilon} (y_\varepsilon - z_d)^2\,\mathrm{d}x = \int_\Omega \chi_\varepsilon (y_\varepsilon - z_d)^2\,\mathrm{d}x \to \int_\Omega \chi_*(y_0 - z_d)^2\,\mathrm{d}x \quad \text{as } \varepsilon\to 0. \tag{8.96}$$

To prove representation (8.95), we have to verify conditions (V2) and (V3). Let $\{(u_\varepsilon, y_\varepsilon)\in \Xi_\varepsilon\}_{\varepsilon\in H}$ be any sequence satisfying condition (V2), that is,

$$(u_\varepsilon, y_\varepsilon) \to (u,y) \in \Xi_0 \qquad \text{weakly in } L^2(\Omega)\times H_0^1(\Omega).$$

Then by (8.96) and the lower semi-continuity property of the norm in $L^2(\Omega)$ with respect to the weak convergence, we have

$$\begin{aligned}
\liminf_{H\ni\varepsilon\to 0} \Big[&\int_\Omega \chi_\varepsilon (y_\varepsilon - z_d)^2\,\mathrm{d}x + \int_\Omega u_\varepsilon^2\,\mathrm{d}x \\
&+ \int_\Omega (y_\varepsilon - \Psi_\varepsilon)^2\,\mathrm{d}\mu_\varepsilon + \int_\Omega |\nabla(y_\varepsilon - \Psi_\varepsilon)|^2\,\mathrm{d}x\Big] \\
&\ge \int_\Omega \chi_*(y-z_d)^2\,\mathrm{d}x + \int_\Omega u^2\,\mathrm{d}x \\
&+ \liminf_{H\ni\varepsilon\to 0} \left[\int_\Omega (y_\varepsilon - \Psi_\varepsilon)^2\,\mathrm{d}\mu_\varepsilon + \int_\Omega |\nabla(y_\varepsilon - \Psi_\varepsilon)|^2\,\mathrm{d}x\right].
\end{aligned} \tag{8.97}$$

Since for every $\varepsilon \in H$, each of the pairs $(u_\varepsilon, y_\varepsilon)$ satisfies the equality

$$\Big\langle -\triangle y_\varepsilon, \varphi \Big\rangle + \int_\Omega (y_\varepsilon - \Psi_\varepsilon)\varphi \, \mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + u_\varepsilon)\varphi \, \mathrm{d}x, \qquad \forall\, \varphi \in V_{\mu_\varepsilon}, \quad (8.98)$$

one obtains, upon putting $\varphi = y_\varepsilon - \Psi_\varepsilon$ into (8.98) and integrating by parts,

$$\int_\Omega |\nabla(y_\varepsilon - \Psi_\varepsilon)|^2 \, \mathrm{d}x + \int_\Omega (y_\varepsilon - \Psi_\varepsilon)^2 \, \mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + u_\varepsilon)(y_\varepsilon - \Psi_\varepsilon) \, \mathrm{d}x + \Big\langle \triangle\Psi_\varepsilon, y_\varepsilon - \Psi_\varepsilon \Big\rangle . \quad (8.99)$$

As a result, taking into account that

$$f_\varepsilon \to f_0 \quad \text{weakly in } L^2(\Omega), \qquad u_\varepsilon \to u \quad \text{weakly in } L^2(\Omega),$$
$$\Psi_\varepsilon \to \Psi_0 \quad \text{weakly in } H_0^1(\Omega), \quad \triangle\Psi_\varepsilon \to \triangle\Psi_0 \quad \text{strongly in } H^{-1}(\Omega),$$

we obtain

$$\begin{cases} \displaystyle\int_\Omega (f_\varepsilon + u_\varepsilon)(y_\varepsilon - \Psi_\varepsilon) \, \mathrm{d}x \to \int_\Omega (f_0 + u)(y - \Psi_0) \, \mathrm{d}x, \\ \Big\langle \triangle\Psi_\varepsilon, y_\varepsilon - \Psi_\varepsilon \Big\rangle \to \Big\langle \triangle\Psi_0, y - \Psi_0 \Big\rangle. \end{cases} \quad (8.100)$$

Now, we use the fact that $(u, y) \in \Xi_0$. We take as a test function $\varphi = y - \Psi_0$ in (8.92) and integrate by parts. Then

$$\int_\Omega |\nabla(y - \Psi_0)|^2 \, \mathrm{d}x + \int_\Omega (y - \Psi_0) \, \mathrm{d}\mu_0 = \int_\Omega (f_0 + u)(y - \Psi_0) \, \mathrm{d}x + \Big\langle \triangle\Psi_0, y - \Psi_0 \Big\rangle. \quad (8.101)$$

Using (8.100) and (8.101), from (8.99) we conclude

$$\begin{aligned} \lim_{\varepsilon\to 0} \Big[\int_\Omega |\nabla(y_\varepsilon - \Psi_\varepsilon)|^2 \, \mathrm{d}x + \int_\Omega (y_\varepsilon - \Psi_\varepsilon)^2 \, \mathrm{d}\mu_\varepsilon \Big] &= \lim_{\varepsilon\to 0} \Big[\int_\Omega (f_\varepsilon + u_\varepsilon)(y_\varepsilon - \Psi_\varepsilon) \, \mathrm{d}x + \big\langle \triangle\Psi_\varepsilon, y_\varepsilon - \Psi_\varepsilon \big\rangle \Big] \\ &= \int_\Omega (f_0 + u)(y - \Psi_0) \, \mathrm{d}x + \big\langle \triangle\Psi_0, y - \Psi_0 \big\rangle \\ &= \int_\Omega |\nabla(y - \Psi_0)|^2 \, \mathrm{d}x + \int_\Omega (y - \Psi_0)^2 \, \mathrm{d}\mu_0. \end{aligned} \quad (8.102)$$

Inserting this equality into (8.97), we immediately obtain

$$\liminf_{H \ni \varepsilon \to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon) \geq I_0(u, y),$$

that is, condition (V2) holds true.

In order to prove condition (V3), we take any pair (u, y) of Ξ_0 and construct the corresponding sequence $\{(\bar{u}_\varepsilon, \bar{y}_\varepsilon)\}_{\varepsilon\in E}$ by the rule

$$\bar{u}_\varepsilon = u \qquad \text{for every } \varepsilon \in E,$$

and $\bar{y}_\varepsilon = y(u, \mu_\varepsilon)$ are the solutions of problems (8.89) with $u_\varepsilon = \bar{u}_\varepsilon = u$.

It is easy to see that $(\bar{u}_\varepsilon, \bar{y}_\varepsilon) \in \Xi_\varepsilon$ for every $\varepsilon \in E$. Moreover, due to Theorem 6.8 in [82] and the fact that $\mu_\varepsilon \to \mu_0$ in the sense of $\gamma^{\triangle}$-convergence, we have

$$\bar{y}_\varepsilon \to y \qquad \text{weakly in } H_0^1(\Omega).$$

Using (8.96) and (8.102), we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0} I_\varepsilon(\bar{u}_\varepsilon, \bar{y}_\varepsilon) &= \limsup_{\varepsilon\to 0} \Big[\int_\Omega \chi_\varepsilon(\bar{y}_\varepsilon - z_d)^2 \,\mathrm{d}x \\ &\quad + \int_\Omega u^2 \,dx + \int_\Omega |\nabla(\bar{y}_\varepsilon - \Psi_\varepsilon)|^2 \,\mathrm{d}x + \int_\Omega (\bar{y}_\varepsilon - \Psi_\varepsilon)^2 \,\mathrm{d}\mu_\varepsilon \Big] \\ &= I_0(u, y),\end{aligned}$$

that is, condition (V3) holds true as well. This concludes the proof. □

In view of this theorem, we will assume that the sequence of OCPs (8.83) is convergent in the sense of Definition 8.33.

Remark 8.36. Conditions (V2) and (V3) coincide with the definition of the Γ-limit for the following sequence of functionals which are uniformly coercive in $L^2(\Omega) \times H_0^1(\Omega)$ (see [10, 45])

$$\Big\{ P_{\Xi_\varepsilon} I_\varepsilon : L^2(\Omega) \times H_0^1(\Omega) \to \bar{\mathbb{R}} \Big\}_{\varepsilon\in E},$$

where the extension operators P_{Ξ_ε} are defined as

$$P_{\Xi_\varepsilon} I(u, y) = \begin{cases} I(u, y) & \text{if } (u, y) \in \Xi_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore, the variational limit (8.91) inherits all the variational properties of Γ-limits, namely if $(u^*, y^*) \in \Xi_0$ is a weak limit in $L^2(\Omega) \times H_0^1(\Omega)$ of the sequence of optimal pairs for the original problem (8.83), then we have the following:

(A1) (u^*, y^*) is an optimal solution for the limit problem (8.91).
(A2) $I_0(u^*, y^*) = \lim_{\varepsilon\to 0} \inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u, y)$.

Indeed, the following result holds.

Corollary 8.37. *The unique optimal pair* (u^0, y^0) *for the limit OCP (8.93)–(8.95) is the weak limit in* $L^2(\Omega)\times H_0^1(\Omega)$ *of the sequence of optimal solutions* $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon\in E}$ *for the original problem (8.83) and, moreover,*

$$\lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) = I_0(u^0, y^0). \tag{8.103}$$

Proof. Using the direct method of Calculus of Variations, it is easy to show that each of the OCPs (8.83) and (8.93)–(8.95) admits a unique solution $(u_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ and $(u^0, y^0) \in \Xi_0$, respectively. Moreover, taking into account the coerciveness property of the cost functional (8.81), the sequence of optimal pairs $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon\in E}$ is uniformly bounded in $L^2(\Omega)\times H_0^1(\Omega)$ with respect to the small parameter ε. So, we may extract from $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon\in E}$ a weakly convergent subsequence $\{(v_\sigma^0, p_\sigma^0)\}_{\sigma\in H}$ ($H \in \mathcal{H}^\#$). Let $(u^*, y^*) \in L^2(\Omega) \times H_0^1(\Omega)$ be its limit. Then, by Lemma 8.32, we have $(u^*, y^*) \in \Xi_0$. Moreover, due to part (V2) of Definition 8.33,

$$\begin{aligned}\liminf_{H\ni\sigma\to 0} \min_{(u,y)\in \Xi_\sigma} I_\sigma(u, y) &= \liminf_{H\ni\sigma\to 0} I_\sigma(v_\sigma^0, p_\sigma^0)\\ &\ge I_0(u^*, y^*) \ge \min_{(u,\,y)\in \Xi_0} I_0(u, y) = I_0(u^0, y^0).\end{aligned} \tag{8.104}$$

At the same time, as follows from part (V3) of Definition 8.33, there exist an index set $H \in \mathcal{H}$ and a sequence $\{(u_\varepsilon, y_\varepsilon)\}$ such that $(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon$ for all values $\varepsilon \in H$,

$$(u_\varepsilon, y_\varepsilon)\longrightarrow(u^0, y^0) \quad \text{as} \quad H \ni \varepsilon \to 0,$$

and

$$I_0(u^0, y^0) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon).$$

Using this fact, we get

$$\begin{aligned}\min_{(u,\,y)\in\Xi_0} I_0(u,\, y) = I_0(u^0, y^0) &\ge \limsup_{H\ni\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon)\\ &\ge \limsup_{\varepsilon\to 0} \min_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u, y) \ge \limsup_{H\ni\sigma\to 0} \min_{(u,y)\in\Xi_\sigma} I_\sigma(u, y)\\ &= \limsup_{H\ni\sigma\to 0} I_\sigma(v_\sigma^0, p_\sigma^0).\end{aligned} \tag{8.105}$$

From (8.104), it follows that

$$\liminf_{H\ni\sigma\to 0} I_\sigma(v_\sigma^0, p_\sigma^0) \ge \limsup_{H\ni\sigma\to 0} I_\sigma(v_\sigma^0, p_\sigma^0).$$

Combining (8.104) and (8.105), we conclude

$$\begin{aligned} I_0(u^*, y^*) = I_0(u^0, y^0) &= \min_{(u,\,y)\in\Xi_0} I_0(u, y),\\ I_0(u^0, y^0) &= \lim_{H\ni\sigma\to 0} \min_{(u,y)\in\Xi_\sigma} I_\sigma(u, y).\end{aligned}$$

Taking into account these relations and the uniqueness of the solution of problem (8.93), we obtain $(u^*, y^*) = (u^0, y^0)$. Since this equality holds for the limits of any converging subsequences of $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon\in E}$, (u^0, y^0) is the weak limit of the sequence $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon\in E}$. Accomplishing for the sequence of minimizers what we did earlier with the subsequence $\{(v_\sigma^0, p_\sigma^0)\}_{H\ni\sigma\to 0}$, we obtain

$$\begin{aligned}\liminf_{E\ni\varepsilon\to 0}\min_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y) &= \liminf_{E\ni\varepsilon\to 0} I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) \geq I_0(u^0, y^0)\\ &= \min_{(u,\,y)\in\Xi_0} I_0(u,y) \geq \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon)\\ &\geq \limsup_{E\ni\varepsilon\to 0}\min_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y)\\ &= \limsup_{E\ni\varepsilon\to 0} I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0).\end{aligned}$$

Thus, relations (8.103) hold. □

8.6 On homogenization of Dirichlet optimal control problems in perforated domains

In this section, we will study the homogenization of an optimal control problem involving a linear elliptic equation with Dirichlet boundary conditions in a perforated domain. The problem is to describe the asymptotic behavior of the sequence of such OCPs as a small parameter tends to 0. We will show that the study of this problem can be traced by the above approach.

Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $\varepsilon \in E = (0; \varepsilon^0]$ be a small parameter and let $\{\Omega_\varepsilon\}_{\varepsilon\in E}$ be a sequence of open sets contained in Ω. For given $z_d \in L^2(\Omega)$ and $f_\varepsilon \in L^2(\Omega)$, we define the OCP in Ω_ε as follows: Find a pair "control state" $(u_\varepsilon^0, y_\varepsilon^0) \in L^2(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon)$ such that

$$I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) = \inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y), \tag{8.106}$$

where

$$I_\varepsilon(u,y) = \int_{\Omega_\varepsilon} |\nabla y|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon} (y - z_d)^2\,\mathrm{d}x + \int_{\Omega_\varepsilon} u^2\,\mathrm{d}x, \tag{8.107}$$

$$\Xi_\varepsilon = \left\{ (u_\varepsilon, y_\varepsilon) \in L^2(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon) \,\middle|\, \begin{array}{l} -\triangle y_\varepsilon = f_\varepsilon + u_\varepsilon \text{ in } \mathcal{D}'(\Omega_\varepsilon) \\ \tilde{u}_\varepsilon \in \mathcal{U}_\varepsilon \end{array} \right\}. \tag{8.108}$$

Here, $\tilde{u}_\varepsilon$ is the trivial extension of $u_\varepsilon \in L^2(\Omega_\varepsilon)$ by 0 to the hole of Ω; $\mathcal{U}_\varepsilon$ is a weakly closed convex subset of $L^2(\Omega)$ such that

$$\mathcal{U}_\varepsilon \longrightarrow \mathcal{U}_0,\ (\mathcal{U}_0 \neq \emptyset) \text{ in the Kuratowski's sense}$$

with respect to the weak topology of $L^2(\Omega)$; χ_ε is the characteristic function of the set Ω_ε, that is, $\chi_\varepsilon = 1$ in Ω_ε and $\chi_\varepsilon = 0$ otherwise.

It is well known for every $\varepsilon \in E$, there exists a unique solution of the problem (8.106)–(8.108) (see [169]). In view of the uniform coerciveness of the cost functionals $I_\varepsilon : \Xi_\varepsilon \to \mathbb{R}$, the sequence of optimal pairs

$$\{(u_\varepsilon^0, y_\varepsilon^0) \in L^2(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon)\}_{\varepsilon \in E}$$

is uniformly bounded, that is,

$$\sup_{\varepsilon \in E} \left[\parallel u_\varepsilon^0 \parallel_{L^2(\Omega_\varepsilon)} + \parallel y_\varepsilon^0 \parallel_{H_0^1(\Omega_\varepsilon)} \right] < \infty.$$

Therefore, it is natural to choose as a basic topology for the homogenization procedure the product of the weak topologies for $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. Since y_ε^0 is equal to 0 on the boundary $\partial\Omega_\varepsilon$, it is reasonable to extend y_ε^0 by 0 to the hole of Ω. We denote this extension by $\tilde{y}_\varepsilon^0$. It belongs to $H_0^1(\Omega)$ and, moreover, by the Poincaré inequality, it is easy to show that

$$\parallel \tilde{y}_\varepsilon^0 \parallel_{H_0^1(\Omega)} \leq C \tag{8.109}$$

for all $f_\varepsilon \in L^2(\Omega)$, $z_d \in L^2(\Omega)$, and $u_\varepsilon^0 \in L^2(\Omega_\varepsilon)$.

Note the contrast with the case of Neumann boundary conditions on $\partial\Omega_\varepsilon$ (see [135]). The extension by 0 of the solution of the Dirichlet problem (8.108) now belongs to the space $H_0^1(\Omega)$, whereas the same extension for the solution of the Neumann problem is only an element of $L^2(\Omega)$. That is why in that case, one needs to construct extension operators.

Further, we note that due to estimate (8.109), one may assume that

$$\tilde{y}_\varepsilon^0 \rightharpoonup y \text{ weakly in } H_0^1(\Omega).$$

Let us denote by χ_* the weak-$*$ limit in $L^\infty(\Omega)$ of the sequence $\{\chi_\varepsilon \in L^\infty(\Omega)\}_{\varepsilon \in E}$.

Remark 8.38. We will always assume that $\text{supp}(\chi_*) = \text{closure}(\Omega)$, that is, $\chi_* > 0$ almost everywhere on Ω. It is clear that in this case, $\Omega \setminus \Omega_\varepsilon$ has to be small enough. Suppose also that $\chi_*^{-1} \in L^\infty(\Omega)$.

Remark 8.39. We also note that if y is a function of $H_0^1(\Omega)$ such that $y = 0$ quasi-everywhere in $\Omega \setminus \Omega_\varepsilon$, then the restriction of y to Ω_ε belongs to $H_0^1(\Omega_\varepsilon)$ (see [15]). Conversely, if we extend a function $y \in H_0^1(\Omega_\varepsilon)$ by setting $y = 0$ in $\Omega \setminus \Omega_\varepsilon$, then y is quasi-continuous and belongs to $H_0^1(\Omega)$. Therefore, if we define the non-negative Borel measure $\mu_\varepsilon \in \mathcal{M}_0^2(\Omega)$ as

$$\mu_\varepsilon(B) = \begin{cases} 0 & \text{if } \text{cap}(B \cap (\Omega \setminus \Omega_\varepsilon), \Omega) = 0, \\ +\infty & \text{if } \text{cap}(B \cap (\Omega \setminus \Omega_\varepsilon), \Omega) > 0, \end{cases} \tag{8.110}$$

then a function $y \in H_0^1(\Omega)$ is the solution of the problem

$$\left.\begin{aligned} &y \in V_{\mu_\varepsilon} = H_0^1(\Omega) \cap L^2_{\mu_\varepsilon}(\Omega), \\ &\big\langle -\triangle y, \varphi \big\rangle + \int_\Omega y\varphi \,\mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + \chi_\varepsilon u)\varphi \,\mathrm{d}x, \qquad \forall\, \varphi \in V_{\mu_\varepsilon} \end{aligned}\right\} \tag{8.111}$$

if and only if the restriction of y to Ω_ε is the solution of the Dirichlet boundary value problem

$$y \in H_0^1(\Omega_\varepsilon), \qquad -\triangle y = f_\varepsilon + u \quad \text{in } \mathcal{D}'(\Omega_\varepsilon)$$

and, in addition, $y = 0$ q.e. in $\Omega \backslash \Omega_\varepsilon$ (see [82]).

Thanks to this result, we may rewrite the original OCP (8.106)–(8.108) as (8.83)–(8.85). We thus consider the sequence

$$\Big\{ \big\langle \inf_{(u,y)\in \tilde{\Xi}_\varepsilon} \tilde{I}_\varepsilon(u,y) \big\rangle,\ \varepsilon \in E \Big\} \tag{8.112}$$

with

$$\begin{aligned} \tilde{I}_\varepsilon(u,y) = &\int_\Omega |\nabla y|^2 \,\mathrm{d}x + \int_\Omega (y - \chi_\varepsilon z_d)^2 \,\mathrm{d}x \\ &+ \int_\Omega \chi_\varepsilon u^2 \,\mathrm{d}x\ ,\ \forall\ (u,y) \in \tilde{\Xi}_\varepsilon, \end{aligned} \tag{8.113}$$

$$\tilde{\Xi}_\varepsilon = \left\{ (u,y) \,\middle|\, \begin{array}{c} \big\langle -\triangle y, \varphi \big\rangle + \displaystyle\int_\Omega y\varphi \,\mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + \chi_\varepsilon u)\varphi \,\mathrm{d}x \\ \forall\, \varphi \in V_{\mu_\varepsilon} \\ y \in V_{\mu_\varepsilon}, \quad u \in L^2(\Omega), \quad \chi_\varepsilon u \in \mathcal{U}_\varepsilon \subseteq L^2(\Omega) \end{array} \right\}, \tag{8.114}$$

where the Borel measures $\mu_\varepsilon \in \mathcal{M}_0^2(\Omega)$ are defined in (8.110).

By virtue of the compactness of $\mathcal{M}_0^2(\Omega)$ with respect to the $\gamma^\triangle$-convergence and the fact that for the sequence of constrained sets $\{\mathcal{U}_\varepsilon\}_{\varepsilon\in E}$ there exists a nonempty limit $\mathcal{U}_0$ (in the sense of Kuratowski with respect to the weak topology of $L^2(\Omega)$), we have the following result.

Lemma 8.40. *Assume that $f_\varepsilon \longrightarrow f_0$ weakly in $L^2(\Omega)$. Then there exists a non-negative Borel measure $\mu_0 \in \mathcal{M}_0^2(\Omega)$ such that the set*

$$\Xi_0 = \left\{ (u,y) \,\middle|\, \begin{array}{l} \big\langle -\triangle y, \varphi \big\rangle + \displaystyle\int_\Omega y\varphi \,\mathrm{d}\mu_0 = f_0 + u, \quad \forall\, \varphi \in V_{\mu_0}, \\ y \in V_{\mu_0}, \quad u \in \mathcal{U}_0 \subseteq L^2(\Omega) \end{array} \right\} \tag{8.115}$$

is the τ-limit of the sequence $\{\tilde{\Xi}_\varepsilon \subset L^2(\Omega) \times V_{\mu_\varepsilon}\}_{\varepsilon\in E}$ in the Kuratowski sense, where τ is the product of the weak topologies for $L^2(\Omega)$ and $H_0^1(\Omega)$.

Proof. Let us denote by $K(\tau)\text{–}Li\,\tilde{\Xi}_\varepsilon$ and $K(\tau)\text{–}Ls\,\tilde{\Xi}_\varepsilon$ the lower and the upper sequential limits of $\{\tilde{\Xi}_\varepsilon\}$ in the Kuratowski's sense, respectively. Let μ_0

be a $\gamma^{\triangle}$-limit of the sequence of measures $\{\mu_\varepsilon\}_{\varepsilon\in E}$. Then due to Remark 8.31 and Lemma 8.32, $K(\tau)\text{–}Ls\,\tilde{\Xi}_\varepsilon \subseteq \Xi_0$. Hence, in order to prove the lemma, it is sufficient to show that $\Xi_0 \subseteq K(\tau)\text{–}Li\,\tilde{\Xi}_\varepsilon$.

Let (u^*, y^*) be any pair of Ξ_0. Then $u^* \in \mathcal{U}_0$; hence, there is a sequence $\{u_\varepsilon\}_{\varepsilon\in E}$ such that $u_\varepsilon \to u^*$ weakly in $L^2(\Omega)$, $u_\varepsilon \in \mathcal{U}_\varepsilon$ for every $\varepsilon \in E$ and $u_\varepsilon = \chi_\varepsilon u_\varepsilon$, by the definition of $\mathcal{U}_\varepsilon$.

We denote by $y_\varepsilon = y(u_\varepsilon, \mu_\varepsilon)$ the corresponding solutions of the problem (8.111) with $u = u_\varepsilon$. Let $y_\varepsilon^* \in V_{\mu_\varepsilon}$ be the solution of (8.111) corresponding to $f_\varepsilon = f_0$ and $u = u^*$. By the definition of $\gamma^{\triangle}$-convergence, $\{y_\varepsilon^*\}$ converges to y^* weakly in $H_0^1(\Omega)$. However, as follows immediately from (8.114), we have

$$\| y_\varepsilon^* - y_\varepsilon \|_{H_0^1(\Omega)} + \left(\int_\Omega (y_\varepsilon^* - y_\varepsilon)^2 \, \mathrm{d}\mu_\varepsilon \right)^{1/2} \le \sqrt{2} \left(\| f_\varepsilon - f_0 \|_{H^{-1}(\Omega)} + \| u_\varepsilon - u^* \|_{H^{-1}(\Omega)} \right).$$

In view of the standing assumptions, we have $y_\varepsilon^* - y_\varepsilon \longrightarrow 0$ strongly in $H_0^1(\Omega)$. Hence, $y_\varepsilon \to y^*$ weakly in $H_0^1(\Omega)$. As a result, the sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon) \in \tilde{\Xi}_\varepsilon\}_{\varepsilon\in E}$ has been constructed such that $(u_\varepsilon, y_\varepsilon) \xrightarrow{\tau} (u^*, y^*)$, that is, by the definition of the Kuratowski limit, we have $(u^*, y^*) \in K(\tau)\text{–}Li\,\tilde{\Xi}_\varepsilon$. This implies that

$$\Xi_0 \subseteq K(\tau)\text{–}Li\,\tilde{\Xi}_\varepsilon \subseteq K(\tau)\text{–}Ls\,\tilde{\Xi}_\varepsilon \subseteq \Xi_0.$$

This completes the proof. □

In order to recover the limit functional $I_0 : \Xi_0 \to \mathbb{R}$ with (V2)–(V3), we note that due to the construction (8.110), we have

$$\int_{\Omega_\varepsilon} y^2 \, \mathrm{d}\mu_\varepsilon = 0 \text{ for any } (u, y) \in \tilde{\Xi}_\varepsilon.$$

Therefore, for every $\varepsilon \in E$, the cost functionals $\tilde{I}_\varepsilon : \tilde{\Xi}_\varepsilon \to \mathbb{R}$ can be represented in the form

$$\tilde{I}_\varepsilon(u, y) = \int_\Omega |\nabla y|^2 \, \mathrm{d}x + \int_\Omega y^2 \, \mathrm{d}\mu_\varepsilon + \int_\Omega (y - \chi_\varepsilon z_d)^2 \, \mathrm{d}x + \int_\Omega \chi_\varepsilon u^2 \, \mathrm{d}x, \ \forall (u, y) \in \tilde{\Xi}_\varepsilon. \tag{8.116}$$

Taking (8.102) into account, we have

$$\lim_{H \ni \varepsilon \to 0} \left[\int_\Omega |\nabla y_\varepsilon|^2 \, \mathrm{d}x + \int_\Omega y_\varepsilon^2 \, \mathrm{d}\mu_\varepsilon \right] = \int_\Omega |\nabla y|^2 \, \mathrm{d}x + \int_\Omega y^2 \, \mathrm{d}\mu_0, \tag{8.117}$$

$$\lim_{h \ni \varepsilon \to 0} \int_\Omega (y_\varepsilon - \chi_\varepsilon z_d)^2 \, \mathrm{d}x = \int_\Omega (y - \chi_* z_d)^2 \, \mathrm{d}x \tag{8.118}$$

for any index set $H \in \mathcal{H}^{\#}$ and any τ-convergent sequence $\{(u_\varepsilon, y_\varepsilon) \in \tilde{\Xi}_\varepsilon\}_{\varepsilon\in H}$.

In order to recover the limit value of the last term in (8.116), we assume that the constrained sets $\mathcal{U}_\varepsilon$ have the form

$$\mathcal{U}_\varepsilon = \left\{ u \in L^2(\Omega) \,\middle|\, \begin{array}{l} \xi_1\chi_\varepsilon \le u \le \xi_2\chi_\varepsilon, \\ u \in \Lambda \subset L^2(\Omega) \end{array} \right\}, \tag{8.119}$$

where $\xi_1, \xi_2 \in L^2(\Omega)$, $\xi_1 \le \xi_2$ almost everywhere, and Λ is a closed convex subset of $L^2(\Omega)$ for which the following property holds: If $u \in \Lambda$, then $\chi_\varepsilon u \in \Lambda$ for every $\varepsilon > 0$.

It is easy to see that for the sequence $\{\mathcal{U}_\varepsilon\}$ there exists a Kuratowski limit $\mathcal{U}_0$ such that

$$\mathcal{U}_0 = \left\{ u = \chi_* v \in L^2(\Omega) \,\middle|\, \xi_1 \le v \le \xi_2,\ v \in \Lambda \right\}. \tag{8.120}$$

As a result, we have (see [135]) the following:

(i) $\liminf\limits_{H\ni\varepsilon\to 0} \int_\Omega \chi_\varepsilon (u_\varepsilon)^2 \,\mathrm{d}x \ge \int_\Omega \chi_*^{-1} u^2 \,\mathrm{d}x$ for every sequence $\{u_\varepsilon \in \mathcal{U}_\varepsilon\}_{\varepsilon\in H}$ weakly converging in $L^2(\Omega)$ to some $u \in \mathcal{U}_0$.

(ii) For every $u \in \mathcal{U}_0$, the sequence $\{\widehat{u}_\varepsilon = \chi_\varepsilon \chi_*^{-1} u\}_{\varepsilon\in E}$ is such that $\widehat{u}_\varepsilon \to u$ weakly in $L^2(\Omega)$, $\widehat{u}_\varepsilon \in \mathcal{U}_\varepsilon$, $\forall\, \varepsilon \in E$, and

$$\lim_{\varepsilon\to 0} \int_\Omega (\widehat{u}_\varepsilon)^2 \,\mathrm{d}x = \lim_{\varepsilon\to 0} \left\langle \chi_\varepsilon, \chi_*^{-2} u^2 \right\rangle_{(L^\infty, L^1)} = \int_\Omega \chi_*^{-1} u^2 \,\mathrm{d}x.$$

Hence, for the sequence of functionals

$$\left\{ \int_\Omega \chi_\varepsilon u^2 \,\mathrm{d}x \right\}_{\varepsilon\in E}$$

properties (V2)–(V3) hold. Combining this fact with relations (8.117) and (8.118), we obtain the following result.

Lemma 8.41. *Assume that the sets $\mathcal{U}_\varepsilon \subset L^2(\Omega)$ have the form (8.119). Then under assumptions of Lemma 8.40, there exists a variational limit*

$$\Gamma(\tau)\text{-}\operatorname*{Lim}_{\varepsilon\to 0} \left\langle \inf_{(u,y)\in\tilde{\Xi}_\varepsilon} \tilde{I}_\varepsilon(u,y) \right\rangle = \left\langle \inf_{(u,y)\in\Xi_0} I_0(u,y) \right\rangle \tag{8.121}$$

such that the set Ξ_0 has the representation (8.115) with $\mathcal{U}_0$ given by (8.120) and

$$\begin{aligned} I_0(u,y) = {} & \int_\Omega |\nabla y|^2 \,\mathrm{d}x + \int_\Omega y^2 \,\mathrm{d}\mu_0 + \int_\Omega (y - \chi_* z_d)^2 \,\mathrm{d}x \\ & + \int_\Omega \chi_*^{-1} u^2 \,\mathrm{d}x. \end{aligned} \tag{8.122}$$

Now, we are in the position to state the main result of this section. We recall that in the framework presented here the variational limit (8.121) is called a limit OCP whenever it corresponds to some control problem that can be recovered from the limiting process.

Theorem 8.42. *Let $\xi_1, \xi_2, z_d \in L^2(\Omega)$ be such that $\xi_1 \le \xi_2$ a.e. in Ω and let $\{f_\varepsilon \in L^2(\Omega)\}_{\varepsilon\in E}$ be a sequence weakly converging to some function $f_0 \in L^2(\Omega)$. Assume that the sequence of open subsets $\{\Omega_\varepsilon \subset \Omega\}_{\varepsilon\in E}$ is such that*

$$\chi_\varepsilon \to \chi_* \text{ weakly-}* \text{ in } L^\infty(\Omega), \quad \chi_*^{-1} \in L^\infty(\Omega).$$

Then there exists a non-negative Borel measure $\mu_0 \in \mathcal{M}_0^2(\Omega)$ such that for the family of OCPs

$$\left.\begin{aligned} &-\triangle y = f_\varepsilon + u \quad \text{in } \mathcal{D}'(\Omega_\varepsilon),\\ &y \in H_0^1(\Omega_\varepsilon),\ \chi_\varepsilon\xi_1 \le u \le \chi_\varepsilon\xi_2,\ u \in \Lambda,\\ &I_\varepsilon(u,y) = \int_{\Omega_\varepsilon} |\nabla y|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon} (y - z_d)^2\,\mathrm{d}x + \int_{\Omega_\varepsilon} u^2\,\mathrm{d}x \to \inf \end{aligned}\right\} \tag{8.123}$$

there exists a limit problem as $\varepsilon \to 0$ with the following representation:

$$\left.\begin{aligned} &-\triangle y + \mu_0 y = f_0 + u \quad \text{in } V_{\mu_0}^*\\ &y \in V_{\mu_0},\ \chi_*\xi_1 \le u \le \chi_*\xi_2 \ \text{ a.e. on } \Omega,\ \chi_*^{-1}u \in \Lambda\\ &I_0(u,y) = \int_\Omega |\nabla y|^2\,\mathrm{d}x + \int_\Omega y^2\,\mathrm{d}\mu_0 + \int_\Omega (y - \chi_* z_d)^2\,\mathrm{d}x\\ &\qquad + \int_\Omega \chi_*^{-1}u^2\,\mathrm{d}x \to \inf \end{aligned}\right\}. \tag{8.124}$$

In addition, if $(u_\varepsilon^0, y_\varepsilon^0)$ and (u^0, y^0) are the optimal solutions of problems (8.123) and (8.124), respectively, then

$$\tilde{y}_\varepsilon^0 \to y^0 \text{ weakly in } H_0^1(\Omega), \quad \tilde{u}_\varepsilon^0 \to u^0 \text{ weakly in } L^2(\Omega), \tag{8.125}$$

$$\chi_\varepsilon(\tilde{u}_\varepsilon^0 - \chi_*^{-1}u^0) \to 0 \text{ strongly in } L^2(\Omega), \tag{8.126}$$

$$I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) \to I_0(u^0, y^0). \tag{8.127}$$

Proof. The validity of these statements including the variational properties (8.125) and (8.127) immediately follows from Lemma 8.40, Lemma 8.41, Definition 8.33, and Theorem 8.30. As for the property (8.126), we note that as $\chi_\varepsilon \to \chi_*$ weakly-$*$ in $L^\infty(\Omega)$ and

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (u_\varepsilon^0)^2\,\mathrm{d}x = \int_\Omega \chi_*^{-1}(u^0)^2\,\mathrm{d}x,$$

one gets

$$\int_\Omega (\chi_\varepsilon(\tilde{u}_\varepsilon^0 - \chi_*^{-1}u^0))^2\,dx$$
$$= \int_\Omega (\tilde{u}_\varepsilon^0)^2\,dx - 2\int_\Omega \tilde{u}_\varepsilon^0\chi_*^{-1}u^0\,dx + \int_\Omega \chi_*^{-2}(u^0)^2\chi_\varepsilon\,dx \to 0.$$

This concludes the proof. □

We conclude this section with the example of an OCP on a perforated domain with Dirichlet conditions on the boundary and show that the $\gamma^{\triangle}$-limit measure $\mu_0 \in \mathcal{M}_2^0(\Omega)$ in the limit problem (8.124) can be recovered. We consider the case where the domain Ω is perforated by an increasing number of holes of critical size. This is the type of perforated domains which was proposed by Cioranescu and Murat in their celebrated paper [65].

We define the perforated domain as

$$\Omega_\varepsilon = \Omega \cap Q_\varepsilon, \quad Q_\varepsilon = \mathbb{R}^n \setminus \cup_{i=1}^{n(\varepsilon)} T_i^\varepsilon.$$

Here, we assume that $\mathbb{R}^n (n \geq 2)$ is covered by cubes P_i^ε of size 2ε and T_i^ε is a ball of radius $a_\varepsilon = \exp(-C_0/\varepsilon^2)$ if $n = 2$ and $a_\varepsilon = C_0\varepsilon^{n/n-2}$ if $n \geq 3$, centered at the very center of the cube P_i^ε.

For given $f \in L^2(\Omega)$ and each value of $\varepsilon \in (0; \varepsilon^0]$, we consider the following OCP on Ω_ε:

$$\left.\begin{aligned} -\triangle y = f + u \ \text{ in } \mathcal{D}'(\Omega_\varepsilon), \\ y \in H_0^1(\Omega_\varepsilon), \quad \| u \|_{L^2(\Omega)} \leq 2, \\ -(\operatorname{meas}\Omega)^{-\frac{1}{2}}\chi_\varepsilon \leq \chi_\varepsilon u \ \text{ a.e. on } \Omega, \\ I_\varepsilon(u,y) = \int_{\Omega_\varepsilon} |\nabla y|^2\,dx + \int_{\Omega_\varepsilon} u^2\,dx \to \inf \end{aligned}\right\}. \tag{8.128}$$

It is easy to see that the weak-$*$ limit in $L^\infty(\Omega)$ of the characteristic functions $\chi_{\Omega_\varepsilon}$ is equal to $\chi_* = 1$ a.e. in Ω. Thanks to Theorem 8.42, there is a limit problem for the family (8.128) which can be represented in the form

$$\begin{aligned} &-\triangle y + \mu_0 y = f + u \qquad \text{in } V_{\mu_0}^*, \\ &y \in V_{\mu_0}, \ \| u \|_{L^2(\Omega)} \leq 2, \ -(\operatorname{meas}\Omega)^{-1/2} \leq u \ \text{ a.e. in } \Omega, \\ &I_0(u,y) = \int_\Omega |\nabla y|^2\,dx + \int_\Omega y^2\,d\mu_0 + \int_\Omega u^2\,dx \to \inf. \end{aligned} \tag{8.129}$$

In order to recover the measure $\mu_0 \in \mathcal{M}_0^2(\Omega)$, we use the following result (see [55, 65]).

Theorem 8.43 (Casado-Diaz [55], Cioranescu and Murat [65]). *If there exists a sequence* $\{\omega_\varepsilon \in H^1(\Omega)\}_{\varepsilon\in E}$ *such that*

$$\omega_\varepsilon = 0 \ \textit{on the holes } T_i^\varepsilon, \ 1 \leq i \leq n(\varepsilon), \quad \omega_\varepsilon \to 1 \ \textit{weakly in } H^1(\Omega),$$

then there is a measure $\mu_* \in \mathcal{M}_0^2(\Omega)$ *satisfying the following: For any* $v_\varepsilon \in H_0^1(\Omega_\varepsilon)$ *and for any* $v \in H_0^1(\Omega)$ *such that*

$$\tilde{v}_\varepsilon = \chi_\varepsilon v \quad \text{and} \quad \tilde{v}_\varepsilon \to v \quad \text{weakly in} \quad H_0^1(\Omega),$$

we have

$$\Big\langle -\triangle\omega_\varepsilon, v_\varepsilon \Big\rangle \to \int_\Omega v \, \mathrm{d}\mu_*. \tag{8.130}$$

The sequence $\{\omega_\varepsilon\}_{\varepsilon\in E}$ was constructed in [65] for the above-mentioned case of perforated domains $\{\Omega_\varepsilon\}$ and it was proven that condition (8.130) holds true if μ_* is defined as

$$\mathrm{d}\mu_* = \frac{\pi}{2}\frac{1}{C_0}\,\mathrm{d}x \text{ if } n = 2, \quad \mathrm{d}\mu_* = S_n(n-2)C_0^{n-2}/2^n \,\mathrm{d}x \text{ if } n \geq 3,$$

where S_n is the surface of the unit sphere in $\mathbb{R}^n$ and C_0 is a positive constant.

A different approach for the construction of the functions $\{\omega_\varepsilon\}_{\varepsilon\in E}$ can be found in [5, 63, 182, 225].

Now, we prove that $\mu_* = \mu_0$ in (8.128). For any fixed $f \in L^2(\Omega)$ and $u \in L^2(\Omega)$, we consider the variational formulation of the original problem taking as the test function $\omega_\varepsilon\varphi$, where $\varphi \in \mathcal{D}(\Omega)$. One gets

$$\begin{aligned}\int_\Omega (f+u)\omega_\varepsilon\varphi \,\mathrm{d}x &= \int_\Omega \varphi(\nabla\tilde{y}_\varepsilon, \nabla\omega_\varepsilon)\,\mathrm{d}x + \int_\Omega \omega_\varepsilon(\nabla\tilde{y}_\varepsilon, \nabla\varphi)\,\mathrm{d}x \\ &= \int_\Omega \omega_\varepsilon(\nabla\tilde{y}_\varepsilon, \nabla\varphi)\,\mathrm{d}x + \Big\langle -\triangle\omega_\varepsilon, \varphi\tilde{y}_\varepsilon\Big\rangle \\ &\quad - \int_\Omega \tilde{y}_\varepsilon(\nabla\omega_\varepsilon, \nabla\varphi)\,\mathrm{d}x.\end{aligned}$$

Then, using (8.130) and the facts that

$$\begin{aligned}\omega_\varepsilon &\to 1 \quad \text{strongly in } L^2(\Omega), \\ \nabla\omega_\varepsilon \to \nabla 1 &= 0 \quad \text{weakly in } (L^2(\Omega))^n, \\ \tilde{y}_\varepsilon &\to y \quad \text{weakly in } H_0^1(\Omega)\end{aligned}$$

and passing to the limit as $\varepsilon \to 0$, we get

$$\int_\Omega (f+u)\varphi \,\mathrm{d}x = \int_\Omega (\nabla y, \nabla\varphi)\,\mathrm{d}x + \int_\Omega y\varphi \,\mathrm{d}\mu_* \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Hence, the limit function $y \in H_0^1(\Omega)$ satisfies the equation

$$-\triangle y + \mu_* y = f + u \qquad \text{in } \mathcal{D}'(\Omega). \tag{8.131}$$

Since the function $y \in H_0^1(\Omega)$ also satisfies the state equation (8.129), it follows from the uniqueness result (see Lemma 5.4 in [82]) that we have

$\mu_0 = \mu_*$. Thus, in order to summarize, the homogenized OCP for the family (8.128) is unique and has the following form (for $n = 2$):

$$\left.\begin{aligned} -\triangle y + \pi/(2C_0)y = f + u \text{ in } \mathcal{D}'(\Omega), \\ y \in H_0^1(\Omega), \quad \| u \|_{L^2(\Omega)} \leq 2, \ u \geq -(\operatorname{meas}\Omega)^{-1/2} \ \text{ a.e. on } \Omega, \\ I_0(u,y) = \int_\Omega |\nabla y|^2 \, \mathrm{d}x + \pi/(2C_0) \int_\Omega y^2 \, \mathrm{d}x + \int_\Omega u^2 \, \mathrm{d}x \to \inf \end{aligned}\right\}. \tag{8.132}$$

In view of the variational properties of limit problems (see Theorem 8.42), we may add that the unique solution $(u^0, y^0) \in L^2(\Omega) \times H_0^1(\Omega)$ of the limit problem (8.132) satisfies the following conditions:

$$\tilde{y}_\varepsilon^0 \to y^0 \text{ weakly in } H_0^1(\Omega), \quad \tilde{u}_\varepsilon^0 \to u^0 \text{ weakly in } L^2(\Omega),$$

$$\int_{\Omega_\varepsilon} (u_\varepsilon^0 - u^0)^2 \, \mathrm{d}x \to 0, \quad I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) \to I_0(u^0, y^0),$$

where $\{(u_\varepsilon^0, y_\varepsilon^0) \in L^2(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon)\}$ is the sequence of the optimal pairs for the family (8.128).

Part II

Optimal Control Problems on Periodic Reticulated Domains: Asymptotic Analysis and Approximate Solutions

9

Suboptimal Control of Linear Steady-State Processes on Thin Periodic Structures with Mixed Boundary Controls

Our main interest in this chapter is in approximate solutions to a class of optimal control problems (OCPs) on thin periodic structures. Typically, thin structures are characterized by two properties: periodicity and small thickness of the material. More precisely, we suppose that the geometry of such structures depends on two small parameters, ε and $h(\varepsilon)$, related to each other by the assumption $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and determining the cell of periodicity and thickness of constituting components, respectively. This is in contrast to the periodically perforated domain for which $h = h(\varepsilon) \to \text{const} \in (0, 1]$ as $\varepsilon \to 0$. In view of this, it should be noted that the asymptotic analysis of boundary value problems in a perforated domain with small holes (without controls) has been intensively studied by many authors. We mainly mention Cioranescu and Saint Jean Paulin [66], Cioranescu and Murat [65], Dal Maso and Murat [82], Marchenko and Khruslov [182], Zhikov, Kozlov, and Oleinik [261]. The interesting effect of homogenization of the Poisson equation with (zero) Dirichlet conditions on the boundary of the holes is well known when a "strange term" appears in the limit equation. This effect first was discovered and studied by Marchenko and Khruslov in [182], and Cioranescu and Murat [65]. Another effect of homogenization of the same equation with a critical size of the holes, when nonhomogeneous Neumann conditions on the boundary of the holes are assumed, was studied by Conca and Donato [70]. In this case, some constant that is proportional to the limit of the total flux of the solution through the boundary of the holes appears in the limit equation. In Cardone, D'Apice, and De Maio [54], and Corbo Esposito, D'Apice, and Gaudiello [73], the authors examined a homogenization problem with both Dirichlet and Neumann conditions on the boundary of the holes. It was shown in [73] for a very simple geometry of holes that an interference phenomenon in the homogenization of the Poisson equation is present when a nonhomogeneous Neumann condition is imposed on one part of the boundary of the holes and Dirichlet condition is imposed on the remaining part.

The asymptotic analysis of boundary value problems on thin periodic structures has been the subject of a large number of publications, among which

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_9,
© Springer Science+Business Media, LLC 2011

we mention the monographs of Ciarlet [62], Nazarov [200], and Panasenko [209]. There are several recent works devoted to the homogenization of thin structures and other singular media. We quote here only the works Argatov and Nazarov [8, 9], Bouchitté and Fragala [26], Braides and Defranceschi [30], Chechkin, Jikov, Lukkassen, and Piatnitski [58], Cioranescu and Saint Jean Paulin [66], Nazarov [199], and Panasenko [207, 208]. Elasticity problems on rod junctures were studied in the interesting works of Bouchitté and Fragala [27], Nazarov and Slutskii [201, 202, 203], Zhikov [257], Pastuchova [211, 213], and Zhikov and Pastuchova [265, 264].

We focus our attention on OCPs with different types of controls as well as control and state constraints. Having provided the asymptotic analysis of such problems as the small parameter ε tends to 0, we construct the so-called asymptotically suboptimal controls for the original objects. For the sake of simplicity, we restrict the analysis to the case when a 2D lattice structure Ω_ε has the representation $\Omega_\varepsilon = \Omega \cap \left(\varepsilon \mathcal{F}^h\right)$, where $\mathcal{F}^h$ denotes a 1-periodic thin grid with a small but same thickness h. It means that hereinafter by thin grid we always mean a 1-periodic planar "fattened" graph after applying the procedure of h-thickening to some singular periodic graph $\mathcal{F}$.

It is well known (see Cioranescu and Saint Jean Paulin [66], Panasenko [209], and Zhikov [257]) that the homogenization limit of most of the boundary value problems on thin structures with Dirichlet and Neumann boundary conditions is always of the following form: Strong convergence of the corresponding solutions holds, the limit function does not contain an oscillating variable, and the limit problem is the same for any $h(\varepsilon) \to 0$, that is, it does not depend on the particular type of the thin structures. However, as we see later, for OCPs with Robin boundary conditions, the structure of the corresponding limit problem depends essentially on how the parameter h tends to 0 as $\varepsilon \to 0$ (the so-called "scaling effect").

Let Ω be a bounded open domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. By $L^2(\Omega, d\mu)$ we denote the Banach space of square integrable functions on Ω with respect to the measure μ. If the measure μ is the Lebesgue, we abbreviate the notation to $L^2(\Omega)$. The space $H^1(\Omega, S)$ is the set of all functions from $H^1(\Omega)$ such that $\gamma(y) = 0$ on S. Let χ_E be the characteristic function of a set E (i.e., $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$).

Assume that a periodic thin structure is contained in an open bounded domain $\Omega \subset \mathbb{R}^2$. This highly contrasting medium in the plane depends on two small related parameters, ε and h, which control the size of a periodicity cell and the thickness of the reinforcement. To describe this structure, we suppose that there is a specified periodic graph $\mathcal{F}$ in the plane such that $\mathcal{F}$ consists of segments. For simplicity, we assume that all the end points of each segment are the intersection points of two or more segments. Let $\square = [0, 1)^2$ be the periodicity cell of $\mathcal{F}$. We call this graph a singular (or an infinitely thin) grid. Following Zhikov [257] (see also Bouchitté and Fragala [27]), we define a thin grid $\mathcal{F}^h$ applying the procedure of h-thickening to the graph $\mathcal{F}$. This means that $\mathcal{F}^h$ can be obtained if any component I of $\mathcal{F}$ is replaced by a strip I^h of

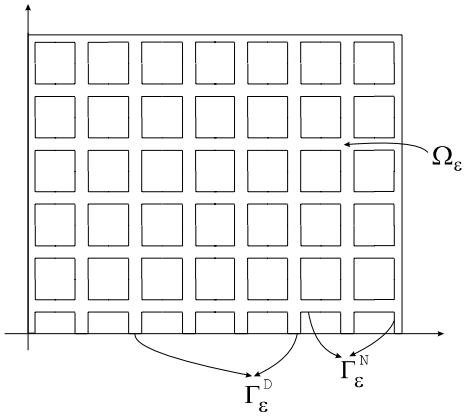


Fig. 9.1. The network of thin wires Ω_ε

width h whose median coincides with I. Sometimes, it is convenient to include into a thin grid $\mathcal{F}^h$, disks of radius $h/2$ centered at the nodes of the grid $\mathcal{F}$. The procedure of h-thickening becomes simpler if the grid $\mathcal{F}$ is composed of several families of parallel straight lines. In this case, we obtain the grid $\mathcal{F}^h$ by replacing any of the lines l of the grid $\mathcal{F}$ by the infinite strip of width h with median l.

For the sake of simplicity, we consider a particular case of 2D reticulated structures, namely the network (or gridworks) that consists of regular arrays of thin wires and is defined by Fig. 9.1 (for the more general case, see Lagnese and Leugering [163], and Leugering and Schmidt [166]). We assume that it is contained in the domain $\Omega \subset \mathbb{R}^2$, where

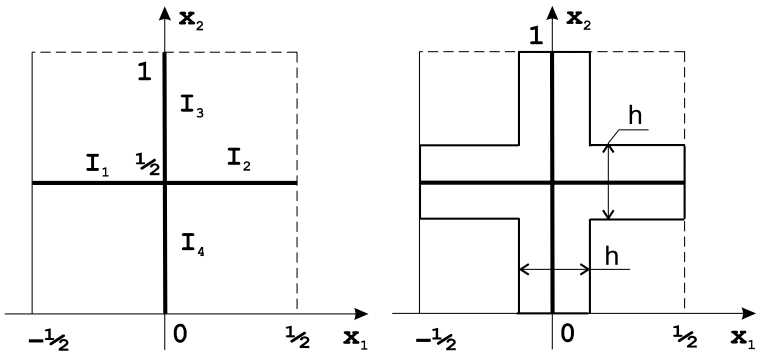
$$\Omega = (0, L_1) \times (0, L_2), \quad \frac{L_1}{L_2} \in \mathbb{Q}.$$

We describe this network as follows. Let $\mathcal{F}$ be the periodic graph in the plane with the cell of periodicity $Y = [-0.5, 0.5) \times [0, 1)$ that consists of the following segments:

$$I_1 = \left\{ x_1 \in \left[-\frac{1}{2}, 0\right],\ x_2 = \frac{1}{2} \right\}, \quad I_2 = \left\{ x_1 \in \left(0, \frac{1}{2}\right),\ x_2 = \frac{1}{2} \right\},$$
$$I_3 = \left\{ x_1 = 0,\ x_2 \in \left[0, \frac{1}{2}\right] \right\}, \quad I_4 = \left\{ x_1 = 0,\ x_2 \in \left(\frac{1}{2}, 1\right) \right\}.$$

We define a thin grid $\mathcal{F}^h$ applying the procedure of h-thickening to the graph $\mathcal{F}$ (see Fig. 9.2). In this case,

$$\mathcal{F}^h = \left\{ (x_1, x_2) \in F^h + \mathbf{k},\ \forall\, \mathbf{k} \in \mathbb{R}^2, \right.$$
$$\left. F^h = \left[-\frac{1}{2} \le x_1 < \frac{1}{2}, \frac{1-h}{2} < x_2 < \frac{1+h}{2} \right] \vee \left[-\frac{h}{2} < x_1 < \frac{h}{2}, 0 \le x_2 < 1 \right] \right\}.$$



(a) The cell of the graph $\mathcal{F}$ (b) The cell of the thin grid $\mathcal{F}^h$

Fig. 9.2. The procedure of h-thickening for the graph F

Let ε be a small parameter. We will always suppose that ε varies in a strictly decreasing sequence of positive numbers which converges to 0 and such that the following numbers are integers: $N_\varepsilon^1 = L_1/\varepsilon$, $N_\varepsilon^2 = \big(2L_2 + 1\big)/\varepsilon$. So, when we write $\varepsilon > 0$, we consider only the elements of this sequence.

Let $\mathcal{F}_\varepsilon^h$ be the homothetic contraction of the grid $\mathcal{F}^h$ with a factor of ε^{-1}(i.e., $\mathcal{F}_\varepsilon^h = \varepsilon\mathcal{F}^h = \big\{\varepsilon x \ : \ x \in \mathcal{F}^h\big\}$). Note that the structure $\mathcal{F}_\varepsilon^h$ consists of strips whose thickness is εh. For every fixed ε, let us define the sets

$$Q_\varepsilon = \big\{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 \in (0, L_1), x_2 \in (0, \varepsilon/2)\big\} \subset \Omega, \tag{9.1}$$

$$\Gamma_\varepsilon^D = \partial\Omega \cap \mathcal{F}_\varepsilon^h \cap \{(x_1, x_2) \ : \ x_2 = 0\}, \quad \Gamma_\varepsilon^N = \partial\mathcal{F}_\varepsilon^h \cap Q_\varepsilon, \tag{9.2}$$

$$S_{ext} = \partial\Omega \setminus (\Gamma_\varepsilon^D \cup \Gamma_\varepsilon^N), \qquad S_{int} = [\Omega \setminus Q_\varepsilon] \cap \partial\mathcal{F}_\varepsilon^h. \tag{9.3}$$

As a result, the network Ω_ε is defined as

$$\Omega_\varepsilon = \Omega \cap \mathcal{F}_\varepsilon^h, \tag{9.4}$$

$$\partial\Omega_\varepsilon = \Gamma_\varepsilon^D \cup \Gamma_\varepsilon^N \cup S_{int}^\varepsilon \cup S_{ext}. \tag{9.5}$$

Essentially, for fixed ε and h, the network Ω_ε can be viewed as a perforated domain with rapidly oscillating part of boundary. However, we suppose that the parameters ε and $h(\varepsilon)$ are related to each other: $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. So, according to Zhikov [257] and Cioranescu and Saint Jean Paulin [66], all thin structures can be divided into three classes:

(A_1) Sufficiently thick, when $\lim_{\varepsilon\to 0}\big(\varepsilon/h(\varepsilon)\big) = 0$;
(A_2) Sufficiently thin, when $\lim_{\varepsilon\to 0}\big(\varepsilon/h(\varepsilon)\big) = +\infty$;
(A_3) Structures of critical thickness, when $\lim_{\varepsilon\to 0}\big(\varepsilon/h(\varepsilon)\big) = \theta \in (0,\infty)$.

Moreover, it should be noted that the domain Ω_ε has such a type of the oscillating boundary that the ratio of the amplitude and the period of the oscillation is bounded as the parameter ε tends to 0. As it will be seen later, this fact is very essential for the research method of various processes in such domains.

9.1 A measure-theoretic approach to the description of the network Ω_ε

To begin, we introduce the periodic framework in which we will work in this section. Note that there are many possibilities for the choice of a representative period Y on the structure Ω_ε. However, as was mentioned in Cioranescu and Saint Jean Paulin [66] and in Bachvalov and Panasenko [14], the limit analysis of many boundary value problems on reticulated structures is independent of the particular choice of this period. For convenience, we consider three different periodicity cells for the network of wires Ω_ε (see Fig. 9.3). Each part of Fig. 9.3 indicates zones where the corresponding boundary conditions will be located, namely, we suppose that the segment (A, B) is the influence zone Λ_D^h of Dirichlet control (see Fig. 9.3(b)). The polyline $ABCD$ (as it is indicated in Fig. 9.3(a)) coincides with the Neumann control zone Λ_N^h. Additionally, the boundary Λ_R^h of the hole $ABCD$ is a zone where Robin condition is concentrated (see Fig. 9.3(c)). Here, the index h indicates the dependence of these sets on the thickness of the thin grid $\mathcal{F}^h$.

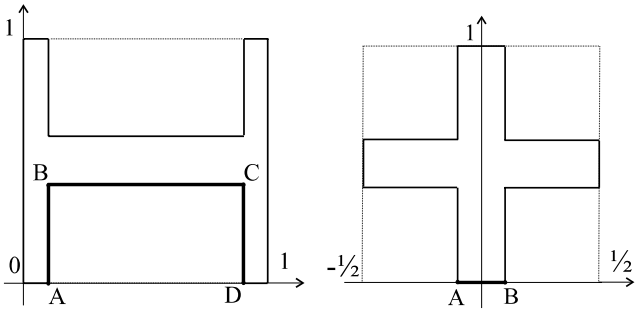
Let us consider three periodic, finite positive Borel measures ν^h, μ^h, and λ^h in $\mathbb{R}^2$ with the toruses of periodicity

$$Y_1 = [0,1)^2, \quad Y_2 = [-1/2,1/2)\times[0,1), \quad \text{and} \quad Y_3 = [0,1)\times[-1/2,1/2),$$

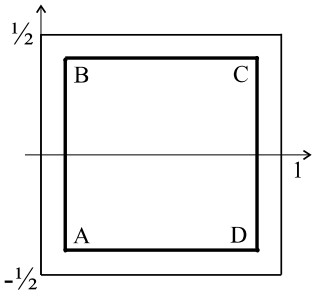
respectively. Let ν^h, μ^h, and λ^h be the probability measures, concentrated, and uniformly distributed on the sets Λ_N^h, Λ_D^h, and Λ_R^h, respectively. So, these measures are proportional to 1D Lebesgue measures and are normalized by the conditions

$$\int_{Y_1} \mathrm{d}\nu^h = \int_{Y_2} \mathrm{d}\mu^h = \int_{Y_3} \mathrm{d}\lambda^h = 1.$$

It is clear that $\nu^h(Y_1 \setminus \Lambda_N^h) = \mu^h(Y_2 \setminus \Lambda_D^h) = \lambda^h(Y_3 \setminus \Lambda_R^h) = 0$, that is, these measure are singular with respect to the Lebesgue measure $\mathcal{L}_2$. Moreover, for any smooth function $\varphi \in C^\infty(\mathbb{R}^2)$, we have



(a) The cell Y_1 (b) The cell Y_2



(c) The cell Y_3

Fig. 9.3. The periodicity cells for Ω_ε

$$\int_{\Lambda_N^h} \varphi \, \mathrm{d}\mathcal{H}^1 = 2(1-h) \int_{Y_1} \varphi \, \mathrm{d}\nu^h, \tag{9.6}$$

$$\int_{\Lambda_D^h} \varphi \, \mathrm{d}x_1 = h \int_{Y_2} \varphi \, \mathrm{d}\mu^h, \tag{9.7}$$

$$\int_{\Lambda_R^h} \varphi \, \mathrm{d}\mathcal{H}^1 = 4(1-h) \int_{Y_3} \varphi \, \mathrm{d}\lambda^h. \tag{9.8}$$

Since the homothetic contraction of the plane with a factor of ε^{-1} takes the grid $\mathcal{F}^h$ to $\mathcal{F}_\varepsilon^h = \varepsilon \mathcal{F}^h$, we introduce the so-called "scaling"measures ν_ε^h, μ_ε^h, and λ_ε^h by the following rules:

$$\left.\begin{aligned} \lambda_\varepsilon^h(B) &= \varepsilon^2 \lambda^h(\varepsilon^{-1} B), \\ \nu_\varepsilon^h(B) &= \begin{cases} \varepsilon \nu^h(\varepsilon^{-1}(B \cap \mathcal{O}_\varepsilon)) & \text{if } B \cap \mathcal{O}_\varepsilon \neq \emptyset, \\ 0 & \text{otherwise} \end{cases} \\ \mu_\varepsilon^h(B) &= \begin{cases} \varepsilon \mu^h(\varepsilon^{-1}(B \cap \mathcal{O}_\varepsilon)) & \text{if } B \cap \mathcal{O}_\varepsilon \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}\right\} \tag{9.9}$$

where B is any planar Borel set and $\mathcal{O}_\varepsilon = \big\{(x_1, x_2) \in \mathbb{R}^2 \ : \ 0 \le x_2 < \varepsilon\big\}$.

Clearly, λ_ε^h is an ε-periodic measure on $\mathbb{R}^2$, whereas the measures ν_ε^h and μ_ε^h are ε-periodic along the x_1-axis. Then

$$\int_{\varepsilon Y_3} \mathrm{d}\lambda_\varepsilon^{h(\varepsilon)} = \varepsilon^2 \int_{\varepsilon Y_3} \mathrm{d}\lambda^{h(\varepsilon)}(x/\varepsilon) = \varepsilon^2 \int_{Y_3} \mathrm{d}\lambda^{h(\varepsilon)} = \varepsilon^2,$$
$$\int_{\varepsilon Y_2} \mathrm{d}\mu_\varepsilon^{h(\varepsilon)} = \varepsilon \int_{Y_2} \mathrm{d}\mu^{h(\varepsilon)} = \varepsilon, \quad \int_{\varepsilon Y_1} \mathrm{d}\nu_\varepsilon^{h(\varepsilon)} = \varepsilon \int_{Y_1} \mathrm{d}\nu^{h(\varepsilon)} = \varepsilon.$$

Hence, the measure $\lambda_\varepsilon^{h(\varepsilon)}$ weakly converges to the planar Lebesgue measure as $\varepsilon \to 0$ (in symbols, $\mathrm{d}\lambda_\varepsilon^{h(\varepsilon)} \rightharpoonup \mathrm{d}x$), that is,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \varphi \, \mathrm{d}\lambda_\varepsilon^{h(\varepsilon)} = \int_{\mathbb{R}^2} \varphi \, \mathrm{d}x \tag{9.10}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ (see Lemma 6.50).

As for the limit properties of the measures $\nu_\varepsilon^{h(\varepsilon)}$ and $\mu_\varepsilon^{h(\varepsilon)}$, we have the following result.

Proposition 9.1.

$$\mathrm{d}\nu_\varepsilon^{h(\varepsilon)} \overset{\varepsilon \to 0}{\rightharpoonup} \delta_{\{x_2=0\}}\mathrm{d}x, \quad \mathrm{d}\mu_\varepsilon^{h(\varepsilon)} \overset{\varepsilon \to 0}{\rightharpoonup} \delta_{\{x_2=0\}}\mathrm{d}x,$$

where by $\delta_{\{x_2=0\}}\mathrm{d}x$ we denote the product of the linear Lebesgue measure $\mathrm{d}x_1$ and the Dirac measure $\delta_{\{x_2=0\}}\mathrm{d}x_2$.

Proof. Let us partition the plane $\mathbb{R}^2$ into squares εY_1 with edges ε and denote these squares by the symbols $\varepsilon Y_1^{i,j}$. Then for any test function $\varphi \in C_0^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \varphi \, \mathrm{d}\nu_\varepsilon^{h(\varepsilon)} &= \sum_i \sum_j \int_{\varepsilon Y_1^{i,j}} \varphi \, \mathrm{d}\nu_\varepsilon^{h(\varepsilon)} \\
&= \sum_i \int_{\varepsilon Y_1^{i,0}} \varphi \, \mathrm{d}\nu_\varepsilon^{h(\varepsilon)} = \sum_i \varphi(x_{i,0}) \int_{\varepsilon Y_1^{i,0}} \mathrm{d}\nu_\varepsilon^{h(\varepsilon)} \\
&= \sum_i \varphi(x_{i,0}) \varepsilon \int_{\varepsilon Y_1} \mathrm{d}\nu^{h(\varepsilon)}(x/\varepsilon) = \sum_i \varphi(x_{i,0}) \varepsilon,
\end{aligned}$$

where $x_{i,0}$ are some points of the corresponding squares $\varepsilon Y_1^{i,0} = \varepsilon(Y_1 + (i,0)) \subset \mathcal{O}_\varepsilon$.

Hence, for a given function $\varphi \in C_0^\infty(\mathbb{R}^2)$, there are a constant $M_\varphi > 0$ independent of i and some points $(x_1^i, 0)$ on the boundary of the torus $\varepsilon Y_1^{i,0}$ such that

$$\varphi(x_{i,0}) = \varphi(x_1^i, 0) + (\varphi(x_{i,0}) - \varphi(x_1^i, 0)),$$
$$|\varphi(x_{i,0}) - \varphi(x_1^i, 0)| \le M_\varphi \|x_{i,0} - (x_1^i, 0)\|_{\mathbb{R}^2} \le M_\varphi \varepsilon.$$

Taking the construction of the Riemann sum into account, we conclude that

$$\lim_{\varepsilon\to 0}\sum_i \varphi(x_{i,0})\varepsilon = \lim_{\varepsilon\to 0}\sum_i \varphi(x_1^i,0)\varepsilon + \lim_{\varepsilon\to 0} D(\varepsilon) = \int_{\mathbb{R}} \varphi(x_1,0)\,\mathrm{d}x_1 + \lim_{\varepsilon\to 0} D(\varepsilon),$$

where $|D(\varepsilon)| \le \sum_i \varepsilon|\varphi(x_{i,0}) - \varphi(x_1^i,0)| \le M_\varphi \sum_i \varepsilon^2 \to 0$ as $\varepsilon \to 0$. Thereby we obtain the required result. It is clear that the same conclusion is true for the measure $\mu_\varepsilon^{h(\varepsilon)}$. □

Let us define the following sets:

$$\Theta_\varepsilon^R = \left\{\mathbf{k} = (k_1,k_2) \in \mathbb{Z}^2 \ : \ (\varepsilon Y_3 + \varepsilon\,\mathbf{k}) \cap \Omega \neq \emptyset\right\}, \tag{9.11}$$

$$\Theta_\varepsilon^N = \left\{\mathbf{k} = (k_1,0) \in \mathbb{Z}^2 \ : \ (\varepsilon Y_1 + \varepsilon\,\mathbf{k}) \cap \Omega \neq \emptyset\right\}. \tag{9.12}$$

Let $z_\varepsilon \in L^2(S_{int}^\varepsilon)$ be a fixed function. Then for any $\varphi \in C_0^\infty(\mathbb{R}^2)$, the integral term $\displaystyle\int_{S_{int}^\varepsilon} z_\varepsilon\varphi\,\mathrm{d}\mathcal{H}^1$ can be rewritten it in the following equivalent form:

$$\begin{aligned}
\int_{S_{int}^\varepsilon} z_\varepsilon\varphi\,\mathrm{d}\mathcal{H}^1 &= \sum_{\mathbf{k}\in\Theta_\varepsilon^R}\int_{\varepsilon(\Lambda_R^h+\mathbf{k})} z_\varepsilon\varphi\,\mathrm{d}\mathcal{H}^1 \\
&= [4\varepsilon\,(1-h(\varepsilon))]\sum_{\mathbf{k}\in\Theta_\varepsilon^R}\int_{\varepsilon(Y_3+\mathbf{k})} \widehat{z}_\varepsilon\varphi\ \mathrm{d}\lambda^h(x/\varepsilon) \\
&= \frac{4}{\varepsilon}\,(1-h(\varepsilon))\sum_{\mathbf{k}\in\Theta_\varepsilon^R}\int_{\varepsilon(Y_3+\mathbf{k})} \widehat{z}_\varepsilon\varphi\ \varepsilon^2\,\mathrm{d}\lambda^h(x/\varepsilon) \\
&= \frac{4}{\varepsilon}\,(1-h(\varepsilon))\sum_{\mathbf{k}\in\Theta_\varepsilon^R}\int_{\varepsilon(Y_3+\mathbf{k})} \widehat{z}_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^h \\
&= \frac{4}{\varepsilon}\,(1-h(\varepsilon))\int_\Omega \widehat{z}_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^h.
\end{aligned} \tag{9.13}$$

Here, $\widehat{z}_\varepsilon$ is a function of $L^2(\Omega, d\lambda_\varepsilon^{h(\varepsilon)})$ taking the same values with the function $z_\varepsilon \in L^2(S_{int}^\varepsilon)$ on S_{int}^ε. We note that the integral $\displaystyle\int_\Omega \widehat{z}_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^{h(\varepsilon)}$ is well defined. Indeed, since the set Ω is bounded and $\widehat{z}_\varepsilon\,\mathrm{d}\lambda_\varepsilon^{h(\varepsilon)}$ is a Radon measure, it follows that $\displaystyle\int_\Omega \widehat{z}_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^{h(\varepsilon)}$ is a linear continuous functional on $C_0^\infty(\mathbb{R}^2)$. Thus,

$$\int_{S_{int}^\varepsilon} z_\varepsilon\varphi\,\mathrm{d}\mathcal{H}^1 = \frac{4}{\varepsilon}\,(1-h(\varepsilon))\int_\Omega \widehat{z}_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^h. \tag{9.14}$$

Remark 9.2. It is easy to see that the scaling measure $\lambda_\varepsilon^{h(\varepsilon)}$ belongs to the class $\mathcal{M}_0^+(\Omega)$. In view of our initial supposition, $\mathrm{cap}(S_{int}) \neq 0$, so $\lambda^h(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\mathrm{cap}(B,\Omega) = 0$. Hence, the first property

of the cone $\mathcal{M}_0^+(\Omega)$ is valid for $\lambda_\varepsilon^{h(\varepsilon)}$. As for the second property, it is true because the measure λ^h is concentrated on S_{int} and proportional there to the linear Lebesgue measure. Thus, $\lambda^h \in \mathcal{M}_0^+(\Omega)$. Since the scaling measure $\lambda_\varepsilon^{h(\varepsilon)}$ inherits these properties from λ^h, it follows that $\lambda_\varepsilon^{h(\varepsilon)} \in \mathcal{M}_0^+(\Omega)$ for every $\varepsilon > 0$. Thus, every function y in $H^1(\Omega)$ is defined $\lambda_\varepsilon^{h(\varepsilon)}$-everywhere and is $\lambda_\varepsilon^{h(\varepsilon)}$-measurable in Ω (see Dal Maso and Murat [82]). Hence, the space $H_0^1(\Omega) \cap L^2(\Omega, d\lambda_\varepsilon^{h(\varepsilon)})$ is well defined.

Taking these facts into account and proceeding by analogy with (9.13), we can obtain the relation

$$\begin{aligned}\int_{\Gamma_\varepsilon^N} w_\varepsilon \varphi \, \mathrm{d}\mathcal{H}^1 &= \sum_{\mathbf{k}\in\Theta_\varepsilon^N} \int_{\varepsilon(\Lambda_N^h+\mathbf{k})} w_\varepsilon \varphi \, \mathrm{d}\mathcal{H}^1 \\ &= 2\,\varepsilon\,(1-h(\varepsilon)) \sum_{\mathbf{k}\in\Theta_\varepsilon^N} \int_{\varepsilon(Y_1+\mathbf{k})} \widehat{w}_\varepsilon \varphi \, \mathrm{d}\nu^h(x/\varepsilon) \\ &= 2\,(1-h(\varepsilon)) \int_\Omega \widehat{w}_\varepsilon \varphi \, \mathrm{d}\nu_\varepsilon^h, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^2). \end{aligned} \tag{9.15}$$

Let $\chi_\varepsilon^h = \chi_{\Omega_\varepsilon}$ be the characteristic function of the network Ω_ε (it is clear that this function is two-parametric, in general). We are now in a position to introduce the non-negative Y-periodic Borel measure η^h such that

$$\mathrm{d}\eta^h = |h(2-h)|^{-1}\chi^h \, \mathrm{d}x, \tag{9.16}$$

where by χ^h we denote the characteristic function of the grid $\mathcal{F}^h$ (here, Y is the sell of periodicity for the grid $\mathcal{F}^h$; see Fig. 9.2(b)). If we define the scaling measure η_ε^h in the usual way, then we immediately obtain

$$|h(\varepsilon)(2-h(\varepsilon))|\mathrm{d}\eta_\varepsilon^h = \chi_\varepsilon^h \, \mathrm{d}x. \tag{9.17}$$

It is clear that this measure is uniformly distributed and concentrated on the thin grid $\mathcal{F}_\varepsilon^h = \varepsilon\mathcal{F}^h$, normalized, and proportional there to the planar Lebesgue measure. Moreover, in this case we have

$$\eta_\varepsilon^h(B) = \frac{\varepsilon^2}{h(2-h)}\mathcal{L}_2(\mathcal{F}^h \cap \varepsilon^{-1}B), \tag{9.18}$$

$$\eta_\varepsilon^h(\varepsilon Y) = \frac{\varepsilon^2}{h(2-h)}\mathcal{L}_2(\mathcal{F}^h \cap Y) = \varepsilon^2 \tag{9.19}$$

for any planar Borel set B. Hence, $\eta_\varepsilon^h(\varepsilon)$ converges weakly to the planar Lebesgue measure as $\varepsilon \to 0$, and for every fixed $\varepsilon > 0$ this measure is absolutely continuous with respect to $\mathcal{L}_2$ (in symbols, $\eta_\varepsilon^h(\varepsilon) \ll \mathcal{L}_2$), that is, $\mathcal{L}_2(B) = 0$ implies $\eta_\varepsilon^h(\varepsilon)(B) = 0$ for all $B \subset \mathbb{R}^2$.

9.2 Statement of the optimal control problem

We consider the following boundary value problem in Ω_ε:

$$\left.\begin{aligned} -\operatorname{div}(A(x/\varepsilon)\nabla y_\varepsilon) + y_\varepsilon &= f_\varepsilon && \text{in } \Omega_\varepsilon,\\ \partial_{\nu_A} y_\varepsilon &= 0 && \text{on } S_{ext},\\ \partial_{\nu_A} y_\varepsilon &= \varepsilon^2(-d y_\varepsilon + g_\varepsilon) && \text{on } S^\varepsilon_{int},\\ y_\varepsilon &= u_\varepsilon && \text{on } \Gamma^D_\varepsilon,\\ \partial_{\nu_A} y_\varepsilon &= \varepsilon p_\varepsilon && \text{on } \Gamma^N_\varepsilon, \end{aligned}\right\} \tag{9.20}$$

where d is a positive constant, $g_\varepsilon \in L^2(S^\varepsilon_{int})$ and $f_\varepsilon \in L^2(\Omega)$ are given functions, $\partial_{\nu_A} y_\varepsilon = (A(x/\varepsilon)\nabla y_\varepsilon)\cdot\nu$, ν is the unit outward normal vector with respect to Ω_ε, and $A(x)$ is a Y-periodic measurable matrix such that

$$\begin{gathered} A(\cdot) \in L^\infty(Y, \mathbb{R}^{2\times 2}),\\ \alpha_0\|\xi\|^2 \le (A(x)\xi,\xi) \le \alpha_0^{-1}\|\xi\|^2 \ \text{ for a.e. } \ x \in Y, \end{gathered} \tag{9.21}$$

for some fixed constant $\alpha_0 > 0$ and every $\xi \in \mathbb{R}^2$.

Hereinafter we interpret the functions u_ε and p_ε as Dirichlet and Neumann boundary controls, respectively. There are two common ways of constraining the control. The first one is to impose an explicit bound on the control. The second way is to add some norm of the control to the cost functional. We will use both of these approaches. We say that the control functions u_ε and p_ε are admissible if the following conditions hold:

$$p_\varepsilon \in L^2(\Gamma^N_\varepsilon), \quad u_\varepsilon \in U_\varepsilon, \tag{9.22}$$

where the set U_ε is defined as

$$U_\varepsilon = \left\{ u|_{\Gamma^D_\varepsilon} \ : \ u \in H^1(\Gamma_0), \quad \|u\|_{H^1(\Gamma_0)} \le \mathbf{C_0} \right\}, \tag{9.23}$$

$\Gamma_0 = \{0 < x_1 < L_1, x_2 = 0\}$, and the constant $\mathbf{C_0}$ is independent of ε.

We suppose that the cost functional takes the form

$$\begin{aligned} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) = \frac{1}{h}\int_{\Omega_\varepsilon} (y_\varepsilon(x) - y^T_\varepsilon)^2 \,\mathrm{d}x\\ + \kappa(\varepsilon)\int_{\Gamma^D_\varepsilon} u^2 \,\mathrm{d}x_1 + \int_{\Gamma^N_\varepsilon} p^2_\varepsilon \,\mathrm{d}\mathcal{H}^1, \end{aligned} \tag{9.24}$$

where $y^T_\varepsilon \in L^2(\Omega)$ is a given function and $\kappa(\varepsilon)$ is a given value.

Thus, the OCP we consider in this section can be stated as follows: Find a triplet $(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) \in H^1(\Gamma^D_\varepsilon) \times L^2(\Gamma^N_\varepsilon) \times H^1(\Omega_\varepsilon)$ such that

$$I_\varepsilon(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) = \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon)\in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon), \tag{9.25}$$

where the set of admissible triplets Ξ_ε is defined as

$$\Xi_\varepsilon = \big\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in H^1(\Gamma_\varepsilon^D) \times L^2(\Gamma_\varepsilon^N) \times H^1(\Omega_\varepsilon) \ : \\ (u_\varepsilon, p_\varepsilon, y_\varepsilon) \text{ satisfies (in some weak sense) (9.20)–(9.23)}\big\}. \tag{9.26}$$

In the sequel, the OCP (9.20)–(9.26) will be called the $\mathbb{P}_\varepsilon$-problem.

It is well known that for every fixed ε and for given

$$f \in L^2(\Omega),\ g_\varepsilon \in L^2(S_{int}^\varepsilon),\ u \in H^1(\Gamma_\varepsilon^D), \text{and } p_\varepsilon \in L^2(\Gamma_\varepsilon^N),$$

the boundary value problem (9.20) admits a unique solution $y_\varepsilon \in H^1(\Omega_\varepsilon)$ such that (see Lions and Magenes [173])

$$y_\varepsilon = u_\varepsilon \ \text{ a.e. on } \ \Gamma_\varepsilon^D \tag{9.27}$$

and the integral identity

$$\int_{\Omega_\varepsilon} A(x/\varepsilon)\nabla y_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x + \int_{\Omega_\varepsilon} y_\varepsilon\varphi \,\mathrm{d}x + d\,\varepsilon^2 \int_{S_{int}^\varepsilon} y_\varepsilon\varphi \,\mathrm{d}\mathcal{H}^1 \\ = \int_{\Omega_\varepsilon} f_\varepsilon\,\varphi \,\mathrm{d}x + \varepsilon^2 \int_{S_{int}^\varepsilon} g_\varepsilon\varphi \,\mathrm{d}\mathcal{H}^1 \\ + \varepsilon \int_{\Gamma_\varepsilon^N} p_\varepsilon\varphi \,\mathrm{d}\mathcal{H}^1, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^2; \Gamma_\varepsilon^D), \tag{9.28}$$

is valid, where we denote by $C_0^\infty(\mathbb{R}^2; \Gamma_\varepsilon^D)$ the set of all functions from $C_0^\infty(\mathbb{R}^2)$ such that $\varphi|_{\Gamma_\varepsilon^D} = 0$.

In the sequel, the function y_ε is called the weak solution to the problem (9.20). It should be emphasized that since the domain Ω_ε is nonconvex with nonsmooth boundary $\partial\Omega_\varepsilon$, any solution of the initial-boundary value problem (9.20) can have only minimal smoothness for ε small enough. It is the main reason for the choice of $H^1(\Omega_\varepsilon)$ as the state space for the original problem. In addition, if u and p_ε are admissible controls, then the weak solution satisfies a priori norm estimate

$$\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le c_1 \big(\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{H^1(\Gamma_\varepsilon^D)} \\ + \sqrt{\varepsilon}\,\|p_\varepsilon\|_{L^2(\Gamma_\varepsilon^N)} + \varepsilon\,\|g_\varepsilon\|_{L^2(S_{int}^\varepsilon)}\big).$$

It is quite obvious that for every $\varepsilon > 0$, the set of admissible solutions Ξ_ε to the $\mathbb{P}_\varepsilon$-problem is a nonempty and convex subset of $H^1(\Gamma_\varepsilon^D) \times L^2(\Gamma_\varepsilon^N) \times H^1(\Omega_\varepsilon)$. As a result, this problem has a unique solution $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ for every $\varepsilon > 0$ (see Sect. 3.3). Moreover, as follows from the a priori norm estimate given above, the $\mathbb{P}_\varepsilon$-problem is uniformly regular:

(H1) $\Xi_\varepsilon \neq \emptyset$ for every $\varepsilon > 0$.

(H2) There is a sequence of admissible solutions $\{(u_\varepsilon^*, p_\varepsilon^*, y_\varepsilon^*) \in \Xi_\varepsilon\}_{\varepsilon>0}$ such that $\limsup_{\varepsilon\to 0} \Big[\|u_\varepsilon^*\|^2_{H^1(\Gamma_\varepsilon^D)} + \|p_\varepsilon^*\|^2_{L^2(\Gamma_\varepsilon^N)} + \|y_\varepsilon^*\|^2_{H^1(\Omega_\varepsilon)}\Big] < +\infty$.

Let us observe that we touch the OCP with two different types of boundary controls. Moreover, the influence zones each of controls are imposed on the different parts of the boundary ($\Gamma_\varepsilon^D \cap \Gamma_\varepsilon^N = \emptyset$). Therefore, one of the distinctive feature of this problem is the fact that the control zones Γ_ε^D and Γ_ε^N have not only a highly oscillating geometrical form and are nonconnected, but also they coincide in the limit, that is,

$$\lim_{\varepsilon\to 0}\Gamma_\varepsilon^D = \Gamma_0 = \lim_{\varepsilon\to 0}\Gamma_\varepsilon^N$$

in the Hausdorff metric. It produces the situation when one of the boundary controls can disappear in the limit as $\varepsilon \to 0$.

On the other hand, the dependence of the $\mathbb{P}_\varepsilon$-problem on the small parameter ε has a rather complicated nature. Indeed, if the parameter ε is changed, then all the components of this control problem (the domain Ω_ε, the constraint set U_ε, the cost functional I_ε, and the set of admissible solutions) and the main functional space

$$H^1(\Gamma_\varepsilon^D) \times L^2(\Gamma_\varepsilon^N) \times H^1(\Omega_\varepsilon)$$

are changed as well. So, in spite of the fact that this problem is rather simple from a control theory point of view, the numerical simulation of its optimal characteristics is very complicated for ε small enough. Therefore, one of the main questions is how to obtain an appropriate approximation for the optimal solutions to the $\mathbb{P}_\varepsilon$-problem for small values of ε.

Taking this into account, we reformulate the original OCP in terms of measures for the thin structure Ω_ε. By analogy with (9.13) and (9.15), it is easy to show that the following representations hold true:

$$d\,\varepsilon^2 \int_{S_{int}^\varepsilon} y_\varepsilon \varphi \, \mathrm{d}\mathcal{H}^1 = 4\varepsilon d\,(1-h(\varepsilon)) \int_\Omega \widehat{y}_\varepsilon \varphi \, \mathrm{d}\lambda_\varepsilon^h, \tag{9.29}$$

$$\varepsilon \int_{\Gamma_\varepsilon^N} p_\varepsilon \varphi \, \mathrm{d}\mathcal{H}^1 = 2\varepsilon\,(1-h(\varepsilon)) \int_\Omega \widehat{p}_\varepsilon \varphi \, \mathrm{d}\nu_\varepsilon^h, \tag{9.30}$$

$$\kappa(\varepsilon) \int_{\Gamma_\varepsilon^D} u^2 \, \mathrm{d}x_1 = \kappa(\varepsilon) h(\varepsilon) \int_\Omega \widehat{u}^2 \, \mathrm{d}\mu_\varepsilon^h, \tag{9.31}$$

$$\int_{\Gamma_\varepsilon^N} p_\varepsilon^2 \, \mathrm{d}\mathcal{H}^1 = 2\,(1-h(\varepsilon)) \int_\Omega \widehat{p}_\varepsilon^2 \, \mathrm{d}\nu_\varepsilon^h, \tag{9.32}$$

$$\varepsilon^2 \int_{S_{int}^\varepsilon} g_\varepsilon \varphi \, \mathrm{d}\mathcal{H}^1 = 4\,\varepsilon\,(1-h(\varepsilon)) \int_\Omega \widehat{g}_\varepsilon \varphi \, \mathrm{d}\lambda_\varepsilon^h. \tag{9.33}$$

Let $\breve{y}_\varepsilon \in L^2(0,T;H^1(\Omega))$ be some extension of the weak solution y_ε to the problem (9.27)–(9.28) on the whole of domain Ω. Then summing up the representations (9.29)–(9.33), the boundary value problem (9.20) can be rewritten in the following variational form:

$$\breve{y}_\varepsilon = \widehat{u}_\varepsilon \quad \mu_\varepsilon^h\text{-a.e. in } \ \Omega, \tag{9.34}$$

$$\int_\Omega \chi_\varepsilon^h (A(x/\varepsilon)\nabla \breve{y}_\varepsilon \cdot \nabla\varphi)\,\mathrm{d}x + \int_\Omega \chi_\varepsilon^h \breve{y}_\varepsilon \varphi\,\mathrm{d}x + 4d\,\varepsilon\,(1-h(\varepsilon))\int_\Omega \breve{y}_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^h$$
$$= \int_\Omega \chi_\varepsilon^h f_\varepsilon\,\varphi\,\mathrm{d}x + 2\,\varepsilon\,(1-h(\varepsilon))\left(\int_\Omega 2\widehat{g}_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^h + \int_\Omega \widehat{p}_\varepsilon\varphi\,\mathrm{d}\nu_\varepsilon^h\right) \quad (9.35)$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2;\Gamma_0)$.

It is easy to see that using the scaling measure η^h with properties (9.16)–(9.19), the corresponding terms in (9.35) can be rewritten as

$$\int_\Omega \chi_\varepsilon^h \breve{y}_\varepsilon\varphi\,\mathrm{d}x = h(\varepsilon)(2-h(\varepsilon))\int_\Omega \widehat{y}_\varepsilon\varphi\,\mathrm{d}\eta_\varepsilon^h,$$
$$\int_\Omega \chi_\varepsilon^h (A(x/\varepsilon)\nabla\breve{y}_\varepsilon\cdot\nabla\varphi)\,\mathrm{d}x = h(\varepsilon)(2-h(\varepsilon))\int_\Omega (A(x/\varepsilon)\nabla\widehat{y}_\varepsilon\cdot\nabla\varphi)\,\mathrm{d}\eta_\varepsilon^h,$$
$$\int_\Omega \chi_\varepsilon^h f_\varepsilon\,\varphi\,\mathrm{d}x = h(\varepsilon)(2-h(\varepsilon))\int_\Omega f_\varepsilon\,\varphi\,\mathrm{d}\eta_\varepsilon^h,$$

where, as usual, we denote by $\widehat{y}_\varepsilon$ a function of $L^2(\Omega, d\eta_\varepsilon^h)$ taking the same values as y_ε on Ω_ε.

Let us define the Sobolev space $H^1(\Omega, d\eta_\varepsilon^h)$ as we did it in Sect. 2.7. We say that a function $\widehat{y}$ belongs to $H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)$ if there exist a vector $\mathbf{z} \in (L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))^2$ and a sequence $\{y_m \in C^\infty(\Omega)\}_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}\int_\Omega (y_m-\widehat{y})^2\,\mathrm{d}\eta_\varepsilon^h = 0, \quad \lim_{m\to\infty}\int_\Omega \left|\nabla y_m - \mathbf{z}\right|^2\mathrm{d}\eta_\varepsilon^h = 0.$$

In this case, one says that $\mathbf{z}$ is a gradient of $\widehat{y}$ and denotes it as $\nabla\widehat{y}$ (i.e., $\mathbf{z}=\nabla\widehat{y}$). In view of the ergodicity property of the measure η_ε^h, this gradient is unique for any $\widehat{y}\in H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)$ (see [254]). Moreover, we may always suppose that for any function $\widehat{y}\in H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)$, one can point out its prototype y in $H^1(\Omega)$ such that $y=\widehat{y}$, $\quad \eta_\varepsilon^h$-a.e. in Ω.

Thus, we may reformulate the $\mathbb{P}_\varepsilon$-optimal control problem as follows: Find a triplet $(\widehat{u}_\varepsilon^{\,0},\widehat{p}_\varepsilon^{\,0},\widehat{y}_\varepsilon^{\,0})$ such that

$$(\widehat{u}_\varepsilon^{\,0},\widehat{p}_\varepsilon^{\,0},\widehat{y}_\varepsilon^{\,0}) \in \mathbb{Z}_\varepsilon \equiv \left[H^1(\Gamma_0)\cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)\right]$$
$$\times\ L^2(\Omega,\mathrm{d}\nu_\varepsilon^h)\times\left[H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)\cap L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)\right],$$

$$\widehat{I}_\varepsilon(\widehat{u}_\varepsilon^{\,0},\widehat{p}_\varepsilon^{\,0},\widehat{y}_\varepsilon^{\,0}) = \inf_{(\widehat{u}_\varepsilon,\widehat{p}_\varepsilon,\widehat{y}_\varepsilon)\in\,\widehat{\Xi}_\varepsilon}\widehat{I}_\varepsilon(\widehat{u}_\varepsilon,\widehat{p}_\varepsilon,\widehat{y}_\varepsilon), \quad (9.36)$$

where the cost functional $\widehat{I}_\varepsilon$ and the set of admissible solutions $\widehat{\Xi}_\varepsilon$ have the following analytical representation:

$$\widehat{I}_\varepsilon(\widehat{u}_\varepsilon,\widehat{p}_\varepsilon,\widehat{y}_\varepsilon) = (2-h(\varepsilon))\int_\Omega (\widehat{y}_\varepsilon - y_\varepsilon^T)^2\,\mathrm{d}\eta_\varepsilon^h$$
$$+\,\kappa(\varepsilon)h(\varepsilon)\int_\Omega \widehat{u}_\varepsilon^2\,\mathrm{d}\mu_\varepsilon^h + 2\,(1-h(\varepsilon))\int_\Omega \widehat{p}_\varepsilon^2\,\mathrm{d}\nu_\varepsilon^h, \quad (9.37)$$

$$\widehat{\Xi}_\varepsilon = \left\{ (\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon) \;\middle|\; \begin{array}{c} \widehat{y}_\varepsilon = \widehat{u}_\varepsilon \quad \mu_\varepsilon^h\text{-a.e. in } \ \Omega, \\ \|\widehat{u}_\varepsilon\|_{H^1(\Gamma_0)} \le \mathbf{C}_0, \\ \displaystyle\int_\Omega (A(x/\varepsilon)\nabla\widehat{y}_\varepsilon \cdot \nabla\varphi)\,\mathrm{d}\eta_\varepsilon^h + \int_\Omega \widehat{y}_\varepsilon\varphi\,\mathrm{d}\eta_\varepsilon^h \\ +4d\,\beta(\varepsilon)\displaystyle\int_\Omega \widehat{y}_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^h = \int_\Omega f_\varepsilon\,\varphi\,\mathrm{d}\eta_\varepsilon^h \\ +4\,\beta(\varepsilon)\displaystyle\int_\Omega \widehat{g}_\varepsilon\varphi\,\mathrm{d}\lambda_\varepsilon^h + 2\,\beta(\varepsilon)\int_\Omega \widehat{p}_\varepsilon\varphi\,\mathrm{d}\nu_\varepsilon^h, \\ \forall\,\varphi \in C_0^\infty(\mathbb{R}^2; \Gamma_\varepsilon^D) \end{array} \right\}. \tag{9.38}$$

Here,

$$\beta(\varepsilon) = \frac{1-h(\varepsilon)}{2-h(\varepsilon)} \cdot \left[\frac{\varepsilon}{h(\varepsilon)}\right].$$

In what follows, the OCP (9.36)–(9.38) will be called the $\widehat{\mathbb{P}}_\varepsilon$-problem. It is quite obvious that this problem has a unique solution $(\widehat{u}_\varepsilon^0, \widehat{p}_\varepsilon^{\,0}, \widehat{y}_\varepsilon^{\,0})$ for every $\varepsilon > 0$ (see Theorem 3.2; see also Fursikov [111] and Lions [169]). This solution can be interpreted as a prototype of an optimal triplet to the $\mathbb{P}_\varepsilon$-problem. Moreover, in this case, the a priori norm estimate takes the form

$$\begin{aligned} \|\widehat{y}_\varepsilon\|_{H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)} + \|\widehat{y}_\varepsilon\|_{L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)} \le \widehat{C}\big[&\|f_\varepsilon\|_{L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)} + \|\widehat{p}_\varepsilon\|_{L^2(\Omega,\mathrm{d}\nu_\varepsilon^h)} \\ &+ \|\widehat{u}_\varepsilon\|_{H^1(\Gamma_0)\cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)} \\ &+ \|\widehat{g}_\varepsilon\|_{L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)}\big], \end{aligned} \tag{9.39}$$

where the constant $\widehat{C}$ is independent of $\varepsilon > 0$.

We proceed to the asymptotic behavior of the problem $\widehat{\mathbb{P}}_\varepsilon$ as $\varepsilon \to 0$. Note that, in contrast to the usual algorithm of homogenization of OCPs (see Conca, Osses, and Saint Jean Paulin [72], Kesavan and Saint Jean Paulin [135], and Saint Jean Paulin and Zoubairi [222]), we do not look for the limit of optimal solutions and do not study the asymptotic behavior of the corresponding necessary optimality conditions as ε tends to 0. Our approach to the homogenization of OCPs is based on the ideas of the epi-convergence theory in variable spaces (see Sect. 8.4). With this aim, we represent the $\widehat{\mathbb{P}}_\varepsilon$-problem for various values of ε in the form of the sequence of corresponding constrained minimization problems

$$\left\{ \left\langle \inf_{(\widehat{u}_\varepsilon,\widehat{p}_\varepsilon,\widehat{y}_\varepsilon)\,\in\,\widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon) \right\rangle ; \varepsilon > 0 \right\}, \tag{9.40}$$

where the cost functional $\widehat{I}_\varepsilon : \Sigma_\varepsilon \to \mathbb{R}$ and the sets of admissible solutions $\widehat{\Xi}_\varepsilon$ are defined by (9.37) and (9.38), respectively. Then the definition of an appropriate limiting OCP to the family (9.36) can be reduced to the analysis of the limit properties of the sequences (9.40) as $\varepsilon \to 0$. To obtain this limit in the form of some constrained minimization problem, we follow Definition 8.22

with respect to the scale of spaces $\{\mathbb{Z}_\varepsilon\}_{\varepsilon>0}$. In view of this, one has to define a concept of convergence in variable spaces $\mathbb{Z}_\varepsilon$ such that the sequence of optimal solutions is relatively compact. This question is discussed in the next section.

9.3 Convergence in the variable space $\mathbb{Z}_\varepsilon$

We recall that the measure η_ε^h introduced earlier is a non-degenerate 2-connected Borel measure in $\mathbb{R}^2$, absolutely continuous with respect to the planar Lebesgue measure (see Definition 6.33). Therefore, we may suppose that for a bounded sequence $\{y_\varepsilon^h \in H^1(\Omega, \mathrm{d}\eta_\varepsilon^h)\}$, there is an element $y \in H^1(\Omega)$ such that $y_\varepsilon^h \rightharpoonup z$ in $L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$ in the sense of Definition 6.14. Moreover, a sequence of the so-called prototypes $\{\widehat{y}_\varepsilon{}^h \in H^1(\Omega)\}_{\varepsilon>0}$ can be found such that

$$\widehat{y}_\varepsilon{}^h \rightharpoonup y \ \text{ in } \ H^1(\Omega) \quad \text{ and } \ y_\varepsilon^h = \widehat{y}_\varepsilon{}^h, \quad \eta_\varepsilon^h\text{-a.e. in } \ \Omega. \tag{9.41}$$

Occasionally, we will identify the elements of $H^1(\Omega, \mathrm{d}\eta_\varepsilon^h)$ with their prototypes in $H^1(\Omega)$.

Remark 9.3. A representation like (9.41) can be explained also by results of Bachvalov and Panasenko [14], Cioranescu and Saint Jean Paulin [66], and Donato and Nabil [96]. It is proved in [14] and [96] that for every fixed h, there exists a uniformly bounded family of extension operators $\{P_\varepsilon^h \in \mathcal{L}(H^1(\Omega, \mathrm{d}\eta_\varepsilon^h), H^1(\Omega))\}$ such that

$$P_\varepsilon^h y_\varepsilon^h \rightharpoonup y^h \quad \text{ weakly in } \ H^1(\Omega) \ \text{ as } \ \varepsilon \to 0.$$

As a result, we have that if $y_\varepsilon^h \rightharpoonup y$ in $L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$ and $y \in H^1(\Omega)$, then $y^h \rightharpoonup y$ weakly in $H^1(\Omega)$ as $h \to 0$.

In view of this, we introduce the following concept.

Definition 9.4. *Let $\{y_\varepsilon^h\}_{\varepsilon>0}$ be a bounded sequence in the variable space*

$$\mathbb{M}_\varepsilon = H^1(\Omega, \mathrm{d}\eta_\varepsilon^h) \cap L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h),$$

that is,

$$\limsup_{\varepsilon\to 0} \left[\|y_\varepsilon^{h(\varepsilon)}\|^2_{H^1(\Omega, \mathrm{d}\eta_\varepsilon^{h(\varepsilon)})} + \|y_\varepsilon^{h(\varepsilon)}\|^2_{L^2(\Omega, \mathrm{d}\lambda_\varepsilon^{h(\varepsilon)})} \right] < +\infty.$$

We say that this sequence is weakly convergent in $\mathbb{M}_\varepsilon$ if there exists a function $y \in H^1(\Omega)$ such that

$$\begin{cases} y_\varepsilon^h \to y & \text{in } \ L^2(\Omega, \mathrm{d}\eta_\varepsilon^h), \\ y_\varepsilon^h \rightharpoonup y & \text{in } \ L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h), \\ \displaystyle\int_\Omega |y_\varepsilon^h - y|^2 \, \mathrm{d}\eta_\varepsilon^h \to 0 \ \text{as } \ \varepsilon \to 0. \end{cases} \tag{9.42}$$

We have the following lemma.

Lemma 9.5. *The inequality*

$$\int_\Omega y^2 \, \mathrm{d}\eta_\varepsilon^h \le C_1 \int_\Omega (y^2 + \varepsilon^2 |\nabla y|^2) \, \mathrm{d}x \tag{9.43}$$

holds for every $y \in C_0^\infty(\mathbb{R}^2)$ *uniformly in* $h > 0$*, where the constant* C_1 *depends on the geometry of the singular grid* $\mathcal{F}$ *and on the diameter of the domain* Ω*.*

Inequality (9.43) can be established in full analogy with Lemma 6.58.

Remark 9.6. Since η_ε^h is absolutely continuous with respect to the planar Lebesgue measure ($\eta_\varepsilon^h(\varepsilon) \ll \mathcal{L}_2$) and $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it follows that inequality (9.43) can be extended to functions $y \in H_0^1(\Omega)$.

To begin, let us apply this inequality in order to establish some specific properties of the convergence in the spaces $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$ and $L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$.

Lemma 9.7. *If* $y \in H^1(\Omega)$*, then*

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_\Omega y^2 \, \mathrm{d}\eta_\varepsilon^h &= \int_\Omega y^2 \, \mathrm{d}x, \\ \lim_{\varepsilon \to 0} \int_\Omega y\varphi \, \mathrm{d}\eta_\varepsilon^h &= \int_\Omega y\varphi \, \mathrm{d}x, \quad \forall\, \varphi \in C_0^\infty(\Omega). \end{aligned} \tag{9.44}$$

Moreover,

$$\text{if} \quad \sup_{\varepsilon > 0} \int_\Omega y^2 \, \mathrm{d}\lambda_\varepsilon^h < +\infty, \quad \text{then} \quad y \rightharpoonup y \quad \text{in} \quad L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h). \tag{9.45}$$

Proof. If $y \in C(\Omega)$, then the weak convergence of the measures $\mathrm{d}\lambda_\varepsilon^h \rightharpoonup \mathrm{d}x$ and $\mathrm{d}\eta_\varepsilon^h \rightharpoonup \mathrm{d}x$ immediately implies relations (9.44) and (9.45) (see Definition 2.29). Let y be an arbitrary element of $H_0^1(\Omega)$. Then due to Lusin's theorem, a sequence $\{y^\delta\}$ can be found such that

$$y^\delta \in C_0^\infty(\mathbb{R}^2), \quad \|y - y^\delta\|_{H_0^1(\Omega)} < \delta. \tag{9.46}$$

We have

$$\begin{aligned} \left| \int_\Omega y^2 \, \mathrm{d}\eta_\varepsilon^h - \int_\Omega y^2 \, \mathrm{d}x \right| &\le \int_\Omega \left| y^2 - (y^\delta)^2 \right| \, \mathrm{d}\eta_\varepsilon^h \\ &\quad + \left| \int_\Omega (y^\delta)^2 \, \mathrm{d}\eta_\varepsilon^h - \int_\Omega (y^\delta)^2 \, \mathrm{d}x \right| \\ &\quad + \left| \int_\Omega y^2 \, dx - \int_\Omega (y^\delta)^2 \, \mathrm{d}x \right| = J_1 + J_2 + J_3. \end{aligned}$$

It is clear that due to the weak convergence of the measures $\mathrm{d}\eta_\varepsilon^h \rightharpoonup \mathrm{d}x$ and condition (9.46), we have $J_2 + J_3 \le d_1 \delta$ for some constant $d_1 > 0$.

As for the value J_1, we apply inequality (9.43) to the function $y - y^\delta$. As a result,

$$\int_\Omega \left(y - y^\delta\right)^2 \, \mathrm{d}\eta_\varepsilon^h \le C_1 \int_\Omega \left(\left(y - y^\delta\right)^2 + \varepsilon^2 |\nabla y - \nabla y^\delta|^2\right) \mathrm{d}x,$$

$$\int_\Omega \left|y^2 - \left(y^\delta\right)^2\right| \mathrm{d}\eta_\varepsilon^h \le \left(\int_\Omega |y + y^\delta|^2 \mathrm{d}\eta_\varepsilon^h\right)^{1/2} \left(\int_\Omega |y - y^\delta|^2 \mathrm{d}\eta_\varepsilon^h\right)^{1/2} < C_2\delta,$$

and hence,

$$\limsup_{\varepsilon \to 0} \left| \int_\Omega y^2 \, \mathrm{d}\eta_\varepsilon^h - \int_\Omega y^2 \, \mathrm{d}x \right| \le C\delta$$

for an arbitrary δ, where the constant C does not depend on δ. This implies $(9.44)_1$. In order to prove $(9.44)_2$, we apply similar arguments:

$$\begin{aligned} \left| \int_\Omega y\varphi \, \mathrm{d}\eta_\varepsilon^h - \int_\Omega y\varphi \, \mathrm{d}x \right| &\le \left| \int_\Omega (y - y^\delta)\varphi \, \mathrm{d}\eta_\varepsilon^h \right| \\ &+ \left| \int_\Omega y^\delta \varphi \, \mathrm{d}\eta_\varepsilon^h - \int_\Omega y^\delta \varphi \, \mathrm{d}x \right| \\ &+ \left| \int_\Omega (y - y^\delta)\varphi \, \mathrm{d}x \right| = I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 \le C \left(\int_\Omega (y - y^\delta)^2 \, \mathrm{d}\eta_\varepsilon^h \right)^{1/2} \overset{\text{by } (9.44)_1}{\le} C\delta,$$

$$I_3 \le C \left(\int_\Omega (y - y^\delta)^2 \, \mathrm{d}x \right)^{1/2} \overset{\text{by } (9.46)}{\le} C\delta,$$

$$I_2 \to 0 \ \text{ is the weak convergence } \ \mathrm{d}\eta_\varepsilon^h \rightharpoonup \mathrm{d}x.$$

In order to replace $H_0^1(\Omega)$ by $H^1(\Omega)$ in relation (9.44), we observe that the weak convergence $\mathrm{d}\eta_\varepsilon^h \rightharpoonup \mathrm{d}x$ implies

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi \, \mathrm{d}\eta_\varepsilon^h = \int_\Omega \varphi \, \mathrm{d}x, \quad \forall\, \varphi \in C(\overline{\Omega})$$

(see Lemma 6.62). It follows that it is enough to consider $y^\delta \in C^\infty(\overline{\Omega})$ in (9.46) to deduce relation (9.44) for $y \in H^1(\Omega)$. The remaining reasoning is preserved.

As for property (9.45), we prove it by contradiction. Indeed, by the criterium of the weak convergence in $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h$, there exists en element $z \in L^2(\Omega)$ such that, up to a subsequence,

$$\lim_{\varepsilon \to 0} \int_\Omega y\varphi \, \mathrm{d}\lambda_\varepsilon^h = \int_\Omega z\varphi \, \mathrm{d}x, \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{9.47}$$

Let us assume that for a fixed $\varphi \in C_0^\infty(\Omega)$, we have

$$\left| \int_\Omega z\varphi \, \mathrm{d}x - \int_\Omega y\varphi \, \mathrm{d}x \right| = \delta > 0. \tag{9.48}$$

Following Ziemer [267], we can identify the function $y \in H^1(\Omega)$ with its quasi-continuous representative (for the details, we refer to Sect. 2.5). Since the set S_{int}^ε, as a support of the measure λ_ε^h, has nonzero capacity but zero Lebesgue measure $\mathcal{L}_2$, it follows that $y \in H^1(\Omega)$ is a continuous function for all $x \in \Omega$ except for a subset with $\mathrm{d}\lambda_\varepsilon^h$-measure zero. Then for a given $\varphi \in C_0^\infty(\Omega)$, there exist an $\varepsilon_1 > 0$ and a function $u^\delta \in C_0^\infty(\Omega)$ such that

$$\left| \int_\Omega y\varphi \, \mathrm{d}\lambda_\varepsilon^h - \int_\Omega u^\delta \varphi \, \mathrm{d}\lambda_\varepsilon^h \right| < \delta/3, \quad \forall\, \varepsilon \in (0, \varepsilon_1). \tag{9.49}$$

Moreover, by the smooth property of u^δ and the weak convergence of the measures $\mathrm{d}\lambda_\varepsilon^h \rightharpoonup \mathrm{d}x$, there exists an $\varepsilon_2 > 0$ such that the estimate

$$\left| \int_\Omega u^\delta \varphi \, \mathrm{d}x - \int_\Omega u^\delta \varphi \, \mathrm{d}\lambda_\varepsilon^h \right| < \delta/3 \tag{9.50}$$

holds true for all $\varepsilon \in (0, \varepsilon_2)$. Further, we note that (9.47) implies the following inequality:

$$\left| \int_\Omega z\varphi \, \mathrm{d}x - \int_\Omega y\varphi \, \mathrm{d}\lambda_\varepsilon^h \right| < \delta/3, \tag{9.51}$$

which holds true for every $\varepsilon \in (0, \varepsilon_3)$ with some $\varepsilon_3 > 0$.

As a result, combining the estimates (9.49)–(9.51), we obtain

$$\begin{aligned} \left| \int_\Omega z\varphi \, \mathrm{d}x - \int_\Omega y\varphi \, \mathrm{d}x \right| &\le \left| \int_\Omega z\varphi \, \mathrm{d}x - \int_\Omega y\varphi \, \mathrm{d}\lambda_\varepsilon^h \right| \\ &+ \left| \int_\Omega u^\delta \varphi \, \mathrm{d}x - \int_\Omega u^\delta \varphi \, \mathrm{d}\lambda_\varepsilon^h \right| \\ &+ \left| \int_\Omega y\varphi \, \mathrm{d}\lambda_\varepsilon^h - \int_\Omega u^\delta \varphi \, \mathrm{d}\lambda_\varepsilon^h \right| \\ &< \delta, \quad \forall\, \varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, \end{aligned}$$

which comes into conflict with (9.48). Thus, $z = y$ a.e. in Ω, and this completes the proof. □

We now consider a more delicate situation.

Lemma 9.8. *Let* $\{y_\varepsilon \in H^1(\Omega)\}_{\varepsilon>0}$ *and* $y \in H^1(\Omega)$ *be such that* $y_\varepsilon \rightharpoonup y$ *in* $H^1(\Omega)$ *and* $\sup_{\varepsilon>0} \int_\Omega y^2 \, \mathrm{d}\lambda_\varepsilon^h < +\infty$. *Then*

$$y_\varepsilon \rightharpoonup y \quad \text{in} \quad L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h), \quad y_\varepsilon \to y \quad \text{in} \quad L^2(\Omega, \mathrm{d}\eta_\varepsilon^h). \tag{9.52}$$

Proof. By the criterium of strong convergence in $L^2(\Omega, d\eta_\varepsilon^h)$ and the property of lower semicontinuity, it is enough to show that

$$\lim_{\varepsilon\to 0}\int_\Omega y_\varepsilon\varphi\,\mathrm{d}\eta_\varepsilon^h = \int_\Omega y\varphi\,\mathrm{d}x, \quad \forall\,\varphi\in C_0^\infty(\Omega), \tag{9.53}$$

$$\lim_{\varepsilon\to 0}\int_\Omega |y_\varepsilon|^2\,\mathrm{d}\eta_\varepsilon^h \le \int_\Omega |y|^2\,\mathrm{d}x. \tag{9.54}$$

Let $\{y^\delta\}$ be the sequence with properties (9.46). Then

$$\begin{aligned}\int_\Omega y_\varepsilon\varphi\,\mathrm{d}\eta_\varepsilon^h - \int_\Omega y\varphi\,\mathrm{d}x &= \int_\Omega (y_\varepsilon - y)\varphi\,\mathrm{d}\eta_\varepsilon^h \\ &+ \int_\Omega (y - y^\delta)\varphi\,\mathrm{d}\eta_\varepsilon^h + \left(\int_\Omega y^\delta\varphi\,\mathrm{d}\eta_\varepsilon^h - \int_\Omega y^\delta\varphi\,\mathrm{d}x\right) \\ &+ \int_\Omega (y^\delta - y)\varphi\,\mathrm{d}x = J_1 + J_2 + J_3 + J_4.\end{aligned}$$

Applying estimate (9.46) and inequality (9.43) to J_2 and J_4, we obtain

$$\left|\int_\Omega (y - y^\delta)\varphi\,\mathrm{d}\eta_\varepsilon^h\right| < C_\varphi\,\delta, \quad \left|\int_\Omega (y^\delta - y)\varphi\,\mathrm{d}x\right| < C_\varphi\,\delta.$$

At the same time, the limits $\lim_{\varepsilon\to 0} J_1(\varepsilon)$ and $\lim_{\varepsilon\to 0} J_3(\varepsilon)$ are equal to 0 since

$$\begin{aligned}\int_\Omega (y_\varepsilon - y)^2\,\mathrm{d}\eta_\varepsilon^h &\le C_1\int_\Omega \left[(y_\varepsilon - y)^2 + \varepsilon|\nabla(y_\varepsilon - y)|^2\right]\mathrm{d}x \\ &= C_1\int_\Omega (y_\varepsilon - y)^2\,\mathrm{d}x + O(\varepsilon^2),\end{aligned} \tag{9.55}$$

and

$$\int_\Omega y^\delta\varphi\,\mathrm{d}\eta_\varepsilon^h \to \int_\Omega y^\delta\varphi\,\mathrm{d}x \quad \text{due to the weak convergence} \quad \mathrm{d}\eta_\varepsilon^h \to \mathrm{d}x.$$

Thus,

$$\limsup_{\varepsilon\to 0}\left|\int_\Omega y_\varepsilon\varphi\,\mathrm{d}\eta_\varepsilon^h - \int_\Omega y\varphi\,\mathrm{d}x\right| \le C_\varphi\,\delta.$$

Since the parameter δ is arbitrary, relation (9.53) is valid. As for (9.54), inequality (9.55) and Lemma 9.7 imply that

$$\lim_{\varepsilon\to 0}\int_\Omega |y_\varepsilon|^2\,\mathrm{d}\eta_\varepsilon^h \le \lim_{\varepsilon\to 0}\int_\Omega |y_\varepsilon - y|^2\,\mathrm{d}\eta_\varepsilon^h + \lim_{\varepsilon\to 0}\int_\Omega |y|^2\,\mathrm{d}\eta_\varepsilon^h = \int_\Omega |y|^2\,\mathrm{d}x,$$

and we obtain the required result. The first relation in (9.52) immediately follows from (9.45) and the strong convergence $y_\varepsilon \to y$ in $L^2(\Omega)$. □

As an obvious consequence of these lemmas, we have the following assertion.

Corollary 9.9. *If $y_\varepsilon \to y$ in $L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$ and $y \in H^1(\Omega)$, then*

$$\lim_{\varepsilon \to 0} \int_\Omega |y_\varepsilon - y|^2 \, \mathrm{d}\eta_\varepsilon^h = 0. \tag{9.56}$$

Indeed, to prove this statement, we note that $(y_\varepsilon - y) \to 0$ in $L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$. Then it is enough to refer to the criterium of the strong convergence in $L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$.

Lemma 9.10. *Let $\{y_\varepsilon \in L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)\}$ and $\{\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^2)\}$ be sequences satisfying the condition $y_\varepsilon \rightharpoonup y$ in $L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$ and $\varphi_\varepsilon \to \varphi$ in $C(\Omega)$. Then*

$$\lim_{\varepsilon \to 0} \int_\Omega y_\varepsilon \varphi_\varepsilon \, \mathrm{d}\lambda_\varepsilon^h = \int_\Omega y\varphi \, \mathrm{d}x. \tag{9.57}$$

Proof. Let us rewrite the left-hand side of (9.57) in the form

$$\int_\Omega y_\varepsilon \varphi_\varepsilon \, \mathrm{d}\lambda_\varepsilon^h = \int_\Omega y_\varepsilon (\varphi_\varepsilon - \varphi) \, \mathrm{d}\lambda_\varepsilon^h + \int_\Omega y_\varepsilon \varphi \, \mathrm{d}x.$$

Since $\mathrm{d}\lambda_\varepsilon^h \rightharpoonup \mathrm{d}x$ and

$$\left\| \int_\Omega y_\varepsilon (\varphi_\varepsilon - \varphi) \, \mathrm{d}\lambda_\varepsilon^h \right\| \le \left(\int_\Omega y_\varepsilon^2 \, \mathrm{d}\lambda_\varepsilon^h \right)^{1/2} \|\varphi_\varepsilon - \varphi\|_{C(\Omega)} \left(\int_\Omega \mathrm{d}\lambda_\varepsilon^h \right)^{1/2},$$

we immediately obtain the required conclusion. □

Note that by analogy one can prove similar results with respect to the measure η_ε^h.

Further, we make use the concept of the two-scale convergence in the variable space $L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$, where $h = h(\varepsilon)$ and $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ (for general properties of the two-scale convergence, we refer to Sect. 6.3). Following the main properties of the two-scale convergence and the representative formula (9.16), we have the following obvious result (see Theorem 7.19 for comparison).

Proposition 9.11. *Let χ_ε^h and χ^h be the characteristic functions of the network Ω_ε and the grid $\mathcal{F}^h$, respectively. Then*

$$\chi^h \rightharpoonup 1 \quad \text{in} \quad L^2(Y, \mathrm{d}\eta^h), \quad \chi_\varepsilon^h \xrightarrow{2} 1, \quad \chi_\varepsilon^h \longrightarrow 1 \quad \text{in} \quad L^2(\Omega, \mathrm{d}\eta_\varepsilon^h).$$

By analogy, similar results can be obtained with respect to the limiting behavior of the characteristic functions $\chi_{\Gamma_\varepsilon^D}$ and $\chi_{\Gamma_\varepsilon^N}$. Indeed, using the facts that the measures ν^h and μ^h are Y_1- and Y_2-periodic, respectively, and $\nu^h \rightharpoonup \nu$ and $\mu^h \rightharpoonup \mu$, where the limit measures ν and μ are defined as

$$\int_{Y_1} \varphi \, \mathrm{d}\nu = \frac{1}{2}\left[\int_0^{1/2} \varphi(0,x_2)\,\mathrm{d}x_2 + \int_0^1 \varphi(x_1,\frac{1}{2})\,\mathrm{d}x_1 + \int_0^{1/2} \varphi(1,x_2)\,\mathrm{d}x_2\right],$$

$$\int_{Y_2} \varphi \, \mathrm{d}\mu = \int_{Y_2} \varphi \, \delta_{\{x_1=0,\ x_2=0\}}\,\mathrm{d}x = \varphi(0,0),$$

we have the following conclusion.

Proposition 9.12. *We have*

$$\chi_{\Gamma_\varepsilon^D} \longrightarrow 1 \quad in \quad L^2(\Omega,\mathrm{d}\mu_\varepsilon^h), \quad \chi_{\Gamma_\varepsilon^N} \longrightarrow 1 \quad in \quad L^2(\Omega,\mathrm{d}\nu_\varepsilon^h).$$

We now observe that since $\mathrm{d}\eta_\varepsilon^h \rightharpoonup \mathrm{d}x$ as $\varepsilon \to 0$, it follows that

$$\int_\Omega \varphi \, \frac{\chi_{\Omega_\varepsilon}}{h(\varepsilon)(2-h(\varepsilon))}\,\mathrm{d}x \equiv \int_\Omega \varphi \, \mathrm{d}\eta_\varepsilon^h \longrightarrow \int_\Omega \varphi \, \mathrm{d}x \text{ as } \varepsilon \to 0, \quad \forall\, \varphi \in C_0(\Omega).$$

However, this fact can be always interpreted as the property of the weak-$*$ convergence in $L^\infty(\Omega)$ of $[h(\varepsilon)(2-h(\varepsilon))]^{-1}\chi_{\Omega_\varepsilon} \in L^\infty(\Omega)$ to 1. Thus, taking this proposition into account, we can supplement it in the following way.

Proposition 9.13.

$$[h(\varepsilon)(2-h(\varepsilon))]^{-1}\chi_{\Omega_\varepsilon} \overset{*}{\rightharpoonup} 1 \qquad in \quad \Omega,$$

$$h^{-1}(\varepsilon)\chi_{\Gamma_\varepsilon^D} \overset{*}{\rightharpoonup} 1 \qquad in \quad \Gamma_0,$$

$$\int_\Omega \varphi \, [2(1-h(\varepsilon))]^{-1}\chi_{\Omega_\varepsilon}\,\mathrm{d}x \longrightarrow \int_{\Gamma_0} \varphi\,\mathrm{d}x_1 \ as \ \ \varepsilon \to 0, \ \forall \in \varphi \in C(\overline{\Omega}),$$

$$\int_{\Gamma_\varepsilon^D \cup \Gamma_\varepsilon^N} \varphi \, \mathrm{d}\mathcal{H}^1 \longrightarrow \int_{\Gamma_0} \varphi\,\mathrm{d}x_1 \ as \ \ \varepsilon \to 0, \ \forall \in \varphi \in C(\overline{\Omega}).$$

Let us introduce the spaces of periodic vector functions with singular Y-periodic measure η: $L^2(\Omega, V_{pot})$ and $L^2(\Omega, V_{sol})$ (see Sect. 6.4.2). In this case, the following orthogonal expansion holds true (see properties (6.70)):

$$L^2\left(\Omega,[L^2(Y,\mathrm{d}\eta)]^2\right) = L^2(\Omega,V_{pot}) \oplus L^2(\Omega,V_{sol}). \tag{9.58}$$

As a result one has the following.

Theorem 9.14. *Let* $\left\{z_\varepsilon^h\right\}_{\varepsilon>0}$ *be a bounded sequence in* $H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)$*, that is,*

$$\sup_{\varepsilon\to 0}\left[\|z_\varepsilon^h\|^2_{L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)} + \|\nabla z_\varepsilon^h\|^2_{[L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)]^2}\right] < \infty.$$

Then there are a subsequence $\left\{z_\varepsilon^h\right\}_{\varepsilon>0}$ *(still indexed by* ε*) and functions* $\mathbf{r}(x,y) \in \left[L^2(\Omega\times Y)\right]^2$ *and* $z(x) \in H^1(\Omega)$ *such that*

$$z_\varepsilon^h \overset{2}{\rightharpoonup} z(x), \tag{9.59}$$

$$\nabla z_\varepsilon^h \overset{2}{\rightharpoonup} \mathbf{r}(x,y), \tag{9.60}$$

$$\mathbf{r}(x,y) - \nabla z(x) \in L^2(\Omega, V_{pot}). \tag{9.61}$$

Proof. In view of the compactness property of two-scale convergence (see Proposition 6.26 and its generalization in Sect. 6.6), we may assume, without loss of generality, the existence of elements $z \in L^2(\Omega \times Y; \mathrm{d}x \times \mathrm{d}\eta)$ and $\mathbf{r} \in \left[L^2(\Omega \times Y; \mathrm{d}x \times \mathrm{d}\eta)\right]^2$ such that

$$z_\varepsilon^h \overset{2}{\rightharpoonup} z(x,y) \quad \text{and} \quad \nabla z_\varepsilon^h \overset{2}{\rightharpoonup} \mathbf{r}(x,y). \tag{9.62}$$

Let $\mathbf{b}$ be a vector-valued function such that $\mathbf{b} \in \left[L^2(Y, \mathrm{d}\eta)\right]^2$. We may assume that for this function there is an element $a \in L^2(Y, \mathrm{d}\eta)$ satisfying the relation

$$-\int_Y a\varphi \,\mathrm{d}\eta = \int_Y (\mathbf{b}, \nabla\varphi)\,\mathrm{d}\eta, \ \forall\, \varphi \in C^\infty_{per}(Y). \tag{9.63}$$

Then taking $\varphi \in C_0^\infty(\Omega)$ as a test function and using the equality $\nabla(\varphi z_\varepsilon^h) = \varphi \nabla z_\varepsilon^h + z_\varepsilon^h \nabla\varphi$, we have the following relations:

$$\begin{aligned}
&\varepsilon \int_\Omega \left(\nabla z_\varepsilon^h \varphi, \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\eta_\varepsilon^h \\
&\qquad = \varepsilon \int_\Omega \left(\nabla(\varphi z_\varepsilon^h), \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\eta_\varepsilon^h - \varepsilon \int_\Omega z_\varepsilon^h \left(\nabla\varphi, \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\eta_\varepsilon^h, \qquad (9.64)\\
&\qquad\quad - \int_\Omega a(\varepsilon^{-1}x)\varphi \,\mathrm{d}\eta_\varepsilon^h = \varepsilon \int_\Omega (\mathbf{b}(\varepsilon^{-1}x), \nabla\varphi)\,\mathrm{d}\eta_\varepsilon^h. \qquad (9.65)
\end{aligned}$$

Therefore, applying (9.65) to (9.64), we obtain

$$\begin{aligned}
&\varepsilon \int_\Omega \varphi \left(\nabla z_\varepsilon^h, \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\eta_\varepsilon^h \\
&\qquad = -\int_\Omega a(\varepsilon^{-1}x) z_\varepsilon^h \varphi \,\mathrm{d}\eta_\varepsilon^h - \varepsilon \int_\Omega z_\varepsilon^h \left(\nabla\varphi, \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\eta_\varepsilon^h. \qquad (9.66)
\end{aligned}$$

However, in view of the initial assumptions and the definition of two-scale convergence, we obtain

$$\lim_{\varepsilon\to 0} \varepsilon \parallel \nabla z_\varepsilon^h \parallel_{(L^2(\Omega, \mathrm{d}\eta_\varepsilon^h))^2} = 0, \quad \lim_{\varepsilon\to 0} \varepsilon \int_\Omega \varphi \left(\nabla z_\varepsilon^h, \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\eta_\varepsilon^h = 0,$$

$$\lim_{\varepsilon\to 0} \varepsilon \int_\Omega z_\varepsilon^h \left(\nabla\varphi, \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\eta_\varepsilon^h = 0.$$

Therefore, passing to the limit in (9.66), we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega a(\varepsilon^{-1}x) z_\varepsilon^h \varphi \,\mathrm{d}\eta_\varepsilon^h = \int_\Omega \int_Y a(y) z(x,y)\,\mathrm{d}\eta(y)\varphi\,\mathrm{d}x = 0. \tag{9.67}$$

Due to Theorem 6.41 and the ergodicity property of the measure η, the set of all functions $a \in L^2(Y, \mathrm{d}\eta)$ satisfying condition (9.63) with $\mathbf{b}$ as an element of $\left[L^2(Y, \mathrm{d}\eta)\right]^2$ is dense in the subspace of functions in $L^2(Y, \mathrm{d}\eta)$ with mean value 0. Thus, from (9.67) we immediately conclude that

$$z(x,y) = z(x),$$

that is, the weak two-scale limit $z(x,y)$ in (9.62) is independent of y.

It remains to show that condition (9.61) is satisfied. To this end, we consider (9.64) with a vector $\mathbf{b} \perp V_{pot}$. This yields

$$\int_\Omega \varphi \left(\nabla z_\varepsilon^h, \mathbf{b}(\varepsilon^{-1}x)\right) \mathrm{d}\eta_\varepsilon = -\int_\Omega z_\varepsilon^h \left(\nabla\varphi, \mathbf{b}(\varepsilon^{-1}x)\right) \mathrm{d}\eta_\varepsilon^h \tag{9.68}$$

for every $\varphi \in C_0^\infty(\Omega)$.

Passing to the limit as $\varepsilon \to 0$ in this relation (in the sense of two-scale convergence), we obtain

$$\int_\Omega \left[\int_Y (\mathbf{r}(x,y), \mathbf{b}(y))\, \mathrm{d}\eta(y)\right] \varphi \,\mathrm{d}x = -\int_\Omega \left(z(x)\int_Y \mathbf{b}(y)\,\mathrm{d}\eta(y), \nabla\varphi\right) \mathrm{d}x. \tag{9.69}$$

Since $\mathbf{r} \in \left[L^2(\Omega, Y)\right]^2$, it follows that $s = \displaystyle\int_Y (\mathbf{r}(x,y), \mathbf{b}(y))\, \mathrm{d}\eta(y)$ belongs to $L^2(\Omega)$. Let us define the function

$$\Upsilon(t,x) = z(x)\int_Y \mathbf{b}(y)\,\mathrm{d}\eta(y).$$

Then rewriting (9.69) in the form

$$\int_\Omega s\varphi \,\mathrm{d}x = -\int_\Omega (\Upsilon, \nabla\varphi)\,\mathrm{d}x$$

and integrating the right-hand side by parts, we deduce

$$s(x) = \left(\nabla z(x), \int_Y \mathbf{b}(y)\,\mathrm{d}\eta(y)\right).$$

Using the non-degeneracy property of the measure η, we see that there exists a vector $\mathbf{c} \in V_{pot}$ $\left(V_{pot} \subset [L^2(Y)]^2\right)$ such that $\displaystyle\int_Y \mathbf{c}(y)\,\mathrm{d}\eta(y) = \vartheta$, where $\vartheta = \displaystyle\int_Y \mathbf{b}(y)\,\mathrm{d}\eta(y)$. Then $s(x) = \left(\nabla z(x), \displaystyle\int_Y \mathbf{c}(y)\,\mathrm{d}\eta(y)\right)$ and, therefore,

(i) $(\nabla z, \displaystyle\int_Y \mathbf{c}\,\mathrm{d}\eta) \in L^2(\Omega)$ in the sense of distributions,

(ii) $\mathbf{c} \in V_{pot}$.

Hence, $\nabla z \in \left[L^2(\Omega)\right]^2$. Summing up the results obtained above, we deduce

$$\left[z \in L^2(\Omega),\ \nabla z \in \left[L^2(\Omega)\right]^2\right] \Longrightarrow z \in H^1(\Omega).$$

Rewriting (9.69) in the form

$$\int_\Omega \left[\int_Y (\mathbf{r}(x,y),\mathbf{b}(y))\,\mathrm{d}\eta(y)\right]\varphi\,\mathrm{d}x = \int_\Omega \left(\nabla z(x), \int_Y \mathbf{b}(y)\,\mathrm{d}\eta(y)\right)\varphi\,\mathrm{d}x,$$

we observe that

$$\int_\Omega\int_Y (r(x,y)-\nabla z(x),\mathbf{b}(y))\ \mathrm{d}\eta(y)\varphi\,\mathrm{d}x = 0$$

for every $\varphi \in C_0^\infty(\Omega)$ and $\mathbf{b} \perp V_{pot}$.

Hence, taking into account that the linear span of the vector-valued functions $\varphi(x)\mathbf{b}(y)$ is dense in $L^2(\Omega, V_{sol})$ and (9.58) holds, we obtain the required property

$$r(x,y)-\nabla z(x) \in L^2(\Omega, V_{pot}).$$

The proof is complete. □

Corollary 9.15. *Let $\{z_\varepsilon^h\}_{\varepsilon>0}$ be a bounded sequence in $H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)$ and let $z \in H^1(\Omega)$ be its weak limit in $L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)$. Then*

$$z_\varepsilon^h \to z \quad in \quad L^2(\Omega,\mathrm{d}\eta_\varepsilon^h), \quad \lim_{\varepsilon\to 0}\int_\Omega |z_\varepsilon^h - z|^2\,\mathrm{d}\eta_\varepsilon^h = 0. \tag{9.70}$$

Proof. As follows from Theorem 9.14,

$$z_\varepsilon^h \overset{2}{\rightharpoonup} z \ \text{ and } \ z_\varepsilon^h \rightharpoonup z \quad \text{ in } \ L^2(\Omega,\mathrm{d}\eta_\varepsilon^h).$$

In order to prove the first assertion in (9.70), it is enough to show that (see the criterium of strong convergence in $L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)$ and the property of lower semicontinuity)

$$\lim_{\varepsilon\to 0}\int_\Omega |z_\varepsilon^h|^2\,\mathrm{d}\eta_\varepsilon^h \le \int_\Omega |z|^2\,\mathrm{d}x. \tag{9.71}$$

In accordance with inequality (9.43), we have

$$\begin{aligned}\int_\Omega (z_\varepsilon^h - z)^2\,\mathrm{d}\eta_\varepsilon^h &\le C_1\int_\Omega \left[(\widehat{z}_\varepsilon{}^h - z)^2 + \varepsilon|\nabla(\widehat{z}_\varepsilon{}^h - z)|^2\right]\,\mathrm{d}x \\ &= C_1\int_\Omega (\widehat{z}_\varepsilon{}^h - z)^2\,\mathrm{d}x + O(\varepsilon^2),\end{aligned} \tag{9.72}$$

where by $\widehat{z}_\varepsilon{}^h$ we denote functions of $H^1(\Omega)$ such that $\widehat{z}_\varepsilon{}^h = z_\varepsilon^h$, η_ε^h-almost everywhere on Ω, and $\widehat{z}_\varepsilon{}^h \rightharpoonup z$ in $H^1(\Omega)$. Note that such a choice is always possible. Indeed, let us assume the converse, that is, $\widehat{z}_\varepsilon{}^h = z_\varepsilon^h$ η_ε^h-almost everywhere on Ω for every $\varepsilon > 0$, but $\widehat{z}_\varepsilon{}^h \rightharpoonup z^*$ in $H^1(\Omega)$ where $z^* \ne z$. Then taking into account the definitions of the measure η_ε^h and of the weak limit in $L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)$, we get

$$\int_\Omega z_\varepsilon^h \varphi \, \mathrm{d}\eta_\varepsilon^h = \frac{1}{h(2-h)} \int_\Omega \chi_{\Omega_\varepsilon} \widehat{z}_\varepsilon^{\,h} \varphi \, \mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega z\varphi \, \mathrm{d}x.$$

However, using the facts that $\widehat{z}_\varepsilon^{\,h} \to z^*$ in $L^2(\Omega)$ and $[h(2-h)]^{-1}\chi_{\Omega_\varepsilon} \to 1$ weakly-$*$ in $L^\infty(\Omega)$, we deduce

$$\frac{1}{h(2-h)} \int_\Omega \chi_{\Omega_\varepsilon} \widehat{z}_\varepsilon^{\,h} \varphi \, \mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega z^* \varphi \, \mathrm{d}x.$$

Hence, $z = z^*$ and inequality (9.72) leads us to the following conclusion:

$$\lim_{\varepsilon\to 0} \int_\Omega (z_\varepsilon^h - z)^2 \, \mathrm{d}\eta_\varepsilon^h = 0. \tag{9.73}$$

Then in view of Lemma 9.7, we have

$$\lim_{\varepsilon\to 0} \int_\Omega |z_\varepsilon^h|^2 \, \mathrm{d}\eta_\varepsilon^h \le \lim_{\varepsilon\to 0} \int_\Omega |z_\varepsilon^h - z|^2 \, \mathrm{d}\eta_\varepsilon^h + \lim_{\varepsilon\to 0} \int_\Omega |z|^2 \, \mathrm{d}\eta_\varepsilon^h = \int_\Omega |z|^2 \, \mathrm{d}x.$$

Thus, $z_\varepsilon^h \to z$ strongly in $L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$. In conclusion, we note that to prove the second statement in (9.70), it remains only to use Corollary 9.9. □

Theorem 9.16. *Every bounded sequence*

$$\left\{ z_\varepsilon^h \in H^1(\Omega, \mathrm{d}\eta_\varepsilon^h) \cap L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h) \right\}_{\varepsilon>0}$$

is relatively compact with respect to the weak convergence in the sense of Definition 9.4.

Proof. Since the sequence $\left\{z_\varepsilon^h\right\}$ is bounded in

$$H^1(\Omega, \mathrm{d}\eta_\varepsilon^h) \cap L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h),$$

we may suppose that there exist elements z and m such that $z, m \in L^2(\Omega)$ and

$$z_\varepsilon^h \to z \quad \text{weakly in} \quad L^2(\Omega, \mathrm{d}\eta_\varepsilon^h),$$
$$z_\varepsilon^h \to m \quad \text{weakly in} \quad L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h).$$

However, from Theorem 9.14, Lemma 9.8, and Corollary 9.15, we conclude that

$$z \in H^1(\Omega), \quad z_\varepsilon^h \rightharpoonup z \quad \text{in} \quad L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h),$$
$$z_\varepsilon^h \to z \quad \text{strongly in} \quad L^2(\Omega, \mathrm{d}\eta_\varepsilon^h), \quad \lim_{\varepsilon\to 0} \int_\Omega |z_\varepsilon^h - z|^2 \, \mathrm{d}\eta_\varepsilon^h = 0.$$

Hence, $z = m$ a.e. in Ω and we obtain the required conclusion. The proof is complete. □

Further, we proceed to the convergence for Dirichlet boundary controls for the $\widehat{\mathbb{P}}_\varepsilon$-optimal control problem. As follows from (9.36)–(9.38), the class of admissible Dirichlet controls for every fixed value of ε can be described as follows:

$$\widehat{u}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon = \left\{ u \in \left[H^1(\Gamma_0) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^h) \right] \ : \ \|u\|_{H^1(\Gamma_0)} \leq \mathbf{C}_0 \right\}. \tag{9.74}$$

Hence, for every $\varepsilon > 0$ and for every admissible control function $\widehat{u}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon$, there exists an extension operator

$$P_\varepsilon : H^1(\Gamma_\varepsilon) \to H^1(\Gamma_0) \ \text{ such that } \ \|P_\varepsilon(\widehat{u}_\varepsilon)\|_{H^1(\Gamma_0)} \leq \mathbf{C}_0.$$

In view of this, we can give two different convergence concepts for such controls.

Definition 9.17. *We say that a function u^* is an s_a-limit for the sequence of Dirichlet controls $\left\{\widehat{u}_\varepsilon \in H^1(\Gamma_\varepsilon)\right\}_{\varepsilon>0}$ if the sequence of its images $\{P_\varepsilon(\widehat{u}_\varepsilon)\}_{\varepsilon>0}$ converges to u^* weakly in $H^1(\Gamma_0)$. So, we say that the sequence of controls $\left\{\widehat{u}_\varepsilon \in L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\right\}_{\varepsilon>0}$ is s_b-convergent to an element $u^{**} \in L^2(\Gamma_0)$ if this sequence is uniformly bounded, that is,*

$$\sup_{\varepsilon \to 0} \int_\Omega |\widehat{u}_\varepsilon|^2 \, \mathrm{d}\mu_\varepsilon^h < +\infty$$

and $\widehat{u}_\varepsilon \rightharpoonup u^{**}$ in the variable space $L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)$.

Below we show that these concepts are the same for admissible controls. Let us fix some sequence of controls $\{\widehat{u}_\varepsilon\}$ and assume that there are two different sequences of its images $\left\{P_\varepsilon^{(1)}(\widehat{u}_\varepsilon)\right\}_{\varepsilon>0}$ and $\left\{P_\varepsilon^{(2)}(\widehat{u}_\varepsilon)\right\}_{\varepsilon>0}$ such that

$$P_\varepsilon^{(1)}(\widehat{u}_\varepsilon) \rightharpoonup u_1^* \ \text{ in } \ H^1(\Gamma_0) \ \text{ and } \ P_\varepsilon^{(2)}(\widehat{u}_\varepsilon) \rightharpoonup u_2^* \ \text{ in } \ H^1(\Gamma_0).$$

Let $\chi_\varepsilon^h \in L^\infty(\Gamma_0)$ be the characteristic function of the set Γ_ε^D, on which the corresponding controls $\widehat{u}_\varepsilon$ are located. By Proposition 9.12, we have that $h^{-1}(\varepsilon)\chi_\varepsilon^h \to 1$ weakly-$*$ in $L^\infty(\Gamma_0)$ as $\varepsilon \to 0$. Then passing to the limit in the following integral identity as $\varepsilon \to 0$,

$$h^{-1}(\varepsilon) \int_{\Gamma_0} \chi_\varepsilon^h P_\varepsilon^{(1)}(\widehat{u}_\varepsilon)\varphi \, \mathrm{d}x_1 = h^{-1}(\varepsilon) \int_{\Gamma_0} \chi_\varepsilon^h P_\varepsilon^{(2)}(\widehat{u}_\varepsilon)\varphi \, \mathrm{d}x_1,$$

which holds true for every $\varphi \in H^1(\Gamma_0)$, we get

$$\int_{\Gamma_0} u_1^* \varphi \, \mathrm{d}x_1 = \int_{\Gamma_0} u_2^* \varphi \, \mathrm{d}x_1, \quad \forall\, \varphi \in H^1(\Gamma_0).$$

Hence, $u_1^* = u_2^*$ almost everywhere in Γ_0. As a result, we arrive at the following conclusion.

Lemma 9.18. *Every sequence of admissible controls* $\{\widehat{u}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon\}_{\varepsilon>0}$ *is relatively compact with respect to the* s_a*-convergence introduced earlier. Moreover, its* s_a*-weak limit* u^* *belongs to the set*

$$U = \left\{u \in H^1(\Gamma_0) \mid \|u\|_{H^1(\Gamma_0)} \le \mathbf{C_0}\right\}.$$

Indeed, the fact $u^* \in U$ is readily seen from the lower semicontinuity property of the norm in $H^1(\Gamma_0)$ with respect to the weak convergence in this space.

Theorem 9.19. *Let* $\{\widehat{u}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon\}_{\varepsilon>0}$ *be a sequence of admissible Dirichlet controls. Then one can extract an* s_b*-convergent subsequence of* $\{\widehat{u}_\varepsilon\}_{\varepsilon>0}$ *for which its* s_a*- and* s_b*-limits coincide.*

Proof. As follows from the definition of the class $\widehat{\mathcal{U}}_\varepsilon$ (see (9.74)), the representation formula (9.31), and Proposition 9.12, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_\Omega |\widehat{u}_\varepsilon|^2\,\mathrm{d}\mu_\varepsilon^h &= \limsup_{\varepsilon\to 0}\frac{1}{h(\varepsilon)}\|\widehat{u}_\varepsilon\|^2_{L^2(\Gamma_\varepsilon^D)}\\ &\le \limsup_{\varepsilon\to 0}\frac{1}{h(\varepsilon)}\chi_{\Gamma_\varepsilon^D}\|P_\varepsilon(\widehat{u}_\varepsilon)\|^2_{H^1(\Gamma_\varepsilon^D)}\\ &= \limsup_{\varepsilon\to 0}\|P_\varepsilon(\widehat{u}_\varepsilon)\|^2_{H^1(\Gamma_\varepsilon^D)} \le \mathbf{C}_0^2.\end{aligned}$$

Consequently, every sequence of admissible controls $\{\widehat{u}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)$. Therefore, taking the compactness criterium of weak convergence in variable spaces into account, we can always suppose without loss of generality that for a given sequence $\{\widehat{u}_\varepsilon\}_{\varepsilon>0}$, there are functions $u^* \in H^1(\Gamma_0)$ and $u^{**} \in L^2(\Gamma_0)$ such that

$$\lim_{\varepsilon\to 0}\int_{\Gamma_0} P_\varepsilon(\widehat{u}_\varepsilon)\varphi\,\mathrm{d}x_1 = \int_{\Gamma_0} u^*\varphi\,\mathrm{d}x_1, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}),$$

$$\lim_{\varepsilon\to 0}\int_\Omega \widehat{u}_\varepsilon\varphi\,\mathrm{d}\mu_\varepsilon^h = \int_{\Gamma_0} u^{**}\varphi\,\mathrm{d}x_1, \quad \forall\,\varphi \in C_0^\infty(\Gamma_0).$$

Since

$$\int_{\Gamma_0} P_\varepsilon(\widehat{u}_\varepsilon)\varphi\,\mathrm{d}x_1 = \int_\Omega \widehat{u}_\varepsilon\varphi\,\mathrm{d}\mu_\varepsilon^h, \quad \forall\,\varepsilon > 0,$$

it follows that $u^* = u^{**}$ almost everywhere on Γ_0. The proof is complete. □

Remark 9.20. Taking into account this result, we can assume that for any sequence of Dirichlet controls $\{\widehat{u}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon\}$, there is a sequence of its prototypes $\{u_\varepsilon \in H^1(\Gamma_0)\}$ such that $\widehat{u}_\varepsilon|_{\Gamma_\varepsilon^D} = u_\varepsilon|_{\Gamma_\varepsilon^D}$ and $u_\varepsilon \rightharpoonup u$ in $H^1(\Gamma_0)$. Then, by analogy with Lemma 9.8, it can be proved that, in this case, $\widehat{u}_\varepsilon$ converges to $u|_{\Gamma_0}$ weakly in $L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)$ and $u|_{\Gamma_0} = u^{**}$ by virtue of the uniqueness of weak

limits in $L^2(\Omega, d\mu_\varepsilon^h)$. Therefore, in what follows, we will assume that every sequence of admissible Dirichlet controls $\{\widehat{u}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon\}$ is compact with respect to the weak convergence in $L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)$ and its limit belongs to $H^1(\Gamma_0)$.

Let $\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \widehat{\Xi}_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible solutions for the $\widehat{\mathbb{P}}_\varepsilon$-problem. Each of the sets $\widehat{\Xi}_\varepsilon$ is defined in (9.38) and $\widehat{\Xi}_\varepsilon \subset \mathbb{Z}_\varepsilon$ for every $\varepsilon > 0$, where

$$\begin{aligned}\mathbb{Z}_\varepsilon \equiv \left[H^1(\Gamma_0) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\right] \times\ & L^2(\Omega, \mathrm{d}\nu_\varepsilon^h) \\ \times \left[H^1(\Omega, \mathrm{d}\eta_\varepsilon^h) \cap L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)\right]&. \end{aligned} \tag{9.75}$$

We assume that this sequence is bounded, that is,

$$\limsup_{\varepsilon>0} \|(u_\varepsilon, p_\varepsilon, y_\varepsilon)\|_{\mathbb{Z}_\varepsilon} < +\infty.$$

Then summing up the above given arguments, we may introduce the following concept of weak convergence in the variable space $\mathbb{Z}_\varepsilon$.

Definition 9.21. *We say that a bounded sequence $\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{Z}_\varepsilon\}_{\varepsilon>0}$ is w-convergent to a triplet $(u, p, y) \in H^1(\Gamma_0) \times L^2(\Gamma_0) \times H^1(\Omega)$ in the variable space $\mathbb{Z}_\varepsilon$ as ε tends to 0 (in symbols, $(u_\varepsilon, p_\varepsilon, y_\varepsilon) \xrightarrow{w} (u, p, y)$), if*

$$\begin{gathered} u_\varepsilon \rightharpoonup u \quad \text{in} \quad L^2(\Omega, \mathrm{d}\mu_\varepsilon^h), \quad u \in H^1(\Gamma_0), \\ p_\varepsilon \rightharpoonup p \quad \text{in} \quad L^2(\Omega, \mathrm{d}\nu_\varepsilon^h), \\ y_\varepsilon^h \to y \ \text{ in } \ L^2(\Omega, \mathrm{d}\eta_\varepsilon^h), \quad y_\varepsilon^h \rightharpoonup y \ \text{ in } \ L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h), \quad y \in H^1(\Omega), \\ \int_\Omega |y_\varepsilon^h - y|^2 \, \mathrm{d}\eta_\varepsilon^h \to 0 \quad \text{as} \quad \varepsilon \to 0. \end{gathered}$$

As an evident consequence of Theorems 9.16 and 9.19, we have the following conclusion.

Theorem 9.22. *Let $\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{Z}_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence of admissible solutions to the OCPs $\widehat{\mathbb{P}}_\varepsilon$. Then this sequence is relatively compact with respect to the w-convergence in the variable space $\mathbb{Z}_\varepsilon$.*

To conclude this section, we provide the following result.

Proposition 9.23. *Let $H^1(\Omega; \Gamma_0) = \{y \in H^1(\Omega) : y = 0 \ \text{ on } \ \Gamma_0\}$. Then for every $y \in H^1(\Omega; \Gamma_0)$, there is a sequence $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^2; \Gamma_\varepsilon^D)$ such that*

$$\varphi_\varepsilon \to \chi_\Omega y \ \text{ strongly in } \ L^2(\mathbb{R}^2), \quad \nabla\varphi_\varepsilon \to \chi_\Omega \nabla y \ \text{ strongly in } \ L^2(\mathbb{R}^2, \mathbb{R}^2).$$

Vice versa, if $\{\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^2, \Gamma_\varepsilon^D)\}$ is a sequence such that $\varphi_\varepsilon|_\Omega$ converges to y weakly in $H^1(\Omega)$, then $y \in H^1(\Omega; \Gamma_0)$.

Proof. Let us introduce the sequence of sets $\{\mathcal{B}_\varepsilon \subset \mathbb{R}^2\}$ with

$$\mathcal{B}_\varepsilon = \Omega \cup \left\{(x_1, x_2) \in \mathbb{R}^2 \,:\, (x_1, -x_2) \in \overline{Q}_\varepsilon \setminus \Omega_\varepsilon\right\}, \ \forall \varepsilon > 0.$$

Here, $\overline{Q}_\varepsilon = \{(x_1, x_2) \,:\, x_1 \in [0, L_1], x_2 \in [0, \varepsilon/2]\}$. Let $X_{\mathcal{B}_\varepsilon}$ and X_Ω be closed linear subspaces of $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2, \mathbb{R}^2)$ such that

$$X_{\mathcal{B}_\varepsilon} = \left\{(\chi_{\mathcal{B}_\varepsilon} y, \chi_{\mathcal{B}_\varepsilon} \nabla y) \,:\, y \in H^1(\mathcal{B}_\varepsilon)\right\},$$
$$X_\Omega = \left\{(\chi_\Omega y, \chi_\Omega \nabla y) \,:\, y \in H^1(\Omega)\right\}.$$

It is clear that $\{\mathcal{B}_\varepsilon\}$ is uniformly bounded sequence of open subsets of $\mathbb{R}^2$ which converges to Ω in the Hausdorff complementary topology, that is, $\lim_{\varepsilon \to 0} d_H(\mathcal{B}_\varepsilon^c, \Omega^c) = 0$, where the Hausdorff distance between the complements of $\mathcal{B}_\varepsilon$ and Ω in $\mathbb{R}^2$ is defined by (see Definition 7.28; see also Falconer [107] for the details)

$$d_H(\mathcal{B}_\varepsilon^c, \Omega^c) = \max\left\{\sup_{x \in \mathcal{B}_\varepsilon^c} \operatorname{dist}(x, \Omega^c), \ \sup_{x \in \Omega^c} \operatorname{dist}(x, \mathcal{B}_\varepsilon^c)\right\}.$$

Then we conclude (see Dal Maso, Ebobisse, and Ponsiglione [80], Theorem 4.2): The following sequence of subspaces $X_{\mathcal{B}_\varepsilon}$ converges to X_Ω in the sense of Mosco (see Definition 7.39):

(M_1) For every $y \in H^1(\Omega)$, there exists a sequence $\{y_\varepsilon \in H^1(\mathcal{B}_\varepsilon)\}$ such that $\chi_{\mathcal{B}_\varepsilon} y_\varepsilon$ converges to $\chi_\Omega y$ strongly in $L^2(\mathbb{R}^2)$ and $\chi_{\mathcal{B}_\varepsilon} \nabla y_\varepsilon$ converges to $\chi_\Omega \nabla y$ strongly in $L^2(\mathbb{R}^2, \mathbb{R}^2)$.

(M_2) If $\{\varepsilon_k\}$ is a sequence of indices converging to 0, $\{y_k\}$ is a sequence such that $y_k \in H^1(\mathcal{B}_{\varepsilon_k})$ for every k and $\chi_{\mathcal{B}_{\varepsilon_k}} y_{\varepsilon_k}$ converges weakly in $L^2(\mathbb{R}^2)$ to a function ϕ, and $\chi_{\mathcal{B}_{\varepsilon_k}} \nabla y_{\varepsilon_k}$ converges weakly in $L^2(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ, then there exists $y \in H^1(\Omega)$ such that

$$\phi = \chi_\Omega y \ \text{ and } \ \psi = \chi_\Omega \nabla y \ \text{ a.e. in } \ \mathbb{R}^2.$$

Let $y \in H^1(\Omega; \Gamma_0)$ be any function. Then $(\chi_\Omega y, \chi_\Omega \nabla y) \in X_\Omega$, and, hence, by property (M_1), there is a sequence $\{y_\varepsilon \in H^1(\mathcal{B}_\varepsilon)\}$ such that $\chi_{\mathcal{B}_\varepsilon} y_\varepsilon$ converges to $\chi_\Omega y$ strongly in $L^2(\mathbb{R}^2)$. For every fixed ε, let us choose a function $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^2, \Gamma_\varepsilon^D)$ such that $\|\varphi_\varepsilon - \chi_{\mathcal{B}_\varepsilon} y_\varepsilon\|_{L^2(\mathbb{R}^2)} \le \varepsilon^2$. Then

$$\begin{aligned}\|\chi_\Omega y - \varphi_\varepsilon\|_{L^2(\mathbb{R}^2)} &\le \|\varphi_\varepsilon - \chi_{\mathcal{B}_\varepsilon} y_\varepsilon\|_{L^2(\mathbb{R}^2)} \\ &\quad + \|\chi_{\mathcal{B}_\varepsilon} y_\varepsilon - \chi_\Omega y\|_{L^2(\mathbb{R}^2)} \longrightarrow 0 \quad \text{as } \varepsilon \to 0,\end{aligned}$$

and hence, $\varphi_\varepsilon \to \chi_\Omega y$ strongly in $L^2(\mathbb{R}^2)$.

To prove the converse case, we observe that by the trace theorem we have the following: If $\varphi_\varepsilon|_\Omega$ converges to y weakly in $H^1(\Omega)$, then $\varphi_\varepsilon|_{\Gamma_0} \in H^{1/2}(\Gamma_0)$ for every $\varepsilon > 0$ and $\varphi_\varepsilon|_{\Gamma_0} \to y|_{\Gamma_0}$ strongly in $L^2(\Gamma_0)$. Consequently,

$$\int_{\Gamma_\varepsilon^D} \left(\varphi_\varepsilon|_{\Gamma_0}\right) \psi \, \mathrm{d}x_1 = 0, \quad \forall\, \psi \in C_0^\infty(\mathbb{R}^2), \ \forall\, \varepsilon > 0.$$

However,

$$0 = \int_{\Gamma_\varepsilon^D} \left(\varphi_\varepsilon|_{\Gamma_0}\right) \psi \, \mathrm{d}x_1 = \int_{\Gamma_0} \left(h^{-1}(\varepsilon)\chi_{\Gamma_\varepsilon^D}\right) \left(\varphi_\varepsilon|_{\Gamma_0}\right) \psi \, \mathrm{d}x_1, \quad \forall\, \varepsilon > 0, \quad (9.76)$$

where $\chi_{\Gamma_\varepsilon^D}$ is the characteristic functions of the sets Γ_ε^D. Using Proposition 9.13 and passing to the limit on the right-hand side of (9.76), which is the product of strongly and weakly converging sequences, we immediately obtain

$$\lim_{\varepsilon \to 0} \int_{\Gamma_0} \left(h^{-1}(\varepsilon)\chi_{\Gamma_\varepsilon^D}\right) \left(\varphi_\varepsilon|_{\Gamma_0}\right) \psi \, \mathrm{d}x_1 = \int_{\Gamma_0} y\psi \, \mathrm{d}x_1 = 0, \quad \forall\, \psi \in C_0^\infty(\mathbb{R}^2).$$

Thus, $y = 0$ on Γ_0 and we establish the required result. □

9.4 Definition of the limit problem and its properties

In this section, we aim to characterize a "limit" minimization problem of the sequence (9.40) as ε tends to 0. The main feature of the $\widehat{\mathbb{P}}_\varepsilon$-problem is the specific construction of the solution space $\mathbb{Z}_\varepsilon$ and the absence of the strong approximation property for the "w-limit space" (see Pastuchova [212])

$$\mathbb{Y} = H^1(\Gamma_0) \times L^2(\Gamma_0) \times H^1(\Omega).$$

This means that not for every triplet $(u, p, y) \in \mathbb{Y}$ there is a sequence $\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{Z}_\varepsilon\}_{\varepsilon > 0}$ such that $(u_\varepsilon, p_\varepsilon, y_\varepsilon) \xrightarrow{w} (u, p, y)$. Because of this, we accept the following concept of a "limit" minimization problem.

Definition 9.24. *We say that a minimization problem*

$$\left\langle \inf_{(u,p,y) \in\, \Xi_0} I_0(u, p, y) \right\rangle \quad (9.77)$$

is the weak variational limit of the sequence (9.40) with respect to the w-convergence in the variable space $\mathbb{Z}_\varepsilon$ (or the variational w-limit) if conditions (d)–(dd) of Definition 8.24 hold true, with

$$\mathbb{X}_\varepsilon = \left[H^1(\Gamma_0) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\right] \times \ L^2(\Omega, \mathrm{d}\nu_\varepsilon^h) \times \left[H^1(\Omega, \mathrm{d}\eta_\varepsilon^h) \cap L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)\right],$$
$$\mathbb{X} = H^1(\Gamma_0) \times L^2(\Gamma_0) \times H^1(\Omega),$$

Remark 9.25. Note that in spite of the lack of the strong approximation property for the "w-limit space" $\mathbb{Y}$, the weak variational limit in the sense of Definition 9.24 possesses the fine variational properties that are similar to that of the Γ-limit (in contrast to Theorem 8.25, we do not assume that the limit problem has a unique solution).

Theorem 9.26. *Assume that the constrained minimization problem (9.77) is the weak variational limit of the sequence (9.40) and assume that this problem is solvable. Assume also that the initial data for the original problem (9.20) satisfy the condition*

$$\sup_{\varepsilon>0} \max\left\{\|f_\varepsilon\|_{L^2(\Omega)}, \|g_\varepsilon\|_{L^2(S^\varepsilon_{int})}\right\} < +\infty.$$

Let $\left\{(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ *be a sequence of optimal solutions to the corresponding* $\widehat{\mathbb{P}}_\varepsilon$*-problems. Then this sequence is relatively w-compact and all w-cluster triplets are optimal solutions to problem (9.77); more precisely, if*

$$(u^0_{\varepsilon_k}, p^0_{\varepsilon_k}, y^0_{\varepsilon_k}) \xrightarrow{\ w\ } (u^0, p^0, y^0), \tag{9.78}$$

then

$$\begin{aligned}\inf_{(u,p,y)\in\Xi_0} I_0(u,p,y) &= I_0\left(u^0, p^0, y^0\right)\\ &= \lim_{k\to\infty} \widehat{I}_{\varepsilon_k}(u^0_{\varepsilon_k}, p^0_{\varepsilon_k}, y^0_{\varepsilon_k}) = \lim_{\varepsilon\to 0} \inf_{(u_\varepsilon,p_\varepsilon,y_\varepsilon)\in\widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon((u_\varepsilon, p_\varepsilon, y_\varepsilon).\end{aligned} \tag{9.79}$$

Proof. Let us assume for a moment that the sequence of optimal solutions $\left\{(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ to the $\mathbb{P}_\varepsilon$-problems is such that

$$\limsup_{\varepsilon>0} \widehat{I}_\varepsilon(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) = \infty.$$

Then for any given value $D > 0$ there is a sequence $\{\varepsilon_k\}$ such that $\lim_{k\to\infty} \varepsilon_k = 0$ and

$$\min_{(u_{\varepsilon_k}, p_{\varepsilon_k}, y_{\varepsilon_k})\in\widehat{\Xi}_{\varepsilon_k}} \widehat{I}_{\varepsilon_k}(u_{\varepsilon_k}, p_{\varepsilon_k}, y_{\varepsilon_k}) = \widehat{I}_{\varepsilon_k}(u^0_{\varepsilon_k}, p^0_{\varepsilon_k}, y^0_{\varepsilon_k}) \geq D, \quad \forall\, k \in \mathbb{N}.$$

On the other hand, it is easy to see that the triplet $(0, 0, y^*_\varepsilon)$ is admissible for every ε (here, y^*_e is the weak solution of the boundary value problem (9.20) under the condition $u_\varepsilon = 0$ and $p_\varepsilon = 0$). Hence, taking the a priori estimate (9.39) into account, we immediately conclude

$$\limsup_{\varepsilon\to 0} \min_{(u_\varepsilon,p_\varepsilon,y_\varepsilon)\in\widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) \leq \limsup_{\varepsilon\to 0} \widehat{I}_\varepsilon(0, 0, y^*_\varepsilon) \leq \widetilde{C} < +\infty.$$

The obtained contradiction proves that there exists a constant $D_0 > 0$ such that

$$\limsup_{\varepsilon\to 0} \widehat{I}_\varepsilon(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) \leq D_0.$$

This fact implies the boundedness of the sequence $\left\{p^0_\varepsilon\right\}$ in $L^2(\Omega, \mathrm{d}\nu^h_\varepsilon)$. Then using the boundedness of the Dirichlet type controls $\left\{u^0_\varepsilon\right\}$ and estimate (9.39), we deduce

$$\limsup_{\varepsilon>0} \|(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon)\|_{\mathbb{Z}_\varepsilon} = \limsup_{\varepsilon>0} \big[\|u^0_\varepsilon\|_{H^1(\Gamma_0)\cap L^2(\Omega,\mathrm{d}\mu^h_\varepsilon)} + \|p^0_\varepsilon\|_{L^2(\Omega,\mathrm{d}\nu^h_\varepsilon)} + \|y^0_\varepsilon\|_{H^1(\Omega,\mathrm{d}\eta^h_\varepsilon)}\big] < +\infty.$$

Then, taking Theorem 9.22 into account, we may suppose that there exist a subsequence $\{(u^0_{\varepsilon_k}, p^0_{\varepsilon_k}, y^0_{\varepsilon_k})\}_{k\in\mathbb{N}}$ of the sequence of optimal solutions and a triplet (u^*, p^*, y^*) such that $(u^0_{\varepsilon_k}, p^0_{\varepsilon_k}, y^0_{\varepsilon_k}) \xrightarrow{w} (u^*, p^*, y^*)$ as $\varepsilon_k \to 0$. Hence, property (d) of Definition 8.24 leads us to $(u^*, p^*, y^*) \in \Xi_0$, and

$$\begin{aligned}\liminf_{k\to\infty} \min_{(u,p,y)\in \widehat{\Xi}_{\varepsilon_k}} \widehat{I}_{\varepsilon_k}(u,p,y) &= \liminf_{k\to\infty} \widehat{I}_{\varepsilon_k}(u^0_{\varepsilon_k}, p^0_{\varepsilon_k}, y^0_{\varepsilon_k}) \ge I_0(u^*, p^*, y^*)\\ &\ge \min_{(u,p,y)\in\,\Xi_0} I_0(u,p,y) = I_0(u^0, p^0, y^0). \end{aligned} \tag{9.80}$$

Let us fix a value $\delta > 0$. Then, by property (d) of Definition 8.24, there exists a δ-realizing sequence $\{(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon) \in \widehat{\Xi}_\varepsilon\}_{\varepsilon>0}$ such that

$$(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{z}_\varepsilon) \xrightarrow{w} (\widehat{u}, \widehat{p}, \widehat{y}), \quad \|(u^0, p^0, y^0) - (\widehat{u}, \widehat{p}, \widehat{y})\|_{\mathbb{Y}} \le \delta,$$

$$I_0(u^0, p^0, y^0) \ge \limsup_{\varepsilon\to 0} \widehat{I}_\varepsilon(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon) - \delta.$$

Using this fact, we have

$$\begin{aligned}\min_{(u,p,y)\in\Xi_0} I_0(u,p,y) + \delta &= I_0(u^0, p^0, y^0) + \delta\\ &\ge \limsup_{\varepsilon\to 0} \widehat{I}_\varepsilon(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon) \ge \limsup_{\varepsilon\to 0} \min_{(u,p,y)\in\widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(u,p,y)\\ &\ge \limsup_{k\to\infty} \min_{(u,p,y)\in\widehat{\Xi}_{\varepsilon_k}} \widehat{I}_{\varepsilon_k}(u,p,y) = \limsup_{k\to\infty} \widehat{I}_{\varepsilon_k}(u^0_{\varepsilon_k}, p^0_{\varepsilon_k}, y^0_{\varepsilon_k}). \end{aligned} \tag{9.81}$$

From this and (9.80), we deduce

$$\liminf_{k\to\infty} \widehat{I}_{\varepsilon_k}(u^0_{\varepsilon_k}, p^0_{\varepsilon_k}, y^0_{\varepsilon_k}) \ge \limsup_{k\to\infty} \widehat{I}_{\varepsilon_k}(u^0_{\varepsilon_k}, p^0_{\varepsilon_k}, p^0_{\varepsilon_k}) - \delta.$$

Since this inequality holds true for all sufficiently small $\delta > 0$, it follows that combining the above obtained relations (9.80) and (9.81), we obtain the required conclusion

$$I_0(u^*, p^*, y^*) = I_0(u^0, p^0, y^0) = \min_{(u,p,y)\in\,\Xi_0} I_0(u,p,y), \tag{9.82}$$

$$I_0(u^0, p^0, y^0) = \lim_{k\to\infty} \min_{(u,p,y)\in\widehat{\Xi}_{\varepsilon_k}} \widehat{I}_{\varepsilon_k}(u,p,y). \tag{9.83}$$

However, we observe that (9.82) and (9.83) hold true for all w-convergent subsequences of $\{(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon>0}$. It follows that these limits in (9.82) and

(9.83) coincide and, therefore, $I_0(u^0, p^0, y^0)$ is the limit of the whole sequence of minimal values

$$\left\{ \widehat{I}_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) = \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon((u_\varepsilon, p_\varepsilon, y_\varepsilon) \right\}_{\varepsilon > 0}.$$

This concludes the proof. □

9.5 Main convergence theorem

The main question that we are going to discuss in this section concerns the limit analysis of the following boundary value problem (see (9.38)):

$$y_\varepsilon = u_\varepsilon \quad \mu_\varepsilon^h\text{-a.e. in } \ \Omega, \ \|u_\varepsilon\|_{H^1(\Gamma_0)} \le \mathbf{C}_0, \tag{9.84}$$

$$\begin{aligned} \int_\Omega (A(x/\varepsilon)\nabla y_\varepsilon \cdot \nabla \varphi)\, \mathrm{d}\eta_\varepsilon^h + \int_\Omega y_\varepsilon \varphi \, \mathrm{d}\eta_\varepsilon^h + 4d\,\beta(\varepsilon) \int_\Omega y_\varepsilon \varphi \, \mathrm{d}\lambda_\varepsilon^h \\ = \int_\Omega f_\varepsilon \, \varphi \, \mathrm{d}\eta_\varepsilon^h + 2\,\beta(\varepsilon) \int_\Omega p_\varepsilon \varphi \, \mathrm{d}\nu_\varepsilon^h \\ + 4\,\beta(\varepsilon) \int_\Omega g_\varepsilon \varphi \, \mathrm{d}\lambda_\varepsilon^h, \ \forall\, \varphi \in C_0^\infty(\mathbb{R}^2; \Gamma_\varepsilon^D), \end{aligned} \tag{9.85}$$

where $\beta(\varepsilon) = (1 - h(\varepsilon))/(2 - h(\varepsilon))\xi(\varepsilon)$, and $\xi(\varepsilon) = \varepsilon/h(\varepsilon)$.

In other words, for a given sequence $\left\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}$, which is supposed to be w-convergent (in this section only), we look for the relations in terms of which a w-limit of this sequence can be determined.

We proceed to the passage to the limit in the integral identity (9.85). We note that in view of Proposition 9.23, the class of test functions $C_0^\infty(\mathbb{R}^2; \Gamma_\varepsilon^D)$ for integral identity (9.85) "tends in the limit" to $C_0^\infty(\mathbb{R}^2; \Gamma_0)$. So, in what follows, we suppose that $\varphi \in C_0^\infty(\mathbb{R}^2; \Gamma_0)$ in (9.85).

Let $\left\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}$ be an equibounded sequence of admissible solutions for the $\widehat{\mathbb{P}}_\varepsilon$-problem. By Theorem 9.22, this sequence is relatively compact with respect to the w-convergence in the variable space $\mathbb{Z}_\varepsilon$. So we may suppose that there exists a triplet $(u, p, y) \in \mathbb{Y}$ such that $(u_\varepsilon, p_\varepsilon, y_\varepsilon) \overset{w}{\rightharpoonup} (u, p, y)$. The main question is: By what kind of relation can this triplet be determined? In order to give an answer, we provide the following result.

Theorem 9.27. *Assume that $A_\varepsilon(x) = A(x/\varepsilon)$ in (9.85) is a Y-periodic measurable matrix satisfying the initial conditions (9.21). Assume also that*

$$\lim_{\varepsilon \to 0} \xi(\varepsilon) = \xi^* < +\infty$$

and that there exist functions $f^0 \in L^2(\Omega)$ and $g \in L^2(\Omega)$ such that

$$f_\varepsilon \rightharpoonup f_0 \quad in \quad L^2(\Omega, \mathrm{d}\eta_\varepsilon^h) \quad and \quad g_\varepsilon \rightharpoonup g \quad in \quad L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h). \tag{9.86}$$

Let

$$\{p_\varepsilon \in L^2(\Omega, \mathrm{d}\nu_\varepsilon^h)\} \ and \ \{u_\varepsilon \in [H^1(\Gamma_0) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]\} \tag{9.87}$$

be any bounded sequences of admissible controls for $\widehat{\mathbb{P}}_\varepsilon$-problems such that

$$p_\varepsilon \rightharpoonup p \quad in \quad L^2(\Omega, \mathrm{d}\nu_\varepsilon^h), \tag{9.88}$$

$$u_\varepsilon \rightharpoonup u \quad in \quad L^2(\Omega, \mathrm{d}\mu_\varepsilon^h). \tag{9.89}$$

Let $\{y_\varepsilon = y_\varepsilon(u_\varepsilon, p_\varepsilon) \in H^1(\Omega, \mathrm{d}\eta_\varepsilon^h) \cap L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)\}_{\varepsilon>0}$ be the corresponding solutions to the problem (9.84)–(9.85). Then $(u_\varepsilon, p_\varepsilon, y_\varepsilon) \overset{w}{\rightharpoonup} (u, p, y)$ as $\varepsilon \to 0$, where y is the unique solution in $H^1(\Omega)$ of the following boundary value problem:

$$z = u \quad a.e. \ in \quad \Gamma_0, \ \ \|u\|_{H^1(\Gamma_0)} \le \mathbf{C}_0, \tag{9.90}$$

$$\int_\Omega (A^{hom} \nabla y \cdot \nabla \varphi)\, \mathrm{d}x + \int_\Omega y\varphi \, \mathrm{d}x + 2d\,\xi^* \int_\Omega y\varphi \, \mathrm{d}x = \int_\Omega f_0\, \varphi \, \mathrm{d}x + 2\,\xi^* \int_\Omega g\varphi \, \mathrm{d}x. \tag{9.91}$$

Here, the matrix A^{hom} is defined by the rule

$$A^{hom}\zeta = \int_Y A(y)(\zeta + \mathbf{r}^0)\, \mathrm{d}\eta(y), \tag{9.92}$$

where $\upsilon^0 \in L^2(\Omega, V_{pot})$ is the unique solution of the minimum problem

$$\min_{\mathbf{r} \in V_{pot}} \int_Y (\zeta + \mathbf{r}, A(\zeta + \mathbf{r}))\, \mathrm{d}\eta = \int_Y (\zeta + \mathbf{r}^0, A(\zeta + \mathbf{r}^0))\, \mathrm{d}\eta. \tag{9.93}$$

Proof. To begin, we note that by the a priori estimate (9.39), the sequence of admissible solutions $\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \widehat{\Xi}_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $\mathbb{Z}_\varepsilon$. Hence, due to Theorem 9.22, we may suppose that there is a function $y \in H^1(\Omega)$ satisfying the conditions

$$\begin{cases} y_\varepsilon \to y \ \ \text{in} \ \ L^2(\Omega, \mathrm{d}\eta_\varepsilon^h), \\ y_\varepsilon \rightharpoonup y \ \ \text{in} \ \ L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h), \\ \nabla y_\varepsilon \overset{2}{\rightharpoonup} \nabla y + \mathbf{r}, \quad \text{where} \ \ \mathbf{r} \in L^2(\Omega, V_{pot}). \end{cases} \tag{9.94}$$

On the other hand, in view of Theorem 9.19 and Remark 9.20, the limit element u in (9.89) is such that $u \in H^1(\Gamma_0)$ and $u_\varepsilon \overset{s_a}{\longrightarrow} u$. Then passing to the limit in (9.84) as $\varepsilon \to 0$, we come to relation (9.90). Our aim now is to prove the correctness of the integral equality (9.91).

For this, we consider the integral identity (9.85) with the test function $\varphi(x) = \varepsilon\Psi(x)\omega(\varepsilon^{-1}x)$, where $\omega \in C^\infty_{per}(Y)$, and $\Psi \in C^\infty_0(\mathbb{R}^2, \Gamma^D_\varepsilon)$.

This yields

$$\begin{aligned}
&\varepsilon \int_\Omega (A(x/\varepsilon)\nabla y_\varepsilon \cdot \nabla\Psi)\omega(\varepsilon^{-1}x)\,\mathrm{d}\eta^h_\varepsilon + \varepsilon \int_\Omega y_\varepsilon \Psi\omega(\varepsilon^{-1}x)\,\mathrm{d}\eta^h_\varepsilon \\
&\quad + \int_\Omega (A(x/\varepsilon)\nabla y_\varepsilon \cdot \nabla\omega)\Psi\,\mathrm{d}\eta^h_\varepsilon + 4\varepsilon d\,\beta(\varepsilon) \int_\Omega y_\varepsilon \Psi\omega(\varepsilon^{-1}x)\,\mathrm{d}\lambda^h_\varepsilon \\
&\quad = \varepsilon \int_\Omega f_\varepsilon\,\Psi\omega(\varepsilon^{-1}x)\,\mathrm{d}\eta^h_\varepsilon + 2\varepsilon\,\beta(\varepsilon) \int_\Omega p_\varepsilon \Psi\omega(\varepsilon^{-1}x)\,\mathrm{d}\nu^h_\varepsilon \\
&\quad + 4\varepsilon\,\beta(\varepsilon) \int_\Omega g_\varepsilon \Psi(x)\omega(\varepsilon^{-1}x)\,\mathrm{d}\lambda^h_\varepsilon. \qquad (9.95)
\end{aligned}$$

Taking the mean value property (see Theorem 6.51), which holds for all periodic Borel measures η^h_ε, λ^h_ε, and ν^h_ε, into account, we pass to the limit in (9.95) as $\varepsilon \to 0$. We infer

$$\lim_{\varepsilon\to 0} \int_\Omega \left(A(\varepsilon^{-1}x)\nabla y_\varepsilon, \nabla\omega(\varepsilon^{-1}x)\right)\Psi\,\mathrm{d}\eta^h_\varepsilon = 0. \qquad (9.96)$$

Then, in view of $(9.94)_3$ and the definition of the weak two-scale limit, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_\Omega \Psi\left[\omega(\varepsilon^{-1}x)A(\varepsilon^{-1}x)\right]\nabla y_\varepsilon\,\mathrm{d}\eta_\varepsilon \\
&\qquad = \int_\Omega \int_Y \Psi(x)\omega(z)A(z)\left[\nabla y(x) + \mathbf{r}(x,z)\right]\,\mathrm{d}\eta(z)\,\mathrm{d}x,
\end{aligned}$$

or

$$A(\varepsilon^{-1}x)\nabla y_\varepsilon \overset{2}{\rightharpoonup} A(z)\left[\nabla y + \mathbf{r}(x,z)\right], \qquad (9.97)$$

$$A(\varepsilon^{-1}x)\nabla y_\varepsilon(x) \rightharpoonup \int_Y A(z)\left[\nabla y(x) + \mathbf{r}(x,z)\right]\,\mathrm{d}\eta(z) \quad \text{in} \quad \left[L^2(\Omega, \mathrm{d}\eta^h_\varepsilon\right]^2, \qquad (9.98)$$

that is, (9.96) can be rewritten in the explicit form as

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_\Omega \left(A(\varepsilon^{-1}x)\nabla y_\varepsilon, \nabla\omega(\varepsilon^{-1}x)\right)\Psi\,\mathrm{d}\eta^h_\varepsilon \\
&\qquad = \int_\Omega \left[\int_Y \left(A(z)\left[\nabla y(x) + \mathbf{r}(x,z)\right], \nabla\omega\right)\,\mathrm{d}\eta(z)\right]\Psi\,\mathrm{d}x = 0.
\end{aligned}$$

Since this equality holds true for every $\Psi \in C^\infty_0(\mathbb{R}^2, \Gamma_0)$, it follows that

$$\int_Y (A(\nabla y + \mathbf{r}), \zeta)\,\mathrm{d}\eta = 0, \qquad \forall\,\zeta \in \left[C^\infty_{per}(Y)\right]^2. \qquad (9.99)$$

However, following Zhikov [256], (9.99) can be viewed as Euler's equation for the minimum problem (9.93). Since this problem has a unique solution

$\mathbf{r}^0 \in L^2(\Omega, V_{pot})$, it follows that $\mathbf{r}^0$ is the unique solution of (9.99) as well. Let us define the matrix A^{hom} as in (9.92). Then, having put $\mathbf{r} = \mathbf{r}^0$ in (9.98) and using formula (9.92), we immediately deduce that

$$A\left(\varepsilon^{-1}x\right)\nabla y_\varepsilon \rightharpoonup A^{hom}\nabla y \qquad \text{in} \quad \left[L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)\right]^2. \tag{9.100}$$

Our next intention is to pass to the limit in (9.85) as $\varepsilon \to 0$. We choose the test functions in (9.85) as follows: $\varphi = \varphi_\varepsilon \in C_0^\infty(\mathbb{R}^2, \Gamma_\varepsilon^D)$ for every $\varepsilon > 0$ and $\varphi_\varepsilon \to \varphi$ in $C(\mathbb{R}^2)$, where $\varphi \in C^1(\Omega, \Gamma_0)$. Note that, due to Proposition 9.12, such choice is always possible. Then the passage to the limit in

$$\begin{aligned}\int_\Omega (A(x/\varepsilon)\nabla y_\varepsilon \cdot \nabla\varphi_\varepsilon)\,\mathrm{d}\eta_\varepsilon^h + \int_\Omega y_\varepsilon\varphi_\varepsilon\,\mathrm{d}\eta_\varepsilon^h + 4d\,\beta(\varepsilon)\int_\Omega y_\varepsilon\varphi_\varepsilon\,\mathrm{d}\lambda_\varepsilon^h \\ = \int_\Omega f_\varepsilon\,\varphi_\varepsilon\,\mathrm{d}\eta_\varepsilon^h + 2\,\beta(\varepsilon)\int_\Omega p_\varepsilon\varphi_\varepsilon\,\mathrm{d}\nu_\varepsilon^h + 4\,\beta(\varepsilon)\int_\Omega g_\varepsilon\varphi_\varepsilon\,\mathrm{d}\lambda_\varepsilon^h\end{aligned}$$

is justified by relations (9.86)–(9.89), (9.92), (9.94),

$$\lim_{\varepsilon\to 0}\beta(\varepsilon) = (1/2)\lim_{\varepsilon\to 0}\xi(\varepsilon) = (1/2)\xi^*, \tag{9.101}$$

and Lemma 9.10. Hence, we obtain

$$\begin{aligned}\int_\Omega (A^{hom}\nabla y\cdot\nabla\varphi)\,\mathrm{d}x + \int_\Omega y\varphi\,\mathrm{d}x + 2d\,\xi^*\int_\Omega y\varphi\,\mathrm{d}x \\ = \int_\Omega f_0\,\varphi\,\mathrm{d}x + \xi^*\int_{\Gamma_0} p\varphi\,\mathrm{d}x_1 + 2\,\xi^*\int_\Omega g\varphi\,\mathrm{d}x\end{aligned}$$

for every $\varphi \in C^1(\Omega, \Gamma_0)$. By the density of $C^1(\Omega, \Gamma_0)$ in $H^1(\Omega, \Gamma_0)$, this relation can be extended to any function $\varphi \in H^1(\Omega, \Gamma_0)$. Since $\int_{\Gamma_0} p\,\varphi\,\mathrm{d}x_1 = 0$ provided $\varphi \in H^1(\Omega, \Gamma_0)$, it follows that (9.91) holds true.

To conclude, we note that relations (9.90) and (9.91) can always be interpreted as the variational formulation of the following boundary value problem:

$$\left.\begin{aligned}-\mathrm{div}\left(A^{hom}\nabla y\right) + (1 + 2d\xi^*)y &= f_0 + 2\xi^* g && \text{in} \quad \Omega,\\ \partial_{\nu_{A^{hom}}} y &= 0 && \text{on} \quad \partial\Omega\setminus\Gamma_0,\\ y &= u && \text{on} \quad \Gamma_0,\end{aligned}\right\} \tag{9.102}$$

for which we have the following: For every $f_0, g \in L^2(\Omega)$ and $u \in H^1(\Gamma_0)$, there exists a unique solution of (9.102) (see Lions [168] and Michajlov [188]). □

The next statement is a direct consequence of well-known results of the theory of boundary value problems (see Lions [168]).

Corollary 9.28. *Let* $(f_{01}, u_1, y_1), (f_{02}, u_2, y_1) \in L^2(\Omega) \times H^1(\Gamma_0) \times H^1(\Omega)$ *be triplets satisfying relation* (9.102). *Then there exists a constant*

$$\hat{C} > 0 \quad (\hat{C} = \hat{C}(\Omega, \Gamma_0, g, \xi^*, \alpha_0))$$

such that

$$\|y_1 - y_2\|_{H^1(\Omega)} \le \hat{C}\left[\|f_{01} - f_{02}\|_{L^2(\Omega)} + \|u_1 - u_2\|_{H^1(\Gamma_0)}\right]. \tag{9.103}$$

9.6 Identification of the limit optimal control problem

In this section, we show that for the sequence of constrained minimization problems (9.40), there exists a weak variational limit with respect to the w-convergence in the variable space $\mathbb{Z}_\varepsilon$, and this limit problem can be recovered in an explicit form. We begin with the following result.

Theorem 9.29. *Let $y_\varepsilon^T \in L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$, $g_\varepsilon \in L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h)$, and $f_\varepsilon \in L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$ be given functions such that*

$$\sup_{\varepsilon\to 0}\left(\int_\Omega \left(y_\varepsilon^T\right)^2 \mathrm{d}\eta_\varepsilon^h\right) < \infty, \quad y_\varepsilon^T \to y^T \ \text{ in } \ L^2(\Omega, \mathrm{d}\eta_\varepsilon^h), \tag{9.104}$$

$$\sup_{\varepsilon\to 0}\left(\int_\Omega (f_\varepsilon)^2 \mathrm{d}\eta_\varepsilon^h\right) < \infty, \quad f_\varepsilon \to f \quad \text{ in } \ L^2(\Omega, \mathrm{d}\eta_\varepsilon^h), \tag{9.105}$$

$$\sup_{\varepsilon\to 0}\left(\int_\Omega (g_\varepsilon)^2 \mathrm{d}\lambda_\varepsilon^h\right) < \infty, \quad g_\varepsilon \rightharpoonup g \quad \text{ in } \ L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h). \tag{9.106}$$

Assume that the correction factor $\kappa(\varepsilon)$ in the cost functional (9.37) is such that

$$\lim_{\varepsilon\to 0} \kappa(\varepsilon)h(\varepsilon) = \kappa^* < +\infty. \tag{9.107}$$

Then for the sequence (9.40), there exists a unique weak variational limit with respect to the w-convergence which has the following representation

$$\left\langle \inf_{(u,p,y)\in\, \Xi_0} I_0(u,p,y) \right\rangle, \tag{9.108}$$

where the cost functional I_0 and the set of admissible solutions Ξ_0 are defined as

$$I_0(u,p,y) = 2\int_\Omega (y(x) - y^T)^2 \,\mathrm{d}x + \kappa^* \int_{\Gamma_0} u^2 \,\mathrm{d}x_1 + 2\int_{\Gamma_0} p^2 \,\mathrm{d}x_1, \tag{9.109}$$

$$\Xi_0 = \left\{ (u,p,y) \;\middle|\; \begin{array}{c} y \in H^1(\Omega), \quad u \in H^1(\Gamma_0), \quad p \in H^1(\Gamma_0), \\ \|u\|_{H^1(\Gamma_0)} \le \mathbf{C}_0, \quad y = u \ \text{a.e. on} \ \ \Gamma_0, \\ \displaystyle\int_\Omega (A^{hom}\nabla y \cdot \nabla\varphi)\,\mathrm{d}x + \int_\Omega y\varphi \,\mathrm{d}x \\ \displaystyle +2d\,\xi^* \int_\Omega y\varphi \,\mathrm{d}x = \int_\Omega f\,\varphi \,\mathrm{d}x \\ \displaystyle +2\,\xi^* \int_\Omega g\varphi \,\mathrm{d}x \quad \forall\, \varphi \in C^\infty(\Omega, \Gamma_0) \end{array} \right\}. \tag{9.110}$$

Proof. The proof of this theorem is divided into two steps concerning the verification of the correspondent items of Definition 8.24.

Step 1: *Statement (d) of Definition 8.24 holds.*

Let $\{(u_k, p_k, y_k) \in \mathbb{Z}_{\varepsilon_k}\}_{\varepsilon>0}$ be a bounded w-convergent sequence, and let $(u, p, y) \in \mathbb{Y}$ be its w-limit. Here, $\{\varepsilon_k\}$ is a subsequence of $\{\varepsilon > 0\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$. We suppose that $(u_k, p_k, y_k) \in \widehat{\Xi}_{\varepsilon_k}$ for all $k \in \mathbb{N}$. Then, due to Theorems 9.22 and 9.27, we have that the w-limit triplet (u, p, y) belongs to the space

$$\mathbb{Y} = H^1(\Gamma_0) \times L^2(\Gamma_0) \times H^1(\Omega)$$

and satisfies conditions (9.90) and (9.91). Hence, the first inclusion in (8.59) holds true.

It remains only to verify inequality (8.59) in which the limit cost functional I_0 has the representation (9.109). Taking into account the property of lower semicontinuity of the weak convergence in variable spaces, we have

$$\begin{aligned}\lim_{k\to\infty} \Bigg[& (2 - h(\varepsilon_k)) \int_\Omega (y_{\varepsilon_k}(x) - y^T_{\varepsilon_k})^2 \, \mathrm{d}\eta^h_{\varepsilon_k} \\ & + \kappa(\varepsilon_k) h(\varepsilon_k) \int_\Omega u^2_{\varepsilon_k} \, \mathrm{d}\mu^h_{\varepsilon_k} + 2 \, (1 - h(\varepsilon_k)) \int_\Omega p^2_{\varepsilon_k} \, \mathrm{d}\nu^h_{\varepsilon_k} \Bigg] \\ & \geq 2 \int_\Omega (y(x) - y^T)^2 \, \mathrm{d}x + \kappa^* \int_{\Gamma_0} u^2 \, \mathrm{d}x_1 + 2 \int_{\Gamma_0} p^2 \, \mathrm{d}x_1. \end{aligned} \tag{9.111}$$

Step 2: *Statement (dd) of Definition 8.24 holds true.*

Let $(u, p, y) \in \Xi_0$ be any admissible solution for the limiting minimization problem (9.108). Let $\widetilde{p} \in H^{3/2}(\Omega)$ be a given function. Then $\widetilde{p}\,|_{\Gamma_0} \in H^1(\Gamma_0)$ and, therefore, $(u, \widetilde{p}\,|_{\Gamma_0}, y) \in \Xi_0$. Moreover, it is readily seen from (9.109) that there is a constant $D_1 > 0$ such that

$$|I_0(u, p, y) - I_0(u, \widetilde{p}\,|_{\Gamma_0}, y)| \leq D_1 \|\widetilde{p}\,|_{\Gamma_0} - p\,\|_{L^2(\Gamma_0)}.$$

Let $0 < \delta < 1$ be a given value. Using the density of the embedding $H^1(B) \hookrightarrow L^2(B)$, we can choose an element $\widetilde{w} \in H^{3/2}(\Omega)$ such that

$$\|w - \widetilde{w}\,\|_{L^2(\Gamma_0)} < \varrho = D_1^{-1} \, \delta.$$

Then, due to the above estimates, we have

$$|I_0(u, p, y) - I_0(u, \widetilde{p}, y\,)| \leq \delta, \tag{9.112}$$

$$\|(u, p, y) - (u, \widetilde{p}, y\,)\|_{\mathbb{Y}} \leq \max\{1, D_1^{-1}\} \, \delta. \tag{9.113}$$

We now construct a δ-realizing sequence $\big\{(u^h_\varepsilon, p^h_\varepsilon, y^h_\varepsilon) \in \mathbb{Z}_\varepsilon\big\}_{\varepsilon>0}$ as follows:

$$u^h_\varepsilon \rightharpoonup u \ \text{ in } L^2(\Omega, \mathrm{d}\mu^h_\varepsilon) \ \text{ and } \ \int_\Omega \big(u^h_\varepsilon\big)^2 \, \mathrm{d}\mu^h_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \int_{\Gamma_0} u^2 \, \mathrm{d}x_1, \tag{9.114}$$

$$p^h_\varepsilon \rightharpoonup \widetilde{p} \ \text{ in } L^2(\Omega, \mathrm{d}\nu^h_\varepsilon) \ \text{ and } \ \int_\Omega \big(p^h_\varepsilon\big)^2 \, \mathrm{d}\nu^h_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \int_{\Gamma_0} \widetilde{p}^2 \, \mathrm{d}x_1, \tag{9.115}$$

$\|u\|_{H^1(\Gamma_0)} \le \mathbf{C}_0$, and y_ε^h is the correspondent solutions of the initial-boundary value problem (9.84)–(9.85). Hence, $(u_\varepsilon^h, p_\varepsilon^h, y_\varepsilon^h) \in \widehat{\Xi}_\varepsilon$ for every $\varepsilon > 0$. It is clear that this sequence is equibounded in $\mathbb{Z}_\varepsilon$.

Then, applying Theorem 9.22, we conclude that the sequence $\{(u_\varepsilon^h, p_\varepsilon^h, y_\varepsilon^h)\}$ is compact with respect to w-convergence. Let $(u, \widetilde{p}, \widetilde{y}^*)$ be its w-limit. Due to Theorem 9.27, we have $(u, \widetilde{p}, \widetilde{y}^*) \in \Xi_0$. Since the boundary value problem (9.102) has a unique solution for every fixed u, it follows that $y = \widetilde{y}^*$ and, hence, $(u_\varepsilon^h, p_\varepsilon^h, y_\varepsilon^h) \overset{w}{\rightharpoonup} (u, \widetilde{p}, y)$ as $\varepsilon \to 0$.

Thus, properties (8.60) of Definition 8.24 are fulfilled. It remains to verify inequality (8.61). To this end, we take into account properties (9.114) and (9.115) and Theorem 9.22. Then $\lim_{\varepsilon\to 0} \widehat{I}_\varepsilon(u_\varepsilon^h, p_\varepsilon^h, y_\varepsilon^h) = I_0(u, \widetilde{p}, y)$. To conclude, we apply inequality (9.112). This yields the required relation

$$I_0(u, p, y) \ge \lim_{\varepsilon\to 0} \widehat{I}_\varepsilon(u_\varepsilon^h, p_\varepsilon^h, y_\varepsilon^h) - \delta.$$

□

Remark 9.30. We emphasize that as follows from (9.109) and (9.110), the Neumann boundary control $p \in L^2(\Gamma_0)$ has a so-called "passive" influence on the limit object (9.108). The limit cost functional (9.109) (in contrast to the limit problem (9.102)) does not depend on this control. Moreover, it can be represented in the form

$$I_0(u, p, y) = 2\int_\Omega (y(x) - y^T)^2 \,\mathrm{d}x + \kappa^* \int_{\Gamma_0} u^2 \,\mathrm{d}x_1 + C(p). \tag{9.116}$$

In fact, taking into account the variational properties of the problem (9.108) stated in Theorem 9.26, the constant $C(p)$ should be defined as $2\displaystyle\int_{\Gamma_0} (p^0)^2 \,\mathrm{d}x_1$, where $p^0 \in L^2(\Gamma_0)$ is a weak limit in $L^2(\Omega, \mathrm{d}\nu_\varepsilon^h)$ of the optimal control $\widehat{p}_\varepsilon^0$ for the $\widehat{\mathbb{P}}_\varepsilon$-problem (see (9.36)–(9.38)). On the other hand, it is clear that an optimal solution (u^0, p^0, y^0) to the problem (9.108)–(9.110) is always such that $p^0 = 0$. So, in the sequel we will omit the constant $C(p)$ in the (9.116). Moreover, in view of the variational properties of the problem (9.108), it means that the Neumann optimal control $\widehat{p}_\varepsilon^0$ for the $\widehat{\mathbb{P}}_\varepsilon$-problem converges to 0 weakly in $L^2(\Omega, \mathrm{d}\nu_\varepsilon^h)$.

Note that since the constrained minimization problem (9.108) admits interpretation in the form of an OCP, this means that for the OCP (9.36)–(9.38), there exists a unique limit with respect to w-convergence as $\varepsilon \to 0$ and it can be represented in the form

$$\left.\begin{aligned} -\operatorname{div}\left(A^{hom}\nabla y\right) + y + 2d\xi^* y &= f + 2\xi^* g \quad \text{in } \Omega, \\ \partial_{\nu_{A^{hom}}} y &= 0 \quad \text{on } \partial\Omega \setminus \Gamma_0, \\ y &= u \quad \text{on } \Gamma_0, \end{aligned}\right\} \tag{9.117}$$

$$u \in H^1(\Gamma_0), \quad \|u\|_{H^1(\Gamma_0)} \le \mathbf{C}_0, \tag{9.118}$$

$$I_{hom}(u,y) = 2\int_\Omega (y(x) - y^T)^2 \,\mathrm{d}x + \kappa^* \int_{\Gamma_0} u^2 \,\mathrm{d}x_1 \longrightarrow \inf. \tag{9.119}$$

In what follows, this OCP will be called the $\mathbb{P}^{hom}$-problem. To examine the variational properties for the $\mathbb{P}^{hom}$-problem, we apply Theorem 9.26 and make use of the following obvious result (see [169]).

Proposition 9.31. *The limit OCP (9.117)–(9.119) has a unique solution.*

Combining the results of Theorem 9.26, Proposition 9.31, and Remark 9.30, we can provide the following conclusion concerning the variational properties of the $\mathbb{P}^{hom}$-problem.

Theorem 9.32. *Let $\left\{(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ be optimal solutions to the $\widehat{\mathbb{P}}_\varepsilon$-problem. Then there exists a pair $(u^0, y^0) \in H^1(\Gamma_0) \times H^1(\Omega)$ such that*

$$\begin{aligned} \lim_{\varepsilon\to 0} \widehat{I}_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) &= \lim_{\varepsilon\to 0} \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon)\in \widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) \\ &= \inf_{(u,p,y)\in\Xi_0} I_0(u,0,y) = I_{hom}(u^0, y^0), \end{aligned} \tag{9.120}$$

$$(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \xrightarrow{w} (u^0, 0, y^0) \quad \textit{in variable space} \quad \mathbb{Z}_\varepsilon. \tag{9.121}$$

In contrast to Theorem 9.26, (9.121) is valid for the whole sequence of optimal triplets. Indeed, since the limit OCP (9.117)–(9.119) has a unique solution, it follows that the limits of all w-convergent subsequences $\left\{(u_{\varepsilon_k}^0, p_{\varepsilon_k}^0, y_{\varepsilon_k}^0)\right\}_{k\in\mathbb{N}}$ of the sequence $\left\{(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ are coincident. Therefore, $(u^0, 0, y^0)$ is the w-limit triplet of the whole sequence. So, it remains to apply the arguments of Theorem 9.26 to the original sequence of optimal solutions $\left\{(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$.

9.7 On suboptimal controls for $\mathbb{P}_\varepsilon$-problems

This section deals with the approximation of optimal solutions to the original problem $\mathbb{P}_\varepsilon$ for small enough values of ε. It is clear that the numerical computation of optimal solutions to $\mathbb{P}_\varepsilon$-problems is very complicated for high accuracy; therefore, we will focus our attention on the definition of suboptimal solutions

$(u_\varepsilon^{sub}, p_\varepsilon^{sub}, y_\varepsilon^{sub})$ which should be admissible triplets guaranteeing the closeness of the corresponding value of the cost functional $I_\varepsilon(u_\varepsilon^{sub}, p_\varepsilon^{sub}, y_\varepsilon^{sub})$ to its minimal one. In view of this, we introduce the following concept.

Definition 9.33. *We say that a sequence of pairs*

$$\left\{(\widetilde{u}_\varepsilon^{\,0}, \widetilde{p}_\varepsilon^{\,0}) \in H^1(\Gamma_\varepsilon^D) \times L^2(\Gamma_\varepsilon^N)\right\}_{\varepsilon>0}$$

is asymptotically suboptimal for the problem $\mathbb{P}_\varepsilon$ if for every $\delta > 0$, there is $\varepsilon_0 > 0$ such that

$$\left| \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) - I_\varepsilon(\widetilde{u}_\varepsilon^{\,0}, \widetilde{p}_\varepsilon^{\,0}, \widetilde{y}_\varepsilon) \right| < \delta, \quad \forall\, \varepsilon < \varepsilon_0. \tag{9.122}$$

Here, by $\widetilde{y}_\varepsilon = \widetilde{y}_\varepsilon(\widetilde{u}_\varepsilon^{\,0}, \widetilde{p}_\varepsilon^{\,0})$ we denote the corresponding solutions of the boundary value problem (9.20).

The main result can be stated as follows.

Theorem 9.34. *Let $u^0 \in H^1(\Gamma_0)$ be an optimal Dirichlet control for the $\mathbb{P}^{hom}$-problem (9.117)–(9.119). Then the sequence of pairs*

$$\left\{(u^0|_{\Gamma_\varepsilon^D}, 0) \in H^1(\Gamma_\varepsilon^D) \times L^2(\Gamma_\varepsilon^N)\right\}_{\varepsilon>0} \tag{9.123}$$

is asymptotically suboptimal for the original optimal control problem $\mathbb{P}_\varepsilon$.

Proof. Let us consider the sequence of triplets $\left\{(u^0|_{\Gamma_\varepsilon^D}, 0, \widetilde{y}_\varepsilon)\right\}_{\varepsilon>0}$, where $\widetilde{y}_\varepsilon = y_\varepsilon(u^0|_{\Gamma_\varepsilon^D}, 0)$ are the corresponding solutions of the boundary value problem (9.20). It is easy to see that each of these triplets is admissible for the original control problem $\mathbb{P}_\varepsilon$. Hence, due to estimate (9.39), we conclude that the sequence of prototypes

$$\left\{(\widehat{u}^{\,0}, 0, \widehat{y}_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$$

is uniformly bounded in $\mathbb{Z}_\varepsilon$ and relatively compact with respect to the w-convergence (see Theorem 9.22). Here, $\widehat{u}^{\,0}$ is a prototype in $L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)$ of the control function $u^0|_{\Gamma_\varepsilon^D}$.

Let $(u, 0, y) \in \mathbb{Y}$ be some w-cluster triplet of this sequence. Applying Definition 9.21, Theorem 9.22, and the fact that $\widehat{u}^{\,0}|_{\Gamma_\varepsilon^D} = u^{\,0}|_{\Gamma_\varepsilon^D}$ for every $\varepsilon > 0$, we have

$$\widehat{y}_{\varepsilon_k} \to y \ \text{ in } \ L^2(\Omega, \mathrm{d}\eta_\varepsilon^h),$$
$$\widehat{u}^{\,0} \rightharpoonup u \ \text{ in } \ L^2(\Omega, \mathrm{d}\mu_\varepsilon^h), \quad \widehat{u}^{\,0} \xrightarrow{s_a} u^{\,0}.$$

Thus, $(u, 0, y) = (u^0, 0, y)$, and due to Theorems 9.27 and 9.32, we obtain

$$(u^0, 0, y) \in \Xi_0 \quad \Longrightarrow \quad (u^0, y) \in \Xi_{hom}, \ u^0 = u,$$

where by Ξ_{hom} we denote the set of admissible solutions for the limit problem (9.117)–(9.119). Hence, $y = y(u^0)$ is the corresponding solution of the limit boundary value problem (9.102). Thus, in view of Theorem 9.29, we have a more precise result: The pair $(u^0, y^0) \equiv (u^0, y) \in \Xi_{hom}$ is optimal for the limit problem $\mathbb{P}^{hom}$.

Let $\left\{(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be optimal solutions to the problems $\mathbb{P}_\varepsilon$. We observe that

$$
\begin{aligned}
&\Big| \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) - I_\varepsilon(u^0|_{\Gamma_\varepsilon^D}, 0, \widetilde{y}_\varepsilon) \Big| \\
&\qquad = \left| I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) - I_\varepsilon(u^0|_{\Gamma_\varepsilon^D}, 0, \widetilde{y}_\varepsilon) \right| \\
&\qquad \le \left| I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) - I_0(u^0, y^0) \right| + \left| I_0(u^0, y^0) - I_\varepsilon(u^0|_{\Gamma_\varepsilon^D}, 0, \widetilde{y}_\varepsilon) \right| \\
&\qquad \le J_1 + J_2 + J_3,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= \left| I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) - I_0(u^0, y^0) \right|, \\
J_2 &= \left| 2 \int_\Omega (y^0(x) - y^T)^2 \, \mathrm{d}x - (2 - h(\varepsilon)) \int_\Omega (y_\varepsilon^0(x) - y^T)^2 \, \mathrm{d}\eta_\varepsilon^h \right|, \\
J_3 &= \left| \kappa^* \int_{\Gamma_0} (u^0)^2 \, \mathrm{d}x_1 - \kappa(\varepsilon) h(\varepsilon) \int_\Omega \left(\widehat{u}_\varepsilon^0\right)^2 \, \mathrm{d}\mu_\varepsilon^h \right|.
\end{aligned}
$$

To conclude the proof, it remains to note that for a given $\delta > 0$, one can find the following:

(1) $\varepsilon_1 > 0$ such that $J_1 < \delta/3$ for all $\varepsilon < \varepsilon_1$ by Theorem 9.32.
(2) $\varepsilon_2 > 0$ such that $J_2 < \delta/3$ for all $\varepsilon < \varepsilon_2$ by Theorem 9.22.
(3) $\varepsilon_3 > 0$ such that $J_3 < \delta/3$ for all $\varepsilon < \varepsilon_3$ by Theorem 9.19 (see also Remark 9.20).

Thus, estimate (9.122) holds true for all $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. □

Our next intention is to derive the optimality conditions for the limit problem (9.117)–(9.119) from which an optimal control for the $\mathbb{P}^{hom}$-problem can be determined. To this end, we use the Lagrange multiplier principle. We obtain the weak form of the optimal system that an optimal pair (u^0, y^0) and Lagrange multipliers must satisfy. As Theorem 9.34 indicates, this optimality system can serve as a basis for the construction of suboptimal solutions to the original problem on the thin periodic structures Ω_ε.

We recall a few notions of the abstract Lagrange multiplier principle. Let $\mathbb{Y}$, $\mathbb{U}$, and $\mathbb{V}$ be Banach spaces. Let $I : \mathbb{Y} \times \mathbb{U} \to \overline{\mathbb{R}}$ be a cost functional and let $F(y, u) : \mathbb{Y} \times \mathbb{U} \to \mathbb{V}$ be a map. Let U_∂ be a closed subset of $\mathbb{U}$ with a nonempty interior. We have the following minimization problem:

$$
I(y, u) \longrightarrow \inf, \quad F(y, u) = 0, \quad u \in U_\partial. \tag{9.124}
$$

The Lagrange functional for the problem (9.124) is defined by

$$\mathcal{L}(y,u,\lambda,\Psi) = \lambda I(y,u) + \langle F(y,u),\Psi\rangle\,, \tag{9.125}$$

where $\lambda \in \mathbb{R}_+$ and $\Psi \in \mathbb{V}^*$.

Then each of the solutions to the constrained minimization problem (9.124) can be characterized as follows (see Theorem 3.41).

Theorem 9.35. *Let $(y^0,u^0) \in \mathbb{Y}\times\mathbb{U}$ be a solution of (9.124). Assume that the mappings $y \to I(y,u)$ and $y \to F(y,u)$ are continuously differentiable at $y \in \mathcal{O}(y^0)$ and $\operatorname{Im} F'_y(y^0,u^0) = \mathbb{V}$. Assume that the mapping $u \to I(y,u)$ is convex, I is differentiable at (y^0,u^0), and the mapping $u \to F(y,u)$ is continuous and affine. Then λ can be taken as 1 and there exists a $\Psi \in \mathbb{V}^*$ such that*

$$\left\langle \mathcal{L}'_y(y^0,u^0,1,\Psi), h\right\rangle = 0, \quad \forall\, h \in \mathbb{V}, \tag{9.126}$$

and

$$\left\langle \mathcal{L}'_u(y^0,u^0,1,\Psi), u-u^0\right\rangle \geq 0, \quad \forall\, u \in U_\partial. \tag{9.127}$$

We are now in a position to apply the Lagrange multiplier principle to optimal control problem (9.117)–(9.119).

Theorem 9.36. *If a pair*

$$(u^0,y^0) \in H^1(\Gamma_0)\times H^1(\Omega)$$

is an optimal solution to the problem (9.117)–(9.119), then there exists a function

$$\Psi \in H^1_0(\Omega) \tag{9.128}$$

such that the triplet (u^0,y^0,Ψ) satisfies the following optimality system:

$$\left.\begin{aligned} -\operatorname{div}\left(A^{hom}\nabla y^0\right) + y^0 + 2d\xi^* y^0 &= f + 2\xi^* g \quad \text{in } \Omega,\\ \partial_{\nu_{A^{hom}}} y^0 &= 0 \quad \text{on } \partial\Omega\setminus\Gamma_0,\\ y^0 &= u^0 \quad \text{on } \Gamma_0, \end{aligned}\right\} \tag{9.129}$$

$$\left.\begin{aligned} -\operatorname{div}\left(\left[A^{hom}\right]^t\nabla\Psi\right) + \Psi + 2d\xi^*\Psi &= 4(y^0-y^T) \quad \text{in } \Omega,\\ \Psi &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{9.130}$$

$$\left.\begin{aligned} \int_{\Gamma_0}\left(2\kappa^* u^0 - \partial_{\nu_{[A^{hom}]^t}}\Psi\right)(u-u^0)\,\mathrm{d}x_1\,\mathrm{d}t \geq 0,\\ \forall\, u \in H^1(\Gamma_0)\,:\,\|u\|_{H^1(\Gamma_0)} \leq \mathbf{C}_0 \end{aligned}\right\}. \tag{9.131}$$

Proof. Let (u^0, y^0) be an optimal solution to the limit problem (9.117)–(9.119). To apply the Lagrange multiplier principle, we set

$$\mathbb{Y} = \left\{ z \in H^1(\Omega) \right\}, \quad \mathbb{U} = H^1(\Gamma_0), \quad \mathbb{V} = (H^1(\Omega))^*,$$

and

$$F(u, y) = \left(-\mathrm{div}\left(A^{hom} \nabla y \right) + y + 2d\xi^* y - f - 2\xi^* g \right).$$

Then $F'_y(u^0, y^0) : \mathbb{Y} \to \mathbb{V}$ is an isomorphism. Hence, all suppositions of Theorem 9.35 are fulfilled.

We define the Lagrange function as

$$\begin{aligned} \mathcal{L}(u, y, \Psi) = 2 \int_\Omega (y(x) - y^T)^2 \, \mathrm{d}x + \kappa^* \int_{\Gamma_0} u^2 \, \mathrm{d}x_1 \\ + \int_\Omega (A^{hom} \nabla y \cdot \nabla \Psi) \, \mathrm{d}x + \int_\Omega y \Psi \, \mathrm{d}x \\ + 2d\,\xi^* \int_\Omega z \Psi \, \mathrm{d}x - \int_\Omega f\,\Psi \, \mathrm{d}x - 2\,\xi^* \int_\Omega g \Psi \, \mathrm{d}x \end{aligned}$$

for all $\Psi \in H_0^1(\Omega)$.

Then in accordance with Theorem 9.35, there exists a function $\Psi \in H_0^1(\Omega)$ such that relations (9.126) and (9.127) are valid. It is easy to see that in this case, (9.126) takes the form (9.129)–(9.130), whereas (9.127) can be written as (9.131). □

We note that this approach to the construction of suboptimal controls for the problem $\mathbb{P}_\varepsilon$ is acceptable if the following hold:

(A) The thin structure Ω_ε is sufficiently thick, that is, $\xi^* = 0$. Then the limit problem from which a suboptimal control can be determined takes the form

$$\left.\begin{aligned} -\mathrm{div}\left(A^{hom} \nabla y \right) &= f \quad \text{in} \quad \Omega, \\ \partial_{\nu_{A^{hom}}} y &= 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma_0, \\ y &= u \quad \text{on} \quad \Gamma_0, \end{aligned}\right\} \tag{9.132}$$

$$u \in H^1(\Gamma_0), \quad \|u\|_{H^1(\Gamma_0)} \le \mathbf{C}_0, \tag{9.133}$$

$$I_{hom}(u, y) = 2 \int_\Omega (y(x) - y^T)^2 \, \mathrm{d}x + \kappa^* \int_{\Gamma_0} u^2 \, \mathrm{d}x_1 \longrightarrow \inf. \tag{9.134}$$

(B) The thin structure Ω_ε has a critical thickness $\xi^* = \theta \in (0, +\infty)$. Then the corresponding limit problem for suboptimal control takes another form which depends on the thickness θ, namely

$$\left.\begin{aligned} -\operatorname{div}\left(A^{hom}\nabla y\right) + y + 2d\theta y &= f + 2\theta g \quad \text{in } \Omega, \\ \partial_{\nu_{A^{hom}}} y &= 0 \quad \text{on } \partial\Omega \setminus \Gamma_0, \\ y &= u \quad \text{on } \Gamma_0, \end{aligned}\right\}$$

$$u \in H^1(\Gamma_0), \quad \|u\|_{H^1(\Gamma_0)} \le \mathbf{C}_0,$$

$$I_{hom}(u, y) = 2\int_\Omega (y(x) - y^T)^2 \,\mathrm{d}x + \kappa^* \int_{\Gamma_0} u^2 \,\mathrm{d}x_1 \longrightarrow \inf.$$

(C) The thin structure Ω_ε is sufficiently thin (i.e., $\xi^* = +\infty$) and instead of the Robin and Neumann conditions along the boundaries S^ε_{int} and Γ^N_ε, respectively, we have the homogeneous Neumann condition

$$\partial_{\nu_A} y_\varepsilon = 0 \quad \text{on } \Gamma^N_\varepsilon \cup \Gamma^N_\varepsilon.$$

In this case, as it immediately follows from the integral identity (9.85) and Theorem 9.27, the limit problem from which a suboptimal Dirichlet control can be determined has the form (9.132)–(9.134).

10

Approximate Solutions of Optimal Control Problems for Ill-Posed Parabolic Problems on Thin Periodic Structures

In this chapter, we focus our attention on control objects which are described by singular parabolic equations with Robin boundary conditions at the boundary of the holes, and with two different types of boundary controls – Dirichlet and Neumann controls – on the external boundary of the thin periodic structure Ω_ε. We allow for a blowing-up phenomenon in the original problem and we provide its asymptotic analysis as the small parameter ε tends to 0. It is shown that the structure of the limit problem depends essentially on how h tends to 0 as $\varepsilon \to 0$ (the so-called "scaling effect"). We derive conditions under which in the limit we do not obtain an optimal control problem (OCP), but rather some initial-boundary value problems with or without controls. Furthermore, we construct asymptotically suboptimal controls for the original problem and show an approximation property of such controls for small enough ε near the optimal characteristics.

Let $\Omega_\varepsilon \subset \mathbb{R}^2$ be a periodic open set with a reticulated structure contained in a fixed bounded open set $\Omega = (0, L_1) \times (0, L_2)$. For the sake of simplicity, we restrict our analysis to the case when set Ω_ε has the representation $\Omega_\varepsilon = \Omega \cap \left(\varepsilon \mathcal{F}^h\right)$, where $\mathcal{F}^h$ denotes a 1-periodic thin grid with a small but same thickness h. So, we preserve all suppositions of Chap. 9 concerning the definition of the thin periodic structure Ω_ε and its measure description (see Fig. 9.1).

The OCPs we consider can be described in a general manner as follows: Seek a triplet $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ such that the functional

$$I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) = \frac{1}{h}\int_{\Omega_\varepsilon} (y_\varepsilon(T,\cdot) - y_\varepsilon^T)^2 \,\mathrm{d}x + \frac{1}{h}\int_0^T \int_{\Omega_\varepsilon} (y_\varepsilon - y_\varepsilon^*)^6 \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \kappa(\varepsilon)\int_0^T \int_{\Gamma_\varepsilon^D} u^2 \,\mathrm{d}x_1\,\mathrm{d}t + \int_0^T \int_{\Gamma_\varepsilon^N} p_\varepsilon^2 \,\mathrm{d}\mathcal{H}^1\mathrm{d}t \longrightarrow \inf \quad (10.1)$$

is minimized, subject to the constraints

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_10,
© Springer Science+Business Media, LLC 2011

$$\left.\begin{aligned}
\partial_t y_\varepsilon - \operatorname{div}\left(A(x/\varepsilon)\nabla y_\varepsilon\right) - y_\varepsilon^3 &= f_\varepsilon && \text{in } (0,T)\times\Omega_\varepsilon,\\
\partial_{\nu_A} y_\varepsilon &= 0 && \text{on } (0,T)\times S_{ext}^\varepsilon,\\
\partial_{\nu_A} y_\varepsilon &= \varepsilon^2(-d\, y_\varepsilon + g_\varepsilon) && \text{on } (0,T)\times S_{int}^\varepsilon,\\
y_\varepsilon &= u_\varepsilon && \text{on } (0,T)\times \Gamma_\varepsilon^D,\\
\partial_{\nu_A} y_\varepsilon &= \varepsilon p_\varepsilon && \text{on } (0,T)\times \Gamma_\varepsilon^N,\\
y_\varepsilon(0,x) &= y_{\varepsilon,0} && \text{for a.a. } x\in\Omega_\varepsilon,
\end{aligned}\right\} \tag{10.2}$$

$$p_\varepsilon \in L^2(0,T;L^2(\Gamma_\varepsilon^N)), \quad u_\varepsilon \in U_\varepsilon, \tag{10.3}$$

$$U_\varepsilon = \left\{u|_{\Gamma_\varepsilon^D} \ :\ u\in L^2(0,T;\, H^1(\Gamma_0)), \quad \|u\|_{L^2(0,T;\, H^1(\Gamma_0))} \le \mathbf{C}_0\right\}, \tag{10.4}$$

Here, $p_\varepsilon \in L^2(0,T;L^2(\Gamma_\varepsilon^N))$ and $u_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon^D))$ are the controls which are realized via the Neumann and Dirichlet boundary conditions posed on different parts Γ_ε^D and Γ_ε^N of the quickly oscillating part of the external boundary $\partial\Omega_\varepsilon$.

The main object of our consideration is the control problem for the singular distributed system (10.2) for which such properties as instabilities, multiple solutions, and blowing-up phenomenon may occur. We provide an asymptotic analysis of the OCP for which not for every admissible controls u_ε and p_ε there exists the corresponding global solution $y_\varepsilon = y_\varepsilon(u_\varepsilon, p_\varepsilon)$ of the original initial-boundary value problem (10.2) in $(0,T)\times\Omega_\varepsilon$, in contrast to fundamental assumptions in classical approaches to the homogenization theory (see, for instance, Attouch [10], Buttazzo and Dal Maso [43], Cioranescu and Saint Jean Paulean [66], Dal Maso [78], Kapustyan and Panik [129], Pastuchova [213], and Zhikov, Kozlov, and Olienik [261]).

Second, the thin domains Ω_ε we consider here are always such that their thickness $h = h(\varepsilon)$ is related with the parameter of periodicity ε by the supposition $h(\varepsilon)\to 0$ as $\varepsilon\to 0$. It is the principal difference between this type of domain and the periodically perforated ones for which the another rule takes place, namely $h = h(\varepsilon)\to \text{const}\in(0,1]$ as $\varepsilon\to 0$.

Third, in this chapter we deal with the OCP rather than with the initial-boundary value problem. In spite of the fact that the distributed system (10.2) is singular, the OCP (10.1)–(10.4), that we are considering here is well defined and solvable for every $\varepsilon > 0$. However, because of the complexity of the numerical computation of the optimal solutions to the original problem for small enough values of ε (through the thick perforation of the domain Ω_ε and mainly because of the singularity of the corresponding optimality system), we proceed to pass to the limit as $\varepsilon\to 0$ and use the limiting problem as a reduced model with preservation of the main variational property: Both optimal solution and minimal value of the cost functional for the original problem converge to the corresponding characteristics of a limit OCP as ε tends to 0.

We propose the so-called "indirect approach" to the limit analysis of the boundary OCP for the singular system. The main idea is to introduce a couple of new OCPs for parabolic equations on the reticulated structure Ω_ε. The first of them is a so-called virtual extension of the original problem $\mathbb{P}_\varepsilon$. Having

entered a fictitious "distributed" control into the $\mathbb{P}_\varepsilon$-problem and modified its cost functional, we prove that the new problem (hereafter referred to as $\widehat{\mathbb{P}}_\varepsilon$) always has a non-empty set of admissible solutions (in contrast to the original one); thus, this one is solvable, and moreover for every $\varepsilon > 0$, there is a one-to-one correspondence between the sets of optimal solutions for the $\mathbb{P}_\varepsilon$- and $\widehat{\mathbb{P}}_\varepsilon$-problems. The second problem is the so-called C-extension of the virtual one. We denote it by $\mathbb{CP}_\varepsilon$. The consideration of this problem is motivated by the following observations: First, the sets of admissible states for the $\widehat{\mathbb{P}}_\varepsilon$- and $\mathbb{CP}_\varepsilon$-problems coincide, whereas their sets of admissible solutions are different. Second, the state equation of the $\mathbb{CP}_\varepsilon$-problem has a linear structure. Finally, there is a one-to-one correspondence between optimal solutions for the $\widehat{\mathbb{P}}_\varepsilon$- and $\mathbb{CP}_\varepsilon$-problems. It is clear that in the linear case the C-extended problem always coincides with the original one. As a result, we propose the following scheme for asymptotic analysis of optimal control problems for the blowing-up parabolic equation:

$$\mathbb{P}_\varepsilon \Longrightarrow \widehat{\mathbb{P}}_\varepsilon \Longrightarrow \mathbb{CP}_\varepsilon - \text{passage to the limit} - \mathbb{CP}^{hom} \Longrightarrow \widehat{\mathbb{P}}^{hom} \Longrightarrow \mathbb{P}^{hom}.$$

We show that the $\mathbb{CP}^{hom}$-problem is a C-extension for $\widehat{\mathbb{P}}^{hom}$ which, in turn, can be interpreted as a virtual extension of the OCP $\mathbb{P}^{hom}$. However, the main advantage of such an approach is the fact that it preserves the well-known variational property: "If $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ is an optimal triplet of the $\mathbb{P}_\varepsilon$-problem and if $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ tends (in some sense) to (u^0, p^0, y^0), then (u^0, p^0, y^0) is an optimal solution for the limit problem."

Thus, in contrast to the well-known approach (see, for instance, Kesavan and Saint Jean Paulin [134, 135], Saint Jean Paulin and Zoubairi [222], and Muthukumar [197]), we do not just look for a limit of optimal solutions and for a limit of minimal values of the cost functionals for the problem (10.1)–(10.4). We propose primarily to look for a $\mathbb{CP}^{hom}$-problem as some variational limit of the C-extended ones and after that to pass from $\mathbb{CP}^{hom}$ for the recovery of the $\mathbb{P}^{hom}$-problem. Of course, this limiting problem should be unique (as a result of some passage to the limit) and should have a clearly defined structure including the limit form of a state equation, control constraints, and a limit cost functional. For this we follow the main ideas of the theory of Γ-convergence and the concept of variational convergence of constrained minimization problems (see Chap. 8; see also Kogut [140], Kogut and Leugering [147, 145, 146, 148, 149, 150], and D'Apice, De Maio, and Kogut [86, 87, 85]).

As we will see later, for the OCP (10.1)–(10.4) with Robin conditions on the boundary of holes, the structure of the corresponding limit problem depends essentially on how h tends to 0 as $\varepsilon \to 0$ (the so-called "scaling effect"), namely we show that such effects hold only for the thin structures of critical thickness. In this case, some additional terms appear in the limiting state equation. At the same time, we do not have any artifacts, if the thin structure Ω_ε is sufficiently thick. However, as for the sufficiently thin domains Ω_ε, the limit problem for (10.1)–(10.4) does not exist. However, the asymptotic anal-

ysis of this object is possible only if, along the boundary S^{ε}_{int}, homogeneous Neumann conditions are given instead of the Robin conditions.

The circumstances we would like to concentrate on concern the so-called "stability of homogenization" with respect to the Dirichlet boundary control. The point is that the original problem $\mathbb{P}_{\varepsilon}$ consists of two types of boundary controls u_{ε} and p_{ε}, respectively. However, the passage to the limit in $\mathbb{P}_{\varepsilon}$ always leads us to an OCP with only one boundary control, namely the Dirichlet control. Moreover, this conclusion holds true in spite of a "volume" of the boundary $\Gamma^{D}_{\varepsilon} \subset \partial\Omega_{\varepsilon}$ occupied by the Dirichlet control zone. We show that the optimal solution for this limit problem can be used for the construction of the so-called asymptotically suboptimal controls. Moreover, we prove an approximation property of such controls for small enough ε.

We also show that an OCP with the Neumann boundary control can appear as a result of the homogenization of $\mathbb{P}_{\varepsilon}$ only if $\Gamma^{D}_{\varepsilon} = \emptyset$ for every $\varepsilon > 0$ and the domain Ω_{ε} is of critical thickness. This result can be explained by the fact that the Dirichlet boundary control is more stable with respect to the homogenization procedure than the Neumann one.

In contrast to this case, we consider also the problem given above on fattened graphs which do not satisfy the definition of thin grids. It is clear that in this case, the $\mathcal{F}^{h}$ may have rather complicated geometry with non-rectilinear form of their edges and with components of different thickness. We show that in this case, the "scaling effect" in the limit may appear not only for the reticulated structures of critical thickness but also for the sufficiently thin domains. Moreover, the homogenization on fattened graphs can be characterized by "non-classical" artifacts. To explain this, we give several concrete examples when the original object (10.1)–(10.4) in the limit leads us to atypical results from the control theory point of view. For instance, in the limit as $\varepsilon \to 0$, we do not have an OCP but only some initial-boundary value problems with or without controls.

10.1 Statement of the problem

Let Ω_{ε} be a thin structure for which description (9.4) and suppose all suppositions of Chap. 9 hold true. For a given $T > 0$, we consider the following controlled initial-boundary value problem in $\Omega_{\varepsilon} \times (0, T)$:

$$\left.\begin{aligned}
\partial_t y_{\varepsilon} - \operatorname{div}\left(A(x/\varepsilon)\nabla y_{\varepsilon}\right) - y^3_{\varepsilon} &= f_{\varepsilon} && \text{in } (0,T)\times\Omega_{\varepsilon},\\
\partial_{\nu_A} y_{\varepsilon} &= 0 && \text{on } (0,T)\times S_{ext},\\
\partial_{\nu_A} y_{\varepsilon} &= \varepsilon^2(-d y_{\varepsilon} + g_{\varepsilon}) && \text{on } (0,T)\times S^{\varepsilon}_{int},\\
y_{\varepsilon} &= u_{\varepsilon} && \text{on } (0,T)\times \Gamma^{D}_{\varepsilon},\\
\partial_{\nu_A} y_{\varepsilon} &= \varepsilon p_{\varepsilon} && \text{on } (0,T)\times \Gamma^{N}_{\varepsilon},\\
y_{\varepsilon}(0,x) &= y_{\varepsilon,0} && \text{for a.a. } x \in \Omega_{\varepsilon},
\end{aligned}\right\} \tag{10.5}$$

where d is a positive constant, $g_{\varepsilon} \in L^2(0,T;L^2(S^{\varepsilon}_{int}))$, $f_{\varepsilon} \in L^2(0,T;L^2(\Omega))$, and $y_{\varepsilon,0} \in H^1(\Omega)$ are given functions, $\partial_{\nu_A} y_{\varepsilon} = (A(x/\varepsilon)\nabla y_{\varepsilon})\cdot\nu$, ν is the

unit outward normal vector with respect to Ω_ε, and $A(x)$ is a Y-periodic measurable matrix such that

$$\begin{gathered} A(\cdot) \in L^\infty(Y, \mathbb{R}^{2\times 2}), \\ \alpha_0 \|\xi\|^2 \le (A(x)\xi, \xi) \le \alpha_0^{-1} \|\xi\|^2 \quad \text{for a.a.} \quad x \in Y, \end{gathered} \tag{10.6}$$

for some fixed constant $\alpha_0 > 0$ and every $\xi \in \mathbb{R}^2$.

Hereinafter we interpret the functions u_ε and p_ε as Dirichlet and Neumann boundary controls, respectively. We say that the control functions u_ε and p_ε are admissible if the following conditions hold:

$$p_\varepsilon \in L^2(0,T; L^2(\Gamma_\varepsilon^N)), \quad u_\varepsilon \in U_\varepsilon, \tag{10.7}$$

where the set U_ε is defined as

$$U_\varepsilon = \left\{ u|_{\Gamma_\varepsilon^D} \ : \ u \in L^2(0,T;\, H^1(\Gamma_0)), \quad \|u\|_{L^2(0,T;\, H^1(\Gamma_0))} \le \mathbf{C_0} \right\}, \tag{10.8}$$

$\Gamma_0 = \{0 < x_1 < L_1, x_2 = 0\}$, and the constant $\mathbf{C_0}$ is independent of ε.

The initial-boundary value problem (10.5) is a singular one in the following sense: Because of the term $-y_\varepsilon^3$, a global solution $y_\varepsilon = y_\varepsilon(u_\varepsilon, p_\varepsilon)$ of (10.5) may not exist for some admissible controls u_ε and p_ε (see Lions [172]).

Taking into account the compactness of the embedding $H^1(\Gamma_\varepsilon^D) \hookrightarrow L^2(\Gamma_\varepsilon^D)$, we associate with (10.5) the following cost functional:

$$\begin{aligned} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) = {} & \frac{1}{h} \int_{\Omega_\varepsilon} (y_\varepsilon(T,x) - y_\varepsilon^T)^2 \, \mathrm{d}x + \frac{1}{h} \int_0^T \int_{\Omega_\varepsilon} (y_\varepsilon - y_\varepsilon^*)^6 \, \mathrm{d}x \, \mathrm{d}t \\ & + \kappa(\varepsilon) \int_0^T \int_{\Gamma_\varepsilon^D} u^2 \, \mathrm{d}x_1 \, \mathrm{d}t + \int_0^T \int_{\Gamma_\varepsilon^N} p_\varepsilon^2 \, \mathrm{d}\mathcal{H}^1 \, \mathrm{d}t \end{aligned} \tag{10.9}$$

defined over all $u_\varepsilon \in L^2(0,T; H^1(\Gamma_\varepsilon^D))$, $p_\varepsilon \in L^2(0,T; L^2(\Gamma_\varepsilon^N))$, and $y_\varepsilon \in L^2(0,T; H^1(\Omega_\varepsilon))$ given by (10.5), subject to the control constraints (10.7). Here, $y_\varepsilon^* \in L^6((0,T) \times \Omega_\varepsilon)$ and $y_\varepsilon^T \in L^2(\Omega)$ are given functions, $\kappa(\varepsilon)$ is a given value, and $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure on Γ_ε^N.

Thus, the OCP we consider in this chapter can be stated as follows: Find a triplet $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in L^2(0,T; H^1(\Gamma_\varepsilon^D)) \times L^2(0,T; L^2(\Gamma_\varepsilon^N)) \times L^2(0,T; H^1(\Omega_\varepsilon))$ such that

$$I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) = \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon), \tag{10.10}$$

where the set of admissible triplets Ξ_ε is defined as

$$\begin{aligned} \Xi_\varepsilon = \big\{ & (u_\varepsilon, p_\varepsilon, y_\varepsilon) \in L^2(0,T; H^1(\Gamma_\varepsilon^D)) \\ & \times L^2(0,T; L^2(\Gamma_\varepsilon^N)) \times L^2(0,T; H^1(\Omega_\varepsilon)) \ : \\ & (u_\varepsilon, p_\varepsilon, y_\varepsilon) \text{ satisfies (in some weak sense) (10.5)–(10.8)} \big\}. \end{aligned} \tag{10.11}$$

In the sequel, the OCP (10.5)–(10.11) will be called the $\mathbb{P}_\varepsilon$-problem.

10.2 On the solvability of $\mathbb{P}_\varepsilon$ and its C-extension

We begin this section with the variational formulation of the initial-boundary value problem (10.5): Find a function $y_\varepsilon = y_\varepsilon(u_\varepsilon, p_\varepsilon) \in L^2(0,T;H^1(\Omega_\varepsilon))$ such that (see Ladyzhenskaya and Ural'tseva [162], and Michajlov [188])

$$y_\varepsilon = u_\varepsilon \ \text{ a.e. on } \ \Gamma_\varepsilon^D, \quad y_\varepsilon(0,\cdot) = y_{\varepsilon,0} \ \text{ a.e. on } \ \Omega_\varepsilon \tag{10.12}$$

and the following identity is valid:

$$\begin{aligned}
&\int_0^T \int_{\Omega_\varepsilon} \left(-y_\varepsilon \varphi\, \psi' + A(x/\varepsilon)\nabla y_\varepsilon \cdot \nabla\varphi\, \psi - y_\varepsilon^3\, \varphi\, \psi\right) \mathrm{d}x\, \mathrm{d}t \\
&\quad + d\,\varepsilon^2 \int_0^T \int_{S_{int}^\varepsilon} y_\varepsilon \varphi\, \psi\, \mathrm{d}\mathcal{H}^1\, \mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon\, \varphi\, \psi\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \varepsilon^2 \int_0^T \int_{S_{int}^\varepsilon} g_\varepsilon \varphi\, \psi\, \mathrm{d}\mathcal{H}^1\, \mathrm{d}t + \varepsilon \int_0^T \int_{\Gamma_\varepsilon^N} p_\varepsilon \varphi\, \psi\, \mathrm{d}\mathcal{H}^1\, \mathrm{d}t, \\
&\qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^2;\Gamma_\varepsilon^D),\ \forall\, \psi \in C_0^\infty(0,T),
\end{aligned} \tag{10.13}$$

where by $C_0^\infty(\mathbb{R}^2;\Gamma_\varepsilon^D)$ we denote the set of all functions from $C_0^\infty(\mathbb{R}^2)$ such that $\varphi|_{\Gamma_\varepsilon^D} = 0$.

In the sequel, the function y_ε is called a weak solution to the problem (10.5). Since the domain Ω_ε is nonconvex with nonsmooth boundary $\partial\Omega_\varepsilon$, a solution of initial-boundary value problem (10.5) can have only minimal smoothness for ε small enough. It is the main reason for the choice of $L^2(0,T;H^1(\Omega_\varepsilon))$ as the main state space for the original problem. Moreover, for given functions

$$f \in L^2((0,T)\times\Omega),\ g_\varepsilon \in L^2(0,T;L^2(S_{int}^\varepsilon)),\ u \in L^2(0,T;H^1(\Gamma_\varepsilon^D)),$$
$$p_\varepsilon \in L^2(0,T;L^2(\Gamma_\varepsilon^N)),$$

the problem (10.5), in general, may not have a global solution at the time, that is, there does not exist a function $y_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon))$ satisfying (10.12)–(10.13). Nevertheless, we suppose that for every $\varepsilon > 0$, there exists at least one triplet

$$(u_\varepsilon^*, p_\varepsilon^*, y_\varepsilon^*) \in U_\varepsilon \times L^2(0,T;L^2(\Gamma_\varepsilon^N)) \times \left[L^6(0,T;L^6(\Omega_\varepsilon)) \cap L^2(0,T;H^1(\Omega_\varepsilon))\right]$$

such that u_ε^*, p_ε^*, and y_ε^* satisfy (10.5)–(10.8), that is,

$$\Xi_\varepsilon \neq \emptyset \quad \text{for every } \varepsilon > 0. \tag{10.14}$$

We make use now of the following result (cf. Dubinskiy [97] and Lions [168, Theorem 5.1, Chap. 1].

Lemma 10.1. *Let $B_0 \subset B_1 \subset B_2$ be Banach spaces such that B_0 and B_1 are reflexive, the injection $B_0 \hookrightarrow B_1$ is compact, and $B_1 \hookrightarrow B_2$ is continuous. Let*

$$W = \{\varphi : \varphi \in L^{p_0}(0,T;B_0), \partial\varphi/\partial t \in L^{p_2}(0,T;B_2)\}, \quad \text{where } 1 < p_0, p_1, p_2 < +\infty.$$

Then W is a Banach space with respect to the graph norm and the injection $W \hookrightarrow L^{p_1}(0,T;B_1)$ is compact.

Having put $B_0 = H^1(\Omega_\varepsilon)$, $B_1 = L^6(\Omega_\varepsilon)$, $B_2 = L^2(\Omega_\varepsilon)$, $p_0 = 2$, $p_1 = 6$, and $p_2 = 2$, the following statement is the direct consequence of the Sobolev embedding theorem and Lemma 10.1.

Lemma 10.2. *The injection*

$$\mathbb{Y}_\varepsilon = \left[H^1(0,T;L^2(\Omega_\varepsilon)) \cap L^2(0,T;H^1(\Omega_\varepsilon))\right] \hookrightarrow L^6(0,T;L^6(\Omega_\varepsilon))$$

is compact.

We are now in a position to prove the following result.

Theorem 10.3. *For every value $\varepsilon > 0$, there exists a solution $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ for the OCP $\mathbb{P}_\varepsilon$.*

Proof. We have assumed (in order to make the problem $\mathbb{P}_\varepsilon$ meaningful) that supposition (10.14) holds true. Then $\inf I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) \le I_\varepsilon(u_\varepsilon^*, p_\varepsilon^*, y_\varepsilon^*)$. Hence, if $\left\{(u_\varepsilon^k, p_\varepsilon^k, y_\varepsilon^k)\right\}_{k=1}^\infty$ is a minimizing sequence, we have

$$I_\varepsilon(u_\varepsilon^k, p_\varepsilon^k, y_\varepsilon^k) \le const, \quad \|u_\varepsilon^k\|_{L^2(0,T;H^1(\Gamma_\varepsilon^D))} \le \mathbf{C}_0 \quad \text{for all} \quad k \in \mathbb{N}.$$

Then, as follows from (10.9), we conclude that

$$\|y_\varepsilon^k\|_{L^6((0,T)\times\Omega_\varepsilon)} \le C, \quad \|p_\varepsilon^k\|_{L^2((0,T)\times\Gamma_\varepsilon^N)} \le C. \tag{10.15}$$

Consequently,

$$\partial_t y_\varepsilon^k - \operatorname{div}\left(A(x/\varepsilon)\nabla y_\varepsilon^k\right) = \left(y_\varepsilon^k\right)^3 + f_\varepsilon \in L^2(0,T;L^2(\Omega)), \quad \forall\, k \in \mathbb{N}.$$

Therefore, using standard techniques, the following a priori estimate can be easily obtained from the integral identity (10.13) (see Lions [172]):

$$\begin{aligned}\|y_\varepsilon^k\|_{L^2(0,T;H^1(\Omega_\varepsilon))} &+ \|y_\varepsilon^k\|_{H^1(0,T;L^2(\Omega_\varepsilon))} \\ &\le C(\varepsilon)\big[\|f_\varepsilon\|_{L^2((0,T)\times\Omega)} + \|u_\varepsilon^k\|_{L^2(0,T;H^1(\Gamma_\varepsilon^D))} \\ &+ \|p_\varepsilon^k\|_{L^2((0,T)\times\Gamma_\varepsilon^N)} + \|y_\varepsilon^k\|_{L^6((0,T)\times\Omega_\varepsilon)} \\ &+ \|y_\varepsilon^0\|_{L^2(\Omega_\varepsilon)}\big].\end{aligned} \tag{10.16}$$

In view of (10.15), we conclude that $\left\{y_\varepsilon^k\right\}$ is bounded in $\mathbb{Y}_\varepsilon$, and for every $k \in \mathbb{N}$, the triplets $(u_\varepsilon^k, p_\varepsilon^k, y_\varepsilon^k)$ are admissible for the $\mathbb{P}_\varepsilon$-problem.

Hence, we can extract a subsequences (still indexed by k) such that

$$p_\varepsilon^k \rightharpoonup p_\varepsilon^0 \ \text{ in } \ L^2((0,T)\times\Gamma_\varepsilon^N), \quad u_\varepsilon^k \rightharpoonup u_\varepsilon^0 \ \text{ in } \ L^2(0,T;H^1(\Gamma_\varepsilon^D)),$$

$$y_\varepsilon^k \rightharpoonup y_\varepsilon^0 \ \text{ in } \ L^2(0,T;H^1(\Omega_\varepsilon)), \quad y_\varepsilon^k \to y_\varepsilon^0 \ \text{ in } \ L^6((0,T)\times\Omega_\varepsilon)$$

(here we have used Lemma 10.2).

Taking these results into account and passing to the limit in (10.13) as $k\to\infty$, we deduce that $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ is admissible for the $\mathbb{P}_\varepsilon$-problem. Finally, using the lower semicontinuity of the cost functional I_ε with respect to the weak topology of $L^2(0,T;H^1(\Gamma_\varepsilon^D))\times L^2((0,T)\times\Gamma_\varepsilon^N)\times L^2(0,T;H^1(\Omega_\varepsilon))$, we obtain

$$I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \le \liminf_{k\to\infty} I_\varepsilon(u_\varepsilon^k, p_\varepsilon^k, y_\varepsilon^k) = \inf_{(u_\varepsilon,p_\varepsilon,y_\varepsilon)\in\,\Xi_\varepsilon} I_\varepsilon(u_\varepsilon,p_\varepsilon,y_\varepsilon),$$

which proves that $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ is an optimal triplet. □

Now, we formulate a couple of new OCPs for parabolic equations on the network Ω_ε. The first of them, the so-called virtual extension $\widehat{\mathbb{P}}_\varepsilon$ of the original $\mathbb{P}_\varepsilon$-problem, is formulated as follows: Find a quadruple

$$(\widehat{q}_\varepsilon{}^0, \widehat{u}_\varepsilon^0, \widehat{p}_\varepsilon^0, \widehat{y}_\varepsilon^0) \in \mathbb{X}_\varepsilon \equiv L^2((0,T)\times\Omega_\varepsilon)\times L^2(0,T;H^1(\Gamma_\varepsilon^D)) \times L^2((0,T)\times\Gamma_\varepsilon^N)\times L^2(0,T;H^1(\Omega_\varepsilon))$$

such that

$$\left(\widehat{\mathbb{P}}_\varepsilon\right): \qquad \widehat{I}_\varepsilon(\widehat{q}_\varepsilon{}^0, \widehat{u}_\varepsilon^0, \widehat{p}_\varepsilon^0, \widehat{y}_\varepsilon^0) = \inf_{(q_\varepsilon,u_\varepsilon,p_\varepsilon,y_\varepsilon)\in\,\widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(q_\varepsilon,u_\varepsilon,p_\varepsilon,y_\varepsilon), \tag{10.17}$$

where the cost functional $\widehat{I}_\varepsilon$ and the set of admissible solutions $\widehat{\Xi}_\varepsilon$ have the following analytical representations:

$$\begin{aligned}\widehat{I}_\varepsilon(q_\varepsilon,u_\varepsilon,p_\varepsilon,y_\varepsilon) &= \frac{1}{h(\varepsilon)}\int_{\Omega_\varepsilon}(y_\varepsilon(T,x)-y_\varepsilon^T)^2\,\mathrm{d}x + \kappa(\varepsilon)\int_0^T\int_{\Gamma_\varepsilon^D} u^2\,\mathrm{d}x_1\,\mathrm{d}t \\ &+ \frac{1}{h(\varepsilon)}\int_0^T\int_{\Omega_\varepsilon}(y_\varepsilon-y_\varepsilon^*)^6\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Gamma_\varepsilon^N} p_\varepsilon^2\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t \\ &+ \frac{\gamma(\varepsilon)}{h(\varepsilon)}\int_0^T\int_{\Omega_\varepsilon}(q_\varepsilon-f_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{10.18}$$

$$\widehat{\Xi}_\varepsilon = \{(q_\varepsilon, u_\varepsilon, p_\varepsilon, y_\varepsilon) :$$
$$\left|\begin{array}{l} y_\varepsilon = u_\varepsilon \ \text{ on } \ \Gamma_\varepsilon^D,\ u_\varepsilon \in U_\varepsilon,\ y_\varepsilon(0,\cdot) = y_{\varepsilon,0} \ \text{ on } \ \Omega_\varepsilon, \\ \quad p_\varepsilon \in L^2((0,T)\times\Gamma_\varepsilon^N),\ q_\varepsilon \in L^2((0,T)\times\Omega_\varepsilon), \\ \quad \displaystyle\int_0^T \int_{\Omega_\varepsilon} (-y_\varepsilon\varphi\,\psi' + A(x/\varepsilon)\nabla y_\varepsilon\cdot\nabla\varphi\,\psi)\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle -\int_0^T \int_{\Omega_\varepsilon} y_\varepsilon^3\,\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t + d\,\varepsilon^2 \int_0^T \int_{S_{int}^\varepsilon} y_\varepsilon\varphi\,\psi\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t \\ \displaystyle = \int_0^T \int_{\Omega_\varepsilon} q_\varepsilon\,\varphi\,\psi\,\mathrm{d}x\mathrm{d}t + \varepsilon^2 \int_0^T \int_{S_{int}^\varepsilon} g_\varepsilon\varphi\,\psi\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t \\ \qquad\qquad \displaystyle +\varepsilon\int_0^T \int_{\Gamma_\varepsilon^N} p_\varepsilon\varphi\,\psi\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t, \\ \qquad\qquad \forall\,\varphi \in C_0^\infty(\mathbb{R}^2;\Gamma_\varepsilon^D),\ \forall\ \psi \in C_0^\infty(0,T) \end{array}\right\}. \quad (10.19)$$

Here, $\gamma(\varepsilon) > 0$ is a penalizing coefficient. It is clear that the corresponding initial-boundary value problem for (10.19) can be formally written in the form

$$\left.\begin{array}{rll} \partial_t y_\varepsilon - \mathrm{div}\,(A(x/\varepsilon)\nabla y_\varepsilon) - y_\varepsilon^3 = q_\varepsilon & & \text{in } (0,T)\times\Omega_\varepsilon, \\ \partial_{\nu_A} y_\varepsilon = 0 & & \text{on } (0,T)\times S_{ext}, \\ \partial_{\nu_A} y_\varepsilon = \varepsilon^2(-d y_\varepsilon + g_\varepsilon) & & \text{on } (0,T)\times S_{int}^\varepsilon, \\ y_\varepsilon = u_\varepsilon & & \text{on } (0,T)\times\Gamma_\varepsilon^D, \\ \partial_{\nu_A} y_\varepsilon = \varepsilon p_\varepsilon & & \text{on } (0,T)\times\Gamma_\varepsilon^N, \\ y_\varepsilon(0,x) = y_{\varepsilon,0} & & \text{for a.a. } x \in \Omega_\varepsilon, \end{array}\right\} \quad (10.20)$$

where $q_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon))$ denotes a virtual control.

Additionally, the second OCP, the so-called C-extension of the $\widehat{\mathbb{P}}_\varepsilon$ problem, takes the following form: Find a quadruple $(a_\varepsilon^0, v_\varepsilon^0, w_\varepsilon^0, z_\varepsilon^0) \in \mathbb{X}_\varepsilon$ such that

$$(\mathbb{CP}_\varepsilon): \qquad J_\varepsilon(a_\varepsilon^0, v_\varepsilon^0, w_\varepsilon^0, z_\varepsilon^0) = \inf_{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon)\in\,\Sigma_\varepsilon} J_\varepsilon(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon), \quad (10.21)$$

where the cost functional J_ε and the set of admissible solutions Σ_ε have the following analytical representations:

$$\begin{aligned} J_\varepsilon(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) = {} & \frac{1}{h(\varepsilon)}\int_{\Omega_\varepsilon} (z_\varepsilon(T,x) - y_\varepsilon^T)^2\,\mathrm{d}x + \kappa(\varepsilon)\int_0^T\int_{\Gamma_\varepsilon^D} v_\varepsilon^2\,\mathrm{d}x_1\,\mathrm{d}t \\ & + \frac{1}{h(\varepsilon)}\int_0^T\int_{\Omega_\varepsilon} (z_\varepsilon - y_\varepsilon^*)^6\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Gamma_\varepsilon^N} w_\varepsilon^2\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t \\ & + \frac{\gamma(\varepsilon)}{h(\varepsilon)}\int_0^T\int_{\Omega_\varepsilon} (a_\varepsilon - z_\varepsilon^3 - f_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \quad (10.22)$$

$$\Sigma_\varepsilon = \{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) :$$
$$\left| \begin{array}{l} z_\varepsilon = v_\varepsilon \ \text{ on } \ \Gamma_\varepsilon^D,\ v_\varepsilon \in U_\varepsilon,\ z_\varepsilon(0,\cdot) = y_{\varepsilon,0} \ \text{ on } \ \Omega_\varepsilon, \\ \quad w_\varepsilon \in L^2((0,T)\times\Gamma_\varepsilon^N),\ a_\varepsilon \in L^2((0,T)\times\Omega_\varepsilon), \\ \quad \displaystyle\int_0^T \int_{\Omega_\varepsilon} (-z_\varepsilon\varphi\,\psi' + A(x/\varepsilon)\nabla z_\varepsilon\cdot\nabla\varphi\,\psi)\,\mathrm{d}x\,\mathrm{d}t \\ \qquad\qquad + d\,\varepsilon^2 \displaystyle\int_0^T \int_{S_{int}^\varepsilon} z_\varepsilon\varphi\,\psi\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t \\ = \displaystyle\int_0^T \int_{\Omega_\varepsilon} a_\varepsilon\,\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 \int_0^T \int_{S_{int}^\varepsilon} g_\varepsilon\varphi\,\psi\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t \\ \qquad\qquad + \varepsilon \displaystyle\int_0^T \int_{\Gamma_\varepsilon^N} w_\varepsilon\varphi\,\psi\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t, \\ \qquad\qquad \forall\,\varphi \in C_0^\infty(\mathbb{R}^2;\Gamma_\varepsilon^D),\ \forall\,\psi \in C_0^\infty(0,T) \end{array} \right\}. \quad (10.23)$$

In this case, the corresponding initial-boundary value problem has the form

$$\left.\begin{array}{rll} \partial_t z_\varepsilon - \mathrm{div}\,(A(x/\varepsilon)\nabla z_\varepsilon) = a_\varepsilon & & \text{in } (0,T)\times\Omega_\varepsilon, \\ \partial_{\nu_A} z_\varepsilon = 0 & & \text{on } \ (0,T)\times S_{ext}, \\ \partial_{\nu_A} z_\varepsilon = \varepsilon^2(-d z_\varepsilon + g_\varepsilon) & & \text{on } \ (0,T)\times S_{int}^\varepsilon, \\ z_\varepsilon = v_\varepsilon & & \text{on } \ (0,T)\times\Gamma_\varepsilon^D, \\ \partial_{\nu_A} z_\varepsilon = \varepsilon w_\varepsilon & & \text{on } \ (0,T)\times\Gamma_\varepsilon^N, \\ z_\varepsilon(0,x) = y_{\varepsilon,0} & & \text{for a.a. } \ x \in \Omega_\varepsilon. \end{array}\right\} \quad (10.24)$$

Remark 10.4. The consideration of the problem $\widehat{\mathbb{P}}_\varepsilon$ with a virtual control $q_\varepsilon \in L^2((0,T)\times\Omega_\varepsilon)$ can be motivated as follows. First, in contrast to the original problem $\mathbb{P}_\varepsilon$, the sets of admissible solutions $\widehat{\Xi}_\varepsilon$ are always nonempty for every $\varepsilon > 0$. Hence, without any additional suppositions, it can be proved (by analogy with Theorem 10.3) that the problem $\widehat{\mathbb{P}}_\varepsilon$ is always solvable. Moreover, due to the structure of the cost functional $\widehat{I}_\varepsilon$, one can take the penalizing coefficient $\gamma(\varepsilon) > 0$ in (10.18) such that any optimal solution of the $\widehat{\mathbb{P}}_\varepsilon$-problem takes the form $(f_\varepsilon, u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$, where $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ is an optimal triplet for the original problem $\mathbb{P}_\varepsilon$. As we will see later, this is the main reason to consider the $\widehat{\mathbb{P}}_\varepsilon$-problem instead of the original one.

Let us denote by $\widehat{\Xi}_\varepsilon^Y$ (Σ_ε^Y, resp.) the projection of the set of admissible solutions $\widehat{\Xi}_\varepsilon$ (Σ_ε^Y, resp.) into the state space $Y = L^2(0,T;H^1(\Omega))$ and call it the set of admissible states for the problem $\mathbb{P}_\varepsilon$ ($\mathbb{CP}_\varepsilon$, resp.).

Definition 10.5. *We say that an optimal control problem is the C-extension of $\widehat{\mathbb{P}}_\varepsilon$, if their sets of admissible states coincide.*

Note that the same property for the control objects was introduced by Serovaĭskiĭ in [227] and this notion was named an equivalence of such objects. However, we accept the notion of "C-extension" or "extension by control" since not every admissible control for the $\mathbb{CP}_\varepsilon$-problem is that for $\widehat{\mathbb{P}}_\varepsilon$. These

problems may have different sets of admissible solutions, in general. For different realizations of this idea, we refer to Zgurovsky and Mel'nik in [253].

Our next assertion justifies the role of the problem $\mathbb{CP}_\varepsilon$ introduced above.

Lemma 10.6. *$\mathbb{CP}_\varepsilon$ is a C-extension of $\widehat{\mathbb{P}}_\varepsilon$.*

Proof. Let $(q_\varepsilon, u_\varepsilon, p_\varepsilon, y_\varepsilon)$ be any admissible solution to the problem (10.17)–(10.19). Let us define $v_\varepsilon = u_\varepsilon$, $w_\varepsilon = p_\varepsilon$, and $a_\varepsilon = q_\varepsilon + y_\varepsilon^3$. By Lemma 10.2, we have $a_\varepsilon \in L^2((0,T)\times\Omega_\varepsilon)$. Hence, a_e, v_ε, and w_ε are admissible controls for the $\mathbb{CP}_\varepsilon$-problem. Let z_ε be the corresponding solution of the problem (10.24). It is well known that for given a_ε, v_ε, w_ε, and ε, this problem admits a unique solution in $L^2(0,T;H^1(\Omega_\varepsilon))$ (see Michajlov [188]). From this, we immediately conclude that $z_\varepsilon = y_\varepsilon$ and hence $(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon)$ is an admissible quadruple to the $\mathbb{CP}_\varepsilon$-problem. Thereby we can deduce that the inclusion $\widehat{\Xi}_\varepsilon^Y \subseteq \Sigma_\varepsilon^Y$ holds.

To prove the reverse inclusion, we take an arbitrary admissible solution $(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon)$ to $\mathbb{CP}_\varepsilon$ and define the corresponding quadruple $(q_\varepsilon, u_\varepsilon, p_\varepsilon, y_\varepsilon)$ as follows: $q_\varepsilon = a_\varepsilon - z_\varepsilon^3$, $u_\varepsilon = v_\varepsilon$, $p_\varepsilon = w_\varepsilon$, and $y_\varepsilon = z_\varepsilon$. Then it is a matter of direct verification to show that $(q_\varepsilon, u_\varepsilon, p_\varepsilon, y_\varepsilon)$ is an admissible solution for $\widehat{\mathbb{P}}_\varepsilon$. Hence, $\Sigma_\varepsilon^Y \subseteq \widehat{\Xi}_\varepsilon^Y$, and we are done. □

Note now that the C-extended problem (10.21)–(10.23) is always solvable; however, its optimal solution is, generally speaking, nonunique (see Fursikov [111] and Lions [169]). Nevertheless, as an obvious consequence of the previous lemma, we have the following conclusion.

Theorem 10.7. *$(a_\varepsilon^0, v_\varepsilon^0, w_\varepsilon^0, z_\varepsilon^0) \in \Sigma_\varepsilon$ is an optimal solution to the $\mathbb{CP}_\varepsilon$-problem if and only if the quadruple $(a_\varepsilon^0 - (z_\varepsilon^0)^3, v_\varepsilon^0, w_\varepsilon^0, z_\varepsilon^0)$ is optimal for $\widehat{\mathbb{P}}_\varepsilon$.*

In view of Remark 10.4, we give the following characteristic of the C-extended problems $\mathbb{CP}_\varepsilon$.

Lemma 10.8. *Assume that the original problem (10.5)–(10.11) is uniformly regular, that is,*

(H1) $\Xi_\varepsilon \neq \emptyset$ for every $\varepsilon > 0$;
(H2) There is a sequence of admissible solutions $\{(u_\varepsilon^*, p_\varepsilon^*, y_\varepsilon^*) \in \Xi_\varepsilon\}_{\varepsilon>0}$ such that

$$\limsup_{\varepsilon\to 0} \int_0^T \left[\|u_\varepsilon^*\|^2_{H^1(\Gamma_\varepsilon^D)} + \|p_\varepsilon^*\|^2_{L^2(\Gamma_\varepsilon^N)} + \|y_\varepsilon^*\|^6_{L^6(\Omega_\varepsilon)}\right] \mathrm{d}t < +\infty. \quad (10.25)$$

Then there is a constant $\gamma^* > 0$ such that, having assumed $\gamma = \gamma^*$ in (10.18) and (10.22), one has for every $\varepsilon > 0$ that $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ is an optimal solution to the original problem (10.5)–(10.11) if and only if the quadruple $(f_\varepsilon + (y_\varepsilon^0)^3, u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ is optimal for $\mathbb{CP}_\varepsilon$.

Proof. First, we note that due to hypothesis (H1), the original OCP (10.5)–(10.11) is solvable for every $\varepsilon > 0$. Let $\{(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of optimal solutions to $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$. Then for every ε, the quadruple $(f_\varepsilon, u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ is

admissible for the corresponding virtual problem $\widehat{\mathbb{P}}_\varepsilon$. Hence, in view of (10.25), there exists a constant $D > 0$ such that

$$\begin{aligned}\inf_{(q_\varepsilon,u_\varepsilon,p_\varepsilon,y_\varepsilon)\in\,\widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(q_\varepsilon,u_\varepsilon,p_\varepsilon,y_\varepsilon) &\equiv \widehat{I}_\varepsilon(\widehat{q}_\varepsilon{}^0,\widehat{u}_\varepsilon^0,\widehat{p}_\varepsilon^0,\widehat{y}_\varepsilon^0)\\ &\le \widehat{I}_\varepsilon(f_\varepsilon,u_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0) = I_\varepsilon(u_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0)\\ &= \inf_{(u_\varepsilon,p_\varepsilon,y_\varepsilon)\in\,\Xi_\varepsilon} I_\varepsilon(u_\varepsilon,p_\varepsilon,y_\varepsilon) \le I_\varepsilon(u_\varepsilon^*,p_\varepsilon^*,y_\varepsilon^*)\\ &\le D < +\infty. \end{aligned} \tag{10.26}$$

Since each of the functionals $\widehat{I}_\varepsilon : \widehat{\Xi}_\varepsilon \to \mathbb{R}$ is strictly monotone increasing with respect to the parameter $\gamma(\varepsilon)$, it follows that, passing to the limit in (10.26) as $\gamma(\varepsilon) \to \infty$, we have

$$I_\varepsilon(u_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0) \le \liminf_{\gamma(\varepsilon)\to\infty} \widehat{I}_\varepsilon(\widehat{q}_\varepsilon{}^0,\widehat{u}_\varepsilon^0,\widehat{p}_\varepsilon^0,\widehat{y}_\varepsilon^0) \le I_\varepsilon(u_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0) \le D, \quad \forall\,\varepsilon > 0.$$

Hence, there exists a constant γ^* such that

$$I_\varepsilon(u_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0) = \inf_{(q_\varepsilon,u_\varepsilon,p_\varepsilon,y_\varepsilon)\in\,\widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(q_\varepsilon,u_\varepsilon,p_\varepsilon,y_\varepsilon)\ \forall\,\gamma(\varepsilon) \ge \gamma^*,\ \forall\,\varepsilon > 0. \tag{10.27}$$

Then from (10.18) we immediately conclude that equalities (10.27) hold true if only $q_\varepsilon^0 = f_\varepsilon$ for every $\varepsilon > 0$.

Thus, from now, we can always suppose that any optimal solution to $\widehat{\mathbb{P}}_\varepsilon$ has the form $(f_\varepsilon,u_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0) \in \widehat{\Xi}_\varepsilon$, where $(u_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0)$ is the solution to the original problem $\mathbb{P}_\varepsilon$. To get the required conclusion, we have to apply Theorem 10.7. The proof is complete. □

As was mentioned earlier, our main intention is to study the asymptotic behavior of the OCP $\mathbb{P}_\varepsilon$ as ε tends to 0 and to identify the limit (or homogenized) problem that should exhibit the following variational property: "If $(u_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0) \in \Xi_\varepsilon$ is an optimal triplet of the $\mathbb{P}_\varepsilon$-problem, and if $(u_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0)$ tends (in some sense) to (u^0,p^0,y^0), then (u^0,p^0,y^0) is an optimal solution for the limit problem." Therefore, taking the above results into account, we may say that the following diagram is quite reasonable for the limit analysis of the singular OCP in our case

$$\begin{array}{ccc} (\mathbb{P}_\varepsilon) \Longrightarrow (\widehat{\mathbb{P}}_\varepsilon) & & (\mathbb{P}^{hom}) \Longleftarrow (\widehat{\mathbb{P}}^{hom}) \\ \Big\downarrow & & \Big\uparrow \\ (\mathbb{CP}_\varepsilon) & & (\mathbb{CP}^{hom}) \\ \Big\downarrow & & \Big\uparrow \\ \Big\langle \inf\limits_{(a_\varepsilon,v_\varepsilon,w_\varepsilon,z_\varepsilon)\in\,\Sigma_\varepsilon} J_\varepsilon(a_\varepsilon,v_\varepsilon,w_\varepsilon,z_\varepsilon) \Big\rangle & \xrightarrow{\varepsilon\to 0} & \Big\langle \inf\limits_{(a,v,w,z)\in\,\Sigma_0} J_0(a,v,w,z) \Big\rangle . \end{array} \tag{10.28}$$

Moreover, in the next sections we prove that this scheme preserves the above-mentioned variational property.

10.3 On the description of $\mathbb{CP}_\varepsilon$ in terms of singular measures

In this section, we deal with the construction of measure-theoretical tools for the description of the class of admissible solutions to the C-extended problem $\mathbb{CP}_\varepsilon$ in terms of singular periodic Borel measures on $\mathbb{R}^2$. It helps us to give a sharp formulation of the limit problem for the considered heat processes on the reticulated structure Ω_ε.

By analogy with Sect. 9.1, we introduce three periodic finite positive Borel measures ν^h, μ^h, and λ^h in $\mathbb{R}^2$, concentrated and uniformly distributed on the sets Λ_N^h, Λ_D^h, and Λ_R^h, respectively, with the correspondent toruses of periodicity

$$Y_1 = [0,1)^2, \quad Y_2 = [-1/2, 1/2) \times [0,1), \quad \text{and} \quad Y_3 = [0,1) \times [-1/2, 1/2)$$

(see Fig. 9.3 and relations (9.6)–(9.8)). Recall that each of the sets Λ_N^h, Λ_D^h, and Λ_R^h indicates a zone where the corresponding boundary conditions for the $\mathbb{CP}_\varepsilon$-problem are located.

We also introduce the so-called "scaling" measures ν_ε^h, μ_ε^h, and λ_ε^h by rules (9.9). Clearly, λ_ε^h is an ε-periodic measure on $\mathbb{R}^2$, whereas the measures ν_ε^h and μ_ε^h are ε-periodic along the x_1-axis. As usual, we relate the parameters h and ε assuming that $h = h(\varepsilon) \to 0$ as $\varepsilon \to 0$. As a result, we have (for the details, we refer to Sect. 9.1) the following.

Proposition 10.9.

$$\mathrm{d}\lambda_\varepsilon^{h(\varepsilon)} \overset{\varepsilon \to 0}{\rightharpoonup} \mathrm{d}x, \quad \mathrm{d}\nu_\varepsilon^{h(\varepsilon)} \overset{\varepsilon \to 0}{\rightharpoonup} \delta_{\{x_2=0\}}\mathrm{d}x, \quad \mathrm{d}\mu_\varepsilon^{h(\varepsilon)} \overset{\varepsilon \to 0}{\rightharpoonup} \delta_{\{x_2=0\}}\mathrm{d}x,$$

where by $\delta_{\{x_2=0\}}\mathrm{d}x$ we denote the product of the linear Lebesgue measure $\mathrm{d}x_1$ and the Dirac measure $\delta_{\{x_2=0\}}\mathrm{d}x_2$.

We are now in a position to give the equivalent formulation of the C-extended OCP (10.21)–(10.23). To begin, we make use of the sets Θ_ε^R and Θ_ε^N defined by (9.11) and (9.12) and consider the second term in the left part of the integral identity (10.23). One has

$$\begin{aligned} d\varepsilon^2 \int_0^T \int_{S_{int}^\varepsilon} z_\varepsilon \varphi \psi \, \mathrm{d}\mathcal{H}^1 \, \mathrm{d}t \\ &= d\varepsilon^2 \int_0^T \psi \left(\sum_{\mathbf{k} \in \Theta_\varepsilon^R} \int_{\varepsilon(\Lambda_R^h + \mathbf{k})} z_\varepsilon \varphi \, \mathrm{d}\mathcal{H}^1 \right) \mathrm{d}t \\ &= d\varepsilon^2 \left[4\varepsilon \left(1 - h(\varepsilon) \right) \right] \int_0^T \psi \left(\sum_{\mathbf{k} \in \Theta_\varepsilon^R} \int_{\varepsilon(Y_3 + \mathbf{k})} \widehat{z}_\varepsilon \varphi \, \mathrm{d}\lambda^h(x/\varepsilon) \right) \mathrm{d}t \end{aligned}$$

$$= 4\varepsilon d\,(1-h(\varepsilon)) \int_0^T \psi \left(\sum_{\mathbf{k}\in\Theta_\varepsilon^R} \int_{\varepsilon(Y_3+\mathbf{k})} \widehat{z}_\varepsilon \varphi\ \varepsilon^2\, \mathrm{d}\lambda^h(x/\varepsilon) \right) \mathrm{d}t$$

$$= 4\varepsilon d\,(1-h(\varepsilon)) \int_0^T \psi \left(\sum_{\mathbf{k}\in\Theta_\varepsilon^R} \int_{\varepsilon(Y_3+\mathbf{k})} \widehat{z}_\varepsilon \varphi\, \mathrm{d}\lambda_\varepsilon^h \right) \mathrm{d}t$$

$$= 4\varepsilon d\,(1-h(\varepsilon)) \int_0^T \int_\Omega \widehat{z}_\varepsilon \varphi\psi\, \mathrm{d}\lambda_\varepsilon^h\, \mathrm{d}t. \tag{10.29}$$

Here, $\widehat{z}_\varepsilon$ is a function of $L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^{h(\varepsilon)}))$ taking the same values as the function $z_\varepsilon \in L^2(0,T;L^2(S_{int}^\varepsilon))$ on S_{int}^ε. We note that the integral $\int_\Omega \widehat{z}_\varepsilon \varphi\, \mathrm{d}\lambda_\varepsilon^{h(\varepsilon)}$ is well defined. Indeed, since the set Ω is bounded and $\widehat{z}_\varepsilon\, \mathrm{d}\lambda_\varepsilon^{h(\varepsilon)}$ is a Radon measure, it follows that $\int_\Omega \widehat{z}_\varepsilon \varphi\, \mathrm{d}\lambda_\varepsilon^{h(\varepsilon)}$ is a linear continuous functional on $C_0^\infty(\mathbb{R}^2;\Gamma_\varepsilon^D)$. Thus,

$$d\,\varepsilon^2 \int_0^T \int_{S_{int}^\varepsilon} z_\varepsilon \varphi\psi\, \mathrm{d}\mathcal{H}^1\, \mathrm{d}t = 4\varepsilon d\,(1-h(\varepsilon)) \int_0^T \int_\Omega \widehat{z}_\varepsilon \varphi\psi\, \mathrm{d}\lambda_\varepsilon^h\, \mathrm{d}t. \tag{10.30}$$

Remark 10.10. It is easy to see that the scaling measure $\lambda_\varepsilon^{h(\varepsilon)}$ belongs to the class $\mathcal{M}_0^+(\Omega)$. In view of our initial supposition, $\mathrm{cap}(S_{int}) \neq 0$, so $\lambda^h(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\mathrm{cap}(B,\Omega) = 0$. Hence, the first property of the cone $\mathcal{M}_0^+(\Omega)$ for the measure $\lambda_\varepsilon^{h(\varepsilon)}$ is valid. As for the second property, the measure λ^h is concentrated on S_{int} and proportional there to the linear Lebesgue measure. Thus, $\lambda^h \in \mathcal{M}_0^+(\Omega)$. Since the scaling measure $\lambda_\varepsilon^{h(\varepsilon)}$ inherits these properties from λ^h, it follows that $\lambda_\varepsilon^{h(\varepsilon)} \in \mathcal{M}_0^+(\Omega)$ for every $\varepsilon > 0$. Thus, every function y in $H^1(\Omega)$ is defined $\lambda_\varepsilon^{h(\varepsilon)}$-everywhere and is $\lambda_\varepsilon^{h(\varepsilon)}$-measurable on Ω (see Dal Maso and Murat [82]). Hence, the space $H_0^1(\Omega) \cap L^2(\Omega,\mathrm{d}\lambda_\varepsilon^{h(\varepsilon)})$ is well defined.

Taking these facts into account and proceeding by analogy with (10.29), we may obtain the following relation:

$$\varepsilon \int_0^T \int_{\Gamma_\varepsilon^N} w_\varepsilon \varphi\psi\, \mathrm{d}\mathcal{H}^1 \mathrm{d}t$$

$$= \varepsilon \int_0^T \psi \left(\sum_{\mathbf{k}\in\Theta_\varepsilon^N} \int_{\varepsilon(\Lambda_N^h+\mathbf{k})} w_\varepsilon \varphi\, \mathrm{d}\mathcal{H}^1 \right) \mathrm{d}t$$

$$= \varepsilon\,[2\,\varepsilon\,(1-h(\varepsilon))] \int_0^T \psi \left(\sum_{\mathbf{k}\in\Theta_\varepsilon^N} \int_{\varepsilon(Y_1+\mathbf{k})} \widehat{w}_\varepsilon \varphi\ \mathrm{d}\nu^h(x/\varepsilon) \right) \mathrm{d}t$$

$$= 2\,\varepsilon\,(1-h(\varepsilon)) \int_0^T \int_\Omega \widehat{w}_\varepsilon \varphi\psi\, \mathrm{d}\nu_\varepsilon^h \mathrm{d}t. \tag{10.31}$$

Hence, applying this approach for the rest of the terms in (10.21)–(10.23), we have

$$\kappa(\varepsilon)\int_0^T\int_{\Gamma_\varepsilon^D} v^2\,\mathrm{d}x_1\,\mathrm{d}t = \kappa(\varepsilon)h(\varepsilon)\int_0^T\int_\Omega \widehat{v}^2\,\mathrm{d}\mu_\varepsilon^h\,\mathrm{d}t, \tag{10.32}$$

$$\int_0^T\int_{\Gamma_\varepsilon^N} w_\varepsilon^2\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t = 2\,(1-h(\varepsilon))\int_0^T\int_\Omega \widehat{w}_\varepsilon^2\,\mathrm{d}\nu_\varepsilon^h\,\mathrm{d}t, \tag{10.33}$$

$$\varepsilon^2\int_0^T\int_{S_{int}^\varepsilon} g_\varepsilon\varphi\,\psi\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t = 4\,\varepsilon\,(1-h(\varepsilon))\int_0^T\int_\Omega \widehat{g}_\varepsilon\varphi\psi\,\mathrm{d}\lambda_\varepsilon^h\,\mathrm{d}t. \tag{10.34}$$

Let $\breve{z}_\varepsilon \in L^2(0,T;H^1(\Omega))$ be some extension of the weak solution z_ε to the problem (10.24) onto the whole domain Ω and let $\chi_\varepsilon^h = \chi_{\Omega_\varepsilon}$ be the characteristic function of the network Ω_ε (it is clear that this function is two-parametric, in general). As a result, summing up representations (10.30)–(10.34), the initial-boundary value problem (10.24) can be rewritten in the following variational form:

$$\breve{z}_\varepsilon = \widehat{v}_\varepsilon \quad \mu_\varepsilon^h\text{-a.e. on } \Omega, \quad \chi_\varepsilon^h\,[\breve{z}_\varepsilon(0,\cdot) - y_{\varepsilon,0}] = 0 \ \text{ a.e. on } \Omega; \tag{10.35}$$

$$\begin{aligned}
&-\int_0^T\int_\Omega \chi_\varepsilon^h\breve{z}_\varepsilon\varphi\,\psi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \chi_\varepsilon^h(A(x/\varepsilon)\nabla\breve{z}_\varepsilon\cdot\nabla\varphi)\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + 4d\,\varepsilon\,(1-h(\varepsilon))\int_0^T\int_\Omega \breve{z}_\varepsilon\varphi\psi\,\mathrm{d}\lambda_\varepsilon^h\,\mathrm{d}t\\
&\quad = \int_0^T\int_\Omega \chi_\varepsilon^h\widehat{a}_\varepsilon\,\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + 2\,\varepsilon\,(1-h(\varepsilon))\int_0^T\left(\int_\Omega 2\widehat{g}_\varepsilon\varphi\psi\,\mathrm{d}\lambda_\varepsilon^h\,\mathrm{d}t + \int_\Omega \widehat{w}_\varepsilon\varphi\psi\,\mathrm{d}\nu_\varepsilon^h\right)\mathrm{d}t
\end{aligned} \tag{10.36}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2;\Gamma_0)$ and $\psi \in C_0^\infty(0,T)$.

Let η_ε^h be the uniformly distributed normalized measure concentrated on the thin grid $\mathcal{F}_\varepsilon^h = \varepsilon\mathcal{F}^h$, which is defined in (9.16) with properties (9.17)–(9.19). It is clear that this measure is absolutely continuous with respect to $\mathcal{L}_2$ (in symbols, $\eta_\varepsilon^h(\varepsilon) \ll \mathcal{L}_2$), that is, $\mathcal{L}_2(B) = 0$ implies $\eta_\varepsilon^h(\varepsilon)(B) = 0$ for all $B \subset \mathbb{R}^2$. By analogy with (10.29) and (10.34), it is easy to show that the first terms in the both parts of the integral equality (10.36) can be rewritten as

$$\int_0^T\int_\Omega \chi_\varepsilon^h\breve{z}_\varepsilon\varphi\,\psi'\,\mathrm{d}x\,\mathrm{d}t = h(\varepsilon)(2-h(\varepsilon))\int_0^T\int_\Omega \widehat{z}_\varepsilon\varphi\,\psi'\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t,$$

$$\int_0^T\int_\Omega \chi_\varepsilon^h\widehat{a}_\varepsilon\,\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t = h(\varepsilon)(2-h(\varepsilon))\int_0^T\int_\Omega \widehat{a}_\varepsilon\,\varphi\,\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t,$$

where, as usual, $\widehat{z}_\varepsilon \in L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$ denotes a function taking the same values as z_ε on Ω_ε.

To define the Sobolev space $H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)$, we say that a function $\hat{z}$ belongs to $H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)$ if there exist a vector $\mathbf{d} \in (L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))^2$ and a sequence $\{z_m \in C^\infty(\Omega)\}_{m\in N}$ such that

$$\lim_{m\to\infty}\int_\Omega (z_m-\hat{z})^2\,\mathrm{d}\eta_\varepsilon^h = 0, \quad \lim_{m\to\infty}\int_\Omega \left|\nabla z_m - \mathbf{d}\right|^2 \mathrm{d}\eta_\varepsilon^h = 0.$$

In this case, one says that $\mathbf{d}$ is a gradient of $\hat{z}$ and denotes it as $\nabla\hat{z}$ (i.e., $\mathbf{d}=\nabla\hat{z}$). In view of the ergodicity property of the measure $\mathrm{d}\eta_\varepsilon^h$, this gradient is unique for any $\hat{z}\in H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)$ (see Sect. 2.7). Moreover, we suppose that for any function $\hat{z}\in H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)$, one can point out its prototype z in $H^1(\Omega)$ such that $z=\hat{z}$ η_ε^h-a.e. in Ω.

Thus, taking the above-mentioned facts into account, we may reformulate the C-extended OCP $\mathbb{CP}_\varepsilon$ as follows: Find a quadruple $(\widehat{a}_\varepsilon{}^0, \widehat{v}_\varepsilon^0, \widehat{w}_\varepsilon{}^0, \widehat{z}_\varepsilon{}^0)$ such that

$$\begin{aligned}(\widehat{a}_\varepsilon{}^0, \widehat{v}_\varepsilon^0, \widehat{w}_\varepsilon{}^0, \widehat{z}_\varepsilon{}^0) &\in \mathbb{Z}_\varepsilon \\ &\equiv L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)) \times L^2(0,T;\left[H^1(\Gamma_0)\cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)\right]) \\ &\times\ L^2(0,T;L^2(\Omega,\mathrm{d}\nu_\varepsilon^h)) \\ &\times \left[L^2(0,T;H^1(\Omega,\mathrm{d}\eta_\varepsilon^h))\cap L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))\right], \qquad (10.37)\end{aligned}$$

$$\widehat{J}_\varepsilon(\widehat{a}_\varepsilon{}^0, \widehat{v}_\varepsilon^0, \widehat{w}_\varepsilon{}^0, \widehat{z}_\varepsilon{}^0) = \inf_{(\widehat{a}_\varepsilon,\widehat{v}_\varepsilon,\widehat{w}_\varepsilon,\widehat{z}_\varepsilon)\in\,\widehat{\Sigma}_\varepsilon} \widehat{J}_\varepsilon(\widehat{a}_\varepsilon,\widehat{v}_\varepsilon,\widehat{w}_\varepsilon,\widehat{z}_\varepsilon), \qquad (10.38)$$

where the cost functional $\widehat{J}_\varepsilon$ and the set of admissible solutions $\widehat{\Sigma}_\varepsilon$ have the following analytical representations:

$$\begin{aligned}\widehat{J}_\varepsilon(\widehat{a}_\varepsilon,\widehat{v}_\varepsilon,\widehat{w}_\varepsilon,\widehat{z}_\varepsilon) &= (2-h(\varepsilon))\int_\Omega (\widehat{z}_\varepsilon(T,x)-y_\varepsilon^T)^2\,\mathrm{d}\eta_\varepsilon^h \\ &+ (2-h(\varepsilon))\int_0^T\int_\Omega (\widehat{z}_\varepsilon - y_\varepsilon^*)^6\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t \\ &+ \kappa(\varepsilon)h(\varepsilon)\int_0^T\int_\Omega \widehat{v}_\varepsilon^2\,\mathrm{d}\mu_\varepsilon^h\,\mathrm{d}t \\ &+ 2\,(1-h(\varepsilon))\int_0^T\int_\Omega \widehat{w}_\varepsilon^2\,\mathrm{d}\nu_\varepsilon^h\,\mathrm{d}t \\ &+ \gamma^*\,(2-h(\varepsilon))\int_0^T\int_\Omega (\widehat{a}_\varepsilon - \widehat{z}_\varepsilon^3 - f_\varepsilon)^2\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t, \qquad (10.39)\end{aligned}$$

$$\widehat{\Sigma}_\varepsilon = \{(\widehat{a}_\varepsilon, \widehat{v}_\varepsilon, \widehat{w}_\varepsilon, \widehat{z}_\varepsilon) :$$

$$\left.\begin{array}{c} \widehat{z}_\varepsilon = \widehat{v}_\varepsilon \quad \mu_\varepsilon^h\text{-a.e. on } (0,T)\times\Omega, \\ \widehat{z}_\varepsilon(0,\cdot) = y_{\varepsilon,0} \quad \eta_\varepsilon^h\text{-a.e. on } \Omega; \\ \|\widehat{v}_\varepsilon\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0, \\ -\displaystyle\int_0^T\int_\Omega \widehat{z}_\varepsilon\varphi\,\psi'\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t + \int_0^T\int_\Omega (A(x/\varepsilon)\nabla\widehat{z}_\varepsilon\cdot\nabla\varphi)\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t \\ +4d\,\beta(\varepsilon)\displaystyle\int_0^T\int_\Omega \widehat{z}_\varepsilon\varphi\psi\,\mathrm{d}\lambda_\varepsilon^h\,\mathrm{d}t = \int_0^T\int_\Omega \widehat{a}_\varepsilon\,\varphi\,\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t \\ +4\,\beta(\varepsilon)\displaystyle\int_0^T\int_\Omega \widehat{g}_\varepsilon\varphi\psi\,\mathrm{d}\lambda_\varepsilon^h\,\mathrm{d}t + 2\,\beta(\varepsilon)\int_0^T\int_\Omega \widehat{w}_\varepsilon\varphi\psi\,\mathrm{d}\nu_\varepsilon^h\,\mathrm{d}t, \\ \forall\,\varphi\in C_0^\infty(\mathbb{R}^2;\Gamma_\varepsilon^D),\ \forall\,\psi\in C_0^\infty(0,T) \end{array}\right\}. \quad (10.40)$$

Here

$$\beta(\varepsilon) = \frac{1-h(\varepsilon)}{2-h(\varepsilon)}\cdot\left[\frac{\varepsilon}{h(\varepsilon)}\right].$$

In what follows, the OCP (10.38)–(10.40) will be called the $\widehat{\mathbb{CP}}_\varepsilon$-problem. It is quite obvious that this problem has a unique solution $(\widehat{a}_\varepsilon{}^0, \widehat{v}_\varepsilon^0, \widehat{w}_\varepsilon{}^0, \widehat{z}_\varepsilon{}^0)$ for every $\varepsilon > 0$ (see Fursikov [111] and Lions [169]). In view of Lemma 10.6 and Remark 10.4, this solution can be interpreted as a prototype of an optimal triplet to $\mathbb{P}_\varepsilon$. Moreover, in this case, an a priori norm estimate (10.16) takes a form (see Kalantarov and Ladizhenskaja [127])

$$\begin{aligned} &\|\widehat{z}_\varepsilon\|_{L^2(0,T;H^1(\Omega,\mathrm{d}\eta_\varepsilon^h))} + \|\partial\widehat{z}_\varepsilon/\partial t\|_{L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))} \\ &\quad + \|\widehat{z}_\varepsilon\|_{L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))} + \|\widehat{z}_\varepsilon\|_{L^6(0,T;L^6(\Omega,\mathrm{d}\eta_\varepsilon^h))} \\ &\quad \le \widehat{C}\left[\|\widehat{a}_\varepsilon\|_{L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))} + \|\widehat{w}_\varepsilon\|_{L^2(0,T;L^2(\Omega,\mathrm{d}\nu_\varepsilon^h))} + \|y_{\varepsilon,0}\|_{H^1(\Omega)}\right. \\ &\quad \left. + \|\widehat{g}_\varepsilon\|_{L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))} + \|\widehat{v}_\varepsilon\|_{L^2(0,T;H^1(\Gamma_0)\cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^h))}\right], \end{aligned} \quad (10.41)$$

where the constant $\widehat{C}$ is independent of $\varepsilon > 0$.

In fact, the $\widehat{\mathbb{P}}_\varepsilon$-problem with a virtual distributed control q_ε (see (10.17)–(10.19)) can be rewritten in the same manner as (10.38)–(10.40).

To investigate the asymptotic behavior of the C-extended OCP, we apply the scheme of the direct variational convergence (see Sect. 8.4). With this aim, we represent the $\widehat{\mathbb{CP}}_\varepsilon$-problem for various values of ε in the form of the sequence of corresponding constrained minimization problems

$$\left\{\left\langle \inf_{(\widehat{a}_\varepsilon,\widehat{v}_\varepsilon,\widehat{w}_\varepsilon,\widehat{z}_\varepsilon)\,\in\,\widehat{\Sigma}_\varepsilon} \widehat{J}_\varepsilon(\widehat{a}_\varepsilon,\widehat{v}_\varepsilon,\widehat{w}_\varepsilon,\widehat{z}_\varepsilon)\right\rangle;\ \varepsilon>0\right\}, \quad (10.42)$$

where the cost functional $\widehat{J}_\varepsilon : \Sigma_\varepsilon \to \mathbb{R}$ and the sets of admissible solutions $\widehat{\Sigma}_\varepsilon$ are defined in (10.39) and (10.40), respectively. Then the definition of an appropriate homogenized OCP to the family (10.38) can be reduced to the anal-

ysis of the limit properties of the sequences (10.42) as $\varepsilon \to 0$. In order to obtain this limit in the form of some constrained minimization problem, we reduce the procedure of the "passage to the limit" in (10.42) to the consecutive identification of the set of admissible solutions for the limit minimization problem and its cost functional. We prove that the definition of the "pass to the limit" given below will guarantee the following property: "If $(\widehat{a}_\varepsilon{}^0, \widehat{v}_\varepsilon^0, \widehat{w}_\varepsilon{}^0, \widehat{z}_\varepsilon{}^0)$ is an optimal quadruple of $\widehat{\mathbb{CP}}_\varepsilon$ and if $(\widehat{a}_\varepsilon{}^0, \widehat{v}_\varepsilon^0, \widehat{w}_\varepsilon{}^0, \widehat{z}_\varepsilon{}^0)$ tends to $(\widehat{a}\,{}^0, \widehat{v}\,{}^0, \widehat{w}\,{}^0, \widehat{z}\,{}^0)$ (in some sense), then $(\widehat{a}\,{}^0, \widehat{v}\,{}^0, \widehat{w}\,{}^0, \widehat{z}\,{}^0)$ is an optimal solution for the limit problem."

Since for every fixed ε, the admissible quadruple $(\widehat{a}_\varepsilon, \widehat{v}_\varepsilon, \widehat{w}_\varepsilon, \widehat{z}_\varepsilon)$ belongs to the variable functional space $\mathbb{Z}_\varepsilon$, one has to define a natural topology for the variational convergence that preserves the above-mentioned variational property and such that the sequence of optimal solutions is relatively compact. These questions are discussed in the next section.

10.4 Convergence in the variable space $\mathbb{Z}_\varepsilon$

We begin this section with the following concept (for the details, we refer to Sect. 6.6).

Definition 10.11. *Let $\left\{z_\varepsilon^h\right\}_{\varepsilon>0}$ be a bounded sequence in the variable space*

$$\mathbb{M}_\varepsilon = L^2(0,T;H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)) \cap L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)),$$

that is,

$$\limsup_{\varepsilon\to 0} \int_0^T \left[\|z_\varepsilon^{h(\varepsilon)}\|^2_{H^1(\Omega,\mathrm{d}\eta_\varepsilon^{h(\varepsilon)})} + \|z_\varepsilon^{h(\varepsilon)}\|^2_{L^2(\Omega,\mathrm{d}\lambda_\varepsilon^{h(\varepsilon)})}\right] \mathrm{d}t < +\infty.$$

We say that this sequence is weakly convergent in the variable space $\mathbb{M}_\varepsilon$ if there is a function $z \in L^2(0,T;H^1(\Omega))$ such that

$$\begin{cases} z_\varepsilon^h \to z \ \textit{in} \ \ L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)), \\ z_\varepsilon^h \rightharpoonup z \ \textit{in} \ \ L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)), \\ \displaystyle\int_0^T \int_\Omega |z_\varepsilon^h - z|^2 \,\mathrm{d}\eta_\varepsilon^h \mathrm{d}t \to 0 \ \textit{as} \ \ \varepsilon \to 0. \end{cases} \tag{10.43}$$

Further, we note that similar to Sect. 9.3, the following results concerning some specific properties of the convergence in the time-dependent variable spaces $L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))$, $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$, and $\mathbb{M}_\varepsilon$ can be established (for the details, we refer to the proofs of Lemmas 9.7, 9.8, 9.10, Theorem 9.14, and Corollaries 9.9 and 9.15).

Lemma 10.12. *If $z \in L^2(0,T;H^1(\Omega))$, then*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega z^2\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t=\int_0^T\int_\Omega z^2\,\mathrm{d}x\,\mathrm{d}t, \tag{10.44}$$

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega z\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t=\int_0^T\int_\Omega z\varphi\psi\,\mathrm{d}x\,\mathrm{d}t, \tag{10.45}$$

$\forall\,\varphi\in C_0^\infty(\Omega)$ *and* $\forall\psi\in C_0^\infty(0,T)$. *Moreover, if*

$$\sup_{\varepsilon>0}\int_0^T\int_\Omega z^2\,\mathrm{d}\lambda_\varepsilon^h\,\mathrm{d}t<+\infty,$$

then

$$z\rightharpoonup z\quad in\quad L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)). \tag{10.46}$$

Lemma 10.13. *Let* $\{z_\varepsilon\in L^2(0,T;H^1(\Omega))\}$ *and* $z\in L^2(0,T;H^1(\Omega))$ *be such that* $z_\varepsilon\rightharpoonup z$ *in* $L^2(0,T;H^1(\Omega))$, $z_\varepsilon\to z$ *in* $L^2((0,T)\times\Omega)$, *and*

$$\sup_{\varepsilon>0}\int_0^T\int_\Omega z^2\,\mathrm{d}\lambda_\varepsilon^h\,dt<+\infty.$$

Then

$$\left.\begin{aligned} z_\varepsilon&\rightharpoonup z\quad in\quad L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)),\\ z_\varepsilon&\to z\quad in\quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)).\end{aligned}\right\} \tag{10.47}$$

Corollary 10.14. *If* $z_\varepsilon\to z$ *in* $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$ *and* $z\in L^2(0,T;H^1(\Omega))$, *then*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega|z_\varepsilon-z|^2\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t=0. \tag{10.48}$$

Lemma 10.15. *Let* $\{z_\varepsilon\in L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))\}$ *and* $\{\varphi_\varepsilon\in C_0^\infty(\mathbb{R}^2)\}$ *be sequences such that* $z_\varepsilon\rightharpoonup z$ *in* $L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))$ *and* $\varphi_\varepsilon\to\varphi$ *in* $C(\Omega)$. *Then*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega z_\varepsilon\varphi_\varepsilon\psi\,\mathrm{d}\lambda_\varepsilon^h\,\mathrm{d}t=\int_0^T\int_\Omega z\varphi\psi\,\mathrm{d}x\,\mathrm{d}t\quad\forall\,\psi\in C_0^\infty(0,T). \tag{10.49}$$

Theorem 10.16. *Let* $\{z_\varepsilon^h\}_{\varepsilon>0}$ *be a bounded sequence in* $L^2(0,T;H^1(\Omega,\mathrm{d}\eta_\varepsilon^h))$, *that is,*

$$\sup_{\varepsilon\to 0}\int_0^T\left[\|z_\varepsilon^h(t,\cdot)\|^2_{L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)}+\|\nabla z_\varepsilon^h(t,\cdot)\|^2_{[L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)]^2}\right]\mathrm{d}t<\infty.$$

Then there are a subsequence $\{z_\varepsilon^h\}_{\varepsilon>0}$ *(still indexed by* ε*) and elements* $\mathbf{r}(t,x,y)\in L^2(0,T;\left[L^2(\Omega,Y)\right]^2)$ *and* $z(t,x)\in L^2(0,T;H^1(\Omega))$ *such that*

$$z_\varepsilon^h\overset{2}{\rightharpoonup}z(t,x)\quad in\quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)), \tag{10.50}$$

$$\nabla z_\varepsilon^h\overset{2}{\rightharpoonup}\mathbf{r}(t,x,y)\quad in\quad L^2(0,T;\left[L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)\right]^2), \tag{10.51}$$

$$\mathbf{r}(t,x,y)-\nabla z(t,x)\in L^2(0,T;L^2(\Omega,V_{pot})). \tag{10.52}$$

Corollary 10.17. *Let* $\{z_\varepsilon^h \in L^2(0,T;H^1(\Omega,\mathrm{d}\eta_\varepsilon^h))\}_{\varepsilon>0}$ *be a bounded sequence and let* $z \in L^2(0,T;H^1(\Omega))$ *be its weak limit in* $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$. *Then*

$$z_\varepsilon^h \to z \quad in \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)), \quad \lim_{\varepsilon\to 0}\int_0^T\int_\Omega |z_\varepsilon^h - z|^2\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t = 0. \tag{10.53}$$

Theorem 10.18. *Every bounded sequence*

$$\{z_\varepsilon^h \in L^2(0,T;H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)) \cap L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))\}_{\varepsilon>0}$$

is relatively compact with respect to the weak convergence in the sense of Definition 10.11.

The following result is crucial for the identification of the limit OCP.

Theorem 10.19. *Let* $\{z_\varepsilon^h\}_{\varepsilon>0}$ *be a bounded sequence in*

$$\mathbb{M}_\varepsilon = L^2(0,T;H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)) \cap L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))$$

and let $z \in L^2(0,T;H^1(\Omega))$ *be its weak limit in the sense of Definition 10.11. Then* $z_\varepsilon^h \to z$ *in* $L^6(0,T;L^6(\Omega,\mathrm{d}\eta_\varepsilon^h))$ *and* $(z_\varepsilon^h)^3 \to z^3$ *in* $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$, *namely*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left(z_\varepsilon^h\right)^3 u_\varepsilon\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t = \int_0^T\int_\Omega z^3 u\,\mathrm{d}x\,\mathrm{d}t \tag{10.54}$$
$$\forall\, u_\varepsilon \rightharpoonup u \quad in \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)),$$

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega |z_\varepsilon^h|^6\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t = \int_0^T\int_\Omega z^6\,\mathrm{d}x\,\mathrm{d}t. \tag{10.55}$$

Proof. First, we note that in accordance with Lemma 10.2, the injection $\mathbb{M}_\varepsilon \hookrightarrow L^8(0,T;L^8(\Omega,d\eta_\varepsilon^h))$ is compact for every fixed $\varepsilon > 0$. To see this, it is enough to take $B_0 = H^1(\Omega_\varepsilon)$, $B_1 = L^8(\Omega_\varepsilon)$, $B_2 = L^2(\Omega_\varepsilon)$, $p_0 = 2$, $p_1 = 8$, and $p_2 = 2$ in Lemma 10.1. Hence, in view of the property of uniform boundedness of $\{z_\varepsilon^h\}_{\varepsilon>0}$ in $\mathbb{M}_\varepsilon$, we have

$$\limsup_{\varepsilon\to 0}\int_0^T\int_\Omega |z_\varepsilon^h|^8\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t < +\infty. \tag{10.56}$$

Thus, the sequence $\{z_\varepsilon^h\}_{\varepsilon>0}$ (resp., $\{(z_\varepsilon^h)^4\}_{\varepsilon>0}$) is compact with respect to the weak convergence in $L^8(0,T;L^8(\Omega,\mathrm{d}\eta_\varepsilon^h))$ (resp., $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$). Let z^* be a weak limit of the original sequence $\{z_\varepsilon^h\}_{\varepsilon>0}$ in $L^8(0,T;L^8(\Omega,\mathrm{d}\eta_\varepsilon^h))$. Then $z_\varepsilon^h \rightharpoonup z^*$ in $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$ as well. However, since the weak limit in $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$ must be unique, it follows that

$$z_\varepsilon^h \rightharpoonup z = z^* \quad \text{in} \quad L^8(0,T;L^8(\Omega,\mathrm{d}\eta_\varepsilon^h)).$$

From (10.56) we may deduce that this sequence is uniformly bounded in $L^p(0,T;L^p(\Omega,\mathrm{d}\eta_\varepsilon^h))$ for any $p \in \{3,\dots,7\}$ and its weak limits in the variable space $L^p(0,T;L^p(\Omega,\mathrm{d}\eta_\varepsilon^h))$ coincide with z. On the other hand, the element $z \in L^2(0,T;H^1(\Omega))$ is the strong limit of the sequence $\{z_\varepsilon^h\}_{\varepsilon>0}$ in $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$ (see Definition 10.11). Then by the definition of the strong convergence in variable spaces, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^T\int_\Omega (z_\varepsilon^h)^2\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t &= \lim_{\varepsilon\to 0}\int_0^T\int_\Omega (z_\varepsilon^h\cdot z_\varepsilon^h)\,\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t\\ &= \int_0^T\int_\Omega z^2\varphi\psi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}\tag{10.57}$$

Indeed, in this case, we have the following estimate:

$$\begin{aligned}&\left|\int_0^T\int_\Omega (z_\varepsilon^h)^2\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\mathrm{d}t - \int_0^T\int_\Omega z^2\varphi\psi\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\le \left|\int_0^T\int_\Omega z_\varepsilon^h(z_\varepsilon^h - z)\,\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t\right| + \left|\int_0^T\int_\Omega (z_\varepsilon^h - z)z\,\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t\right|\\ &+ \left|\int_0^T\int_\Omega z^2\,\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t - \int_0^T\int_\Omega z^2\varphi\psi\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\le 2\sup_{\varepsilon\to 0}\max\left\{\|z_\varepsilon^h\|_{L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))}, \|z\|_{L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))}\right\}\\ &\times \|\varphi\|_{C(\Omega)}\|\psi\|_{C(0,T)}\sqrt{\mathbb{J}_\varepsilon}\\ &+ \left|\int_0^T\int_\Omega z^2\,\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t - \int_0^T\int_\Omega z^2\varphi\psi\,\mathrm{d}x\,\mathrm{d}t\right|,\end{aligned}$$

where $\mathbb{J}_\varepsilon = \displaystyle\int_0^T\int_\Omega (z_\varepsilon^h - z)^2\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t$.

Passing to the limit as $\varepsilon\to 0$, we see that $\lim_{\varepsilon\to 0}\mathbb{J}_\varepsilon = 0$ and

$$\left|\int_0^T\int_\Omega z^2\,\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t - \int_0^T\int_\Omega z^2\varphi\psi\,\mathrm{d}x\,\mathrm{d}t\right| \longrightarrow 0$$

(by Lemma 10.12). Thereby, we have shown that relation (10.57) holds true. However, this can be interpreted as a definition of the weak convergence of the sequence $\{(z_\varepsilon^h)^2\}_{\varepsilon>0}$ to z^2 in $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$.

As a matter of fact, we can establish a stronger result, namely $(z_\varepsilon^h)^3 \rightharpoonup z^3$ in $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$. Indeed, in order to prove this, it is enough to use the definition of strong convergence in $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$. Making an a priori estimate as above, we can obtain the assertion

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega (z_\varepsilon^h)^3\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_0^T\int_\Omega (z_\varepsilon^h)^2\cdot z_\varepsilon^h\,\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t$$
$$= \int_0^T\int_\Omega z^3\varphi\psi\,\mathrm{d}x\,\mathrm{d}t.$$

By analogy with the previous motivation, we conclude from this that the sequence $\{(z_\varepsilon^h)^3\}$ is weakly convergent to z^3 in $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$. Therefore,

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left(z_\varepsilon^h\right)^4\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left(z_\varepsilon^h\right)^3 z_\varepsilon^h\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t = \int_0^T\int_\Omega z^4\,\mathrm{d}x\mathrm{d}t,$$
$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left(z_\varepsilon^h\right)^4\varphi\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left(z_\varepsilon^h\right)^3 z_\varepsilon^h\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t = \int_0^T\int_\Omega z^4\varphi\psi\,\mathrm{d}x\,\mathrm{d}t$$

(as the limit of the product of weakly and strongly convergent sequences in $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$). Hence, by the criterium of the strong convergence in variable spaces, we deduce that

$$z_\varepsilon^h \to z \ \text{ in } \ L^4(0,T;L^4(\Omega,\mathrm{d}\eta_\varepsilon^h)),$$
$$\left(z_\varepsilon^h\right)^2 \to z^2 \ \text{ in } \ L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)),$$
$$\left(z_\varepsilon^h\right)^4 \rightharpoonup z^4 \ \text{ in } \ L^8(0,T;L^8(\Omega,\mathrm{d}\eta_\varepsilon^h)).$$

Using a similar procedure, we obtain

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left(z_\varepsilon^h\right)^6\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left(z_\varepsilon^h\right)^4\left(z_\varepsilon^h\right)^2\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t = \int_0^T\int_\Omega z^6\,\mathrm{d}x\,\mathrm{d}t$$

(as the limit of the product of weakly and strongly convergent sequences in $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$). Thus,

$$z_\varepsilon^h \rightharpoonup z \ \text{ in } \ L^6(0,T;L^6(\Omega,\mathrm{d}\eta_\varepsilon^h)),$$
$$\int_0^T\int_\Omega \left(z_\varepsilon^h\right)^6\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t \longrightarrow \int_0^T\int_\Omega z^6\,\mathrm{d}x\,\mathrm{d}t.$$

and we come to the required conclusion. □

10.4.1 Convergence formalism for Dirichlet boundary controls

As follows from the statement of the C-extended OCP (10.38)–(10.40), the class of admissible Dirichlet controls for every fixed value of ε can be described as

$$\widehat{v}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon = \left\{v \in L^2(0,T;\left[H^1(\Gamma_0)\cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)\right]) \ : \right.$$
$$\left. \|v\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0\right\}. \quad (10.58)$$

Hence, for every $\varepsilon > 0$ and for every admissible control function $\widehat{v}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon$, there exists an extension operator

$$P_\varepsilon : L^2(0,T;H^1(\Gamma_\varepsilon)) \to L^2(0,T;H^1(\Gamma_0))$$

such that $\|P_\varepsilon(\widehat{v}_\varepsilon)\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0$. In view of this, we can give two different convergence concepts for such controls.

Definition 10.20. *We say that a function u^* is an s_a-limit for the sequence of Dirichlet controls $\{\widehat{v}_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon))\}_{\varepsilon>0}$ if some sequence of its images $\{P_\varepsilon(\widehat{v}_\varepsilon)\}_{\varepsilon>0}$ converges to u^* weakly in $L^2(0,T;H^1(\Gamma_0))$. We also say that the sequence $\{\widehat{v}_\varepsilon \in L^2(0,T;L^2(\Omega,\mathrm{d}\mu^h_\varepsilon))\}_{\varepsilon>0}$ is s_b-convergent to an element $u^{**} \in L^2(0,T;L^2(\Gamma_0))$ if this sequence is uniformly bounded, that is,*

$$\sup_{\varepsilon\to 0} \int_0^T \int_\Omega |\widehat{v}_\varepsilon|^2 \,\mathrm{d}\mu^h_\varepsilon \,dt < +\infty,$$

and $\widehat{v}_\varepsilon \rightharpoonup u^{**}$ in the variable space $L^2(0,T;L^2(\Omega,\mathrm{d}\mu^h_\varepsilon))$.

Below we show that these concepts are the same for the admissible controls. However, to begin, we prove the correctness of the first one. Indeed, let us fix some sequence of the controls $\{\widehat{v}_\varepsilon\}$ and assume that there are two different sequences of its images $\{P^{(1)}_\varepsilon(\widehat{v}_\varepsilon)\}_{\varepsilon>0}$ and $\{P^{(2)}_\varepsilon(\widehat{v}_\varepsilon)\}_{\varepsilon>0}$ such that

$$P^{(1)}_\varepsilon(\widehat{v}_\varepsilon) \rightharpoonup u^*_1 \quad \text{and} \quad P^{(2)}_\varepsilon(\widehat{v}_\varepsilon) \rightharpoonup u^*_2 \ \text{ in } \ L^2(0,T;H^1(\Gamma_0)).$$

Let $\chi^h_\varepsilon \in L^\infty(\Gamma_0)$ be the characteristic function of the set Γ^D_ε where the corresponding controls $\widehat{v}_\varepsilon$ are located. By Proposition 9.12, we have that $h^{-1}(\varepsilon)\chi^h_\varepsilon \to 1$ weakly-$*$ in $L^\infty(\Gamma_0)$ as $\varepsilon \to 0$. Then, passing to the limit in the integral identity

$$h^{-1}(\varepsilon)\int_0^T \int_{\Gamma_0} \chi^h_\varepsilon P^{(1)}_\varepsilon(\widehat{v}_\varepsilon)\varphi\psi \,\mathrm{d}x_1 \,\mathrm{d}t = h^{-1}(\varepsilon)\int_0^T \int_{\Gamma_0} \chi^h_\varepsilon P^{(2)}_\varepsilon(\widehat{v}_\varepsilon)\varphi\psi \,\mathrm{d}x_1 \,\mathrm{d}t,$$

as $\varepsilon \to 0$, which holds true for every $\varphi \in H^1(\Gamma_0)$ and $\psi \in C^\infty_0(0,T)$, we get

$$\int_0^T \int_{\Gamma_0} u^*_1\varphi\psi \,\mathrm{d}x_1 \,\mathrm{d}t = \int_0^T \int_{\Gamma_0} u^*_2\varphi\psi \,\mathrm{d}x_1 \,\mathrm{d}t, \quad \forall\,\varphi \in H^1(\Gamma_0),\ \forall\,\psi \in C^\infty_0(0,T).$$

Hence, $u^*_1 = u^*_2$ almost everywhere in $(0,T)\times\Gamma_0$. Thus, if for a given sequence of controls its s_a-limit exists, then it is unique and does not depend on the choice of the extension operators. As an obvious consequence of this remark, we have the following result.

Lemma 10.21. *Every sequence of admissible controls $\{\widehat{v}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon\}_{\varepsilon>0}$ is relatively compact with respect to the s_a-convergence introduced above. Moreover, if u^* is its s_a-weak limit point, then u^* belongs to the set*

$$U = \left\{u \in L^2(0,T;H^1(\Gamma_0)) \mid \|u\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C_0}\right\}.$$

Indeed, the fact $u^* \in U$ is apparent from the lower semicontinuity property of the norm in $L^2(0,T;H^1(\Gamma_0))$ with respect to the weak convergence in this space.

The following result can be proved in much the same way as Theorem 9.19.

Theorem 10.22. *Let $\{\widehat{v}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible Dirichlet controls. Then one can extract an s_b-convergent subsequence of $\{\widehat{v}_\varepsilon\}_{\varepsilon>0}$ for which its s_a- and s_b-limits coincide.*

Remark 10.23. Hereinafter we suppose that for any sequence of Dirichlet controls $\{\widehat{v}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon\}$, there is a sequence of its prototypes $\{v_\varepsilon \in L^2(0,T;H^1(\Gamma_0))\}$ such that $\widehat{v}_\varepsilon|_{\Gamma_\varepsilon^D} = v_\varepsilon|_{\Gamma_\varepsilon^D}$ and $v_\varepsilon \rightharpoonup u$ in $L^2(0,T;H^1(\Gamma_0))$. Then, by analogy with Lemma 10.13, it can be proved that $\widehat{v}_\varepsilon$ converging to $u|_{\Gamma_0}$ weakly in $L^2(0,T;L^2(\Omega,\mathrm{d}\mu_\varepsilon^h))$ and $u|_{\Gamma_0} = u^{**}$ by virtue of the uniqueness of weak limits in $L^2(0,T;L^2(\Omega,\mathrm{d}\mu_\varepsilon^h))$. Therefore, in what follows, we will suppose that every sequence of admissible Dirichlet controls $\{\widehat{v}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon\}$ is compact with respect to the weak convergence in $L^2(0,T;L^2(\Omega,\mathrm{d}\mu_\varepsilon^h))$ and its limit belongs to $L^2(0,T;H^1(\Gamma_0))$.

10.4.2 w-Convergence of admissible solutions of the C-extended problems

Let $\{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \in \widehat{\Sigma}_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible solutions for the $\widehat{\mathbb{CP}}_\varepsilon$-problem. Each of the sets $\widehat{\Sigma}_\varepsilon$ is defined in (10.40) and $\widehat{\Sigma}_\varepsilon \subset \mathbb{Z}_\varepsilon$ for every $\varepsilon > 0$, where

$$\begin{aligned}\mathbb{Z}_\varepsilon \equiv\ & L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)) \\ & \times L^2(0,T;\left[H^1(\Gamma_0) \cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)\right]) \times\ L^2(0,T;L^2(\Omega,\mathrm{d}\nu_\varepsilon^h)) \\ & \times \left[L^2(0,T;H^1(\Omega,\mathrm{d}\eta_\varepsilon^h)) \cap L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))\right]. \end{aligned} \tag{10.59}$$

We assume that this sequence is bounded, that is,

$$\limsup_{\varepsilon>0} \|(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon)\|_{\mathbb{Z}_\varepsilon}] < +\infty.$$

Now, summing up the above given arguments, we may introduce the following concept of the weak convergence in the variable space $\mathbb{Z}_\varepsilon$.

Definition 10.24. *We say that a bounded sequence $\{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \in \mathbb{Z}_\varepsilon\}_{\varepsilon>0}$ is w-convergent to a quadruple*

$$\begin{aligned}(a,v,w,z) \in\ & L^2((0,T)\times\Omega) \\ & \times L^2(0,T;H^1(\Gamma_0)) \times L^2((0,T)\times\Gamma_0) \times L^2(0,T;H^1(\Omega))\end{aligned}$$

in the variable space $\mathbb{Z}_\varepsilon$ *as* ε *tends to* 0 *(in symbols,* $(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \xrightarrow{w} (a, v, w, z)$ *) if*

$$
\begin{aligned}
&a_\varepsilon \rightharpoonup a \quad \text{in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)),\\
&v_\varepsilon \rightharpoonup v \quad \text{in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)), \quad v \in L^2(0,T;H^1(\Gamma_0)),\\
&w_\varepsilon \rightharpoonup w \quad \text{in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\nu_\varepsilon^h)),\\
&z_\varepsilon^h \to z \ \text{in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)),\\
&z_\varepsilon^h \rightharpoonup z \ \text{in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)),\\
&\int_0^T \int_\Omega |z_\varepsilon^h - z|^2 \,\mathrm{d}\eta_\varepsilon^h \,\mathrm{d}t \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}
$$

Then, taking into account Theorems 10.18 and 10.22, we come to the following conclusion.

Theorem 10.25. *Let* $\{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \in \mathbb{Z}_\varepsilon\}_{\varepsilon>0}$ *be a bounded sequence of admissible solutions to the* C*-extended problems* $\widehat{\mathbb{CP}}_\varepsilon$*. Then this sequence is relatively compact with respect to the* w*-convergence in the variable space* $\mathbb{X}_\varepsilon$ *and, moreover, if* $(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \xrightarrow{w} (a, v, w, z)$*, then*

$$z_\varepsilon \to z \quad \text{in} \quad L^6(0,T;L^6(\Omega,\mathrm{d}\eta_\varepsilon^h)), \tag{10.60}$$

$$(z_\varepsilon)^3 \to z^3 \quad \text{in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)). \tag{10.61}$$

10.5 The limiting optimal control problem and its properties

By analogy with the case of steady-state processes on thin periodic structures, we begin this section with the definition of a "limit" minimization problem for the sequence (10.42) as ε tends to 0. To this end, we make use of the concept of variational convergence of constrained minimization problems in variable spaces. Note that, in view of the specific construction of the solution space $\mathbb{Z}_\varepsilon$, the strong approximation property for the "w-limit space"

$$\mathbb{Y} = L^2((0,T)\times\Omega) \times L^2(0,T;H^1(\Gamma_0)) \times L^2((0,T)\times\Gamma_0) \times L^2(0,T;H^1(\Omega))$$

does not hold, in general. The last sentence means that not for every quadruple $(a, v, w, z) \in \mathbb{Y}$ can a sequence $\{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \in \mathbb{Z}_\varepsilon\}_{\varepsilon>0}$ be found such that $(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \xrightarrow{w} (a, v, w, z)$. Because of this, we give the following concept of a "limit" minimization problem.

Definition 10.26. *We say that a minimization problem*

$$\left\langle \inf_{(a,v,w,z)\in\,\Sigma_0} J_0(a,v,w,z) \right\rangle \tag{10.62}$$

is the weak variational limit of the sequence (10.42) with respect to the w-convergence in the variable space $\mathbb{Z}_\varepsilon$ (or variational w-limit) if the conditions (d)–(dd) of Definition 8.24 hold true with

$$\begin{aligned}\mathbb{X}_\varepsilon &= L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))\\ &\times\ L^2(0,T;\left[H^1(\Gamma_0)\cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)\right])\times\ L^2(0,T;L^2(\Omega,\mathrm{d}\nu_\varepsilon^h))\\ &\times\left[L^2(0,T;H^1(\Omega,\mathrm{d}\eta_\varepsilon^h))\cap L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))\right],\end{aligned}$$

$$\mathbb{X}=L^2((0,T)\times\Omega)\times L^2(0,T;H^1(\Gamma_0))\times L^2((0,T)\times\Gamma_0)\times L^2(0,T;H^1(\Omega)).$$

Theorem 10.27. *Assume that the constrained minimization problem (10.62) is the weak variational limit of the sequence (10.42), and is solvable. Assume also that the initial data for the original problem (10.5) satisfy the condition*

$$\sup_{\varepsilon>0}\max\left\{\|y_{\varepsilon,0}\|_{H^1(\Omega)},\|g_\varepsilon\|_{L^2(0,T;L^2(S^\varepsilon_{int}))}\right\}<+\infty.$$

Let $\left\{(a^0_\varepsilon,v^0_\varepsilon,w^0_\varepsilon,z^0_\varepsilon)\in\widehat{\Sigma}_\varepsilon\right\}_{\varepsilon>0}$ be a sequence of optimal solutions for the $\widehat{\mathbb{CP}}_\varepsilon$-problems. Then this sequence is relatively w-compact and its every w-cluster quadruple is an optimal solution to the problem (10.62); more precisely, if

$$(a^0_{\varepsilon_k},v^0_{\varepsilon_k},w^0_{\varepsilon_k},z^0_{\varepsilon_k})\ \xrightarrow{\ w\ }\ (a^0,v^0,w^0,z^0),\tag{10.63}$$

then

$$\begin{aligned}\inf_{(a,v,w,z)\in\,\Sigma_0}J_0(a,v,w,z)&=J_0\left(a^0,v^0,w^0,z^0\right)\\ &=\lim_{k\to\infty}\widehat{J}_{\varepsilon_k}(a^0_{\varepsilon_k},v^0_{\varepsilon_k},w^0_{\varepsilon_k},z^0_{\varepsilon_k})\\ &=\lim_{\varepsilon\to0}\ \inf_{(a_\varepsilon,v_\varepsilon,w_\varepsilon,z_\varepsilon)\in\,\widehat{\Sigma}_\varepsilon}\widehat{J}_\varepsilon(a_\varepsilon,v_\varepsilon,w_\varepsilon,z_\varepsilon).\end{aligned}\tag{10.64}$$

Proof. Let us assume that the sequence of optimal solutions

$$\left\{(a^0_\varepsilon,v^0_\varepsilon,w^0_\varepsilon,z^0_\varepsilon)\in\widehat{\Sigma}_\varepsilon\right\}_{\varepsilon>0}$$

to the C-extended problems $\mathbb{CP}_\varepsilon$ is such that

$$\limsup_{\varepsilon>0}\widehat{J}_\varepsilon(a^0_\varepsilon,v^0_\varepsilon,w^0_\varepsilon,z^0_\varepsilon)=\infty.$$

Then for any given value $D>0$, there is a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k\to0$ and

$$\begin{aligned}\min_{(a_{\varepsilon_k},v_{\varepsilon_k},w_{\varepsilon_k},z_{\varepsilon_k})\in\,\widehat{\Sigma}_{\varepsilon_k}}&\widehat{J}_{\varepsilon_k}(a_{\varepsilon_k},v_{\varepsilon_k},w_{\varepsilon_k},z_{\varepsilon_k})\\ &=\widehat{J}_{\varepsilon_k}(a^0_{\varepsilon_k},v^0_{\varepsilon_k},w^0_{\varepsilon_k},z^0_{\varepsilon_k})\ge D,\quad\forall\,k\in\mathbb{N}.\end{aligned}$$

On the other hand, it is easy to see that the quadruple $(0,0,0,z_\varepsilon^*)$ is admissible for every ε (here, z_e^* is the weak solution of the initial-boundary value problem (10.24) under the condition $a_\varepsilon = 0$, $v_\varepsilon = 0$, and $w_\varepsilon = 0$). Hence, taking the a priori estimate (10.41) into account, we immediately conclude

$$\limsup_{\varepsilon\to 0} \min_{(a_\varepsilon,v_\varepsilon,w_\varepsilon,z_\varepsilon)\in\,\widehat{\Sigma}_\varepsilon} \widehat{J}_\varepsilon(a_\varepsilon,v_\varepsilon,w_\varepsilon,z_\varepsilon) \le \limsup_{\varepsilon\to 0} \widehat{J}_\varepsilon(0,0,0,z_\varepsilon^*) \le \widetilde{C} < +\infty.$$

The obtained contradiction proves that there is a constant $D_0 > 0$ such that

$$\limsup_{\varepsilon\to 0} \widehat{J}_\varepsilon(a_\varepsilon^0, v_\varepsilon^0, w_\varepsilon^0, z_\varepsilon^0) \le D_0.$$

This fact implies the boundedness of the sequences $\{a_\varepsilon^0\}$ and $\{w_\varepsilon^0\}$ in $L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$ and $L^2(0,T;L^2(\Omega,\mathrm{d}\nu_\varepsilon^h))$, respectively. Then using the boundedness of the Dirichlet type controls $\{v_\varepsilon^0\}$ and estimate (10.41), we deduce

$$\begin{aligned}
&\limsup_{\varepsilon>0} \|(a_\varepsilon^0, v_\varepsilon^0, w_\varepsilon^0, z_\varepsilon^0)\|_{\mathbb{Z}_\varepsilon} \\
&\qquad = \limsup_{\varepsilon>0} \big[\|a_\varepsilon^0\|_{L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))} + \|v_\varepsilon^0\|_{L^2(0,T;H^1(\Gamma_0)\cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)} \\
&\qquad + \|w_\varepsilon^0\|_{L^2(0,T;L^2(\Omega,\mathrm{d}\nu_\varepsilon^h))} + \|z_\varepsilon^0\|_{L^2(0,T;H^1(\Omega,\mathrm{d}\eta_\varepsilon^h))}\big] < +\infty.
\end{aligned}$$

Then, taking Theorem 10.25 into account, we assume that there exist a subsequence $\{(a_{\varepsilon_k}^0, v_{\varepsilon_k}^0, w_{\varepsilon_k}^0, z_{\varepsilon_k}^0)\}_{k\in\mathbb{N}}$ of the sequence of optimal solutions and a quadruple (a^*, v^*, w^*, z^*) such that $(a_{\varepsilon_k}^0, v_{\varepsilon_k}^0, w_{\varepsilon_k}^0, z_{\varepsilon_k}^0) \xrightarrow{w} (a^*, v^*, w^*, z^*)$ as $\varepsilon_k \to 0$. In order to conclude this proof, it remains to apply the rest of the arguments of the proof of Theorem 9.26. □

10.6 Main convergence theorem

The main question that we are going to discuss in this section concerns the asymptotic behavior as $\varepsilon \to 0$ of the initial-boundary value problem

$$z_\varepsilon = v_\varepsilon \quad \mu_\varepsilon^h\text{-a.e. on } (0,T)\times\Omega, \ \|v_\varepsilon\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0, \tag{10.65}$$

$$z_\varepsilon(0,\cdot) = y_{\varepsilon,0} \quad \eta_\varepsilon^h\text{-a.e. in } \Omega, \tag{10.66}$$

$$\begin{aligned}
&-\int_0^T\int_\Omega z_\varepsilon\varphi\,\psi'\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t + \int_0^T\int_\Omega (A(x/\varepsilon)\nabla z_\varepsilon\cdot\nabla\varphi)\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t \\
&\qquad + 4d\,\beta(\varepsilon)\int_0^T\int_\Omega z_\varepsilon\varphi\psi\,\mathrm{d}\lambda_\varepsilon^h\,\mathrm{d}t \\
&\qquad = \int_0^T\int_\Omega a_\varepsilon\,\varphi\,\psi\,\mathrm{d}\eta_\varepsilon^h\,\mathrm{d}t + 2\,\beta(\varepsilon)\int_0^T\int_\Omega w_\varepsilon\varphi\psi\,\mathrm{d}\nu_\varepsilon^h\,\mathrm{d}t \\
&\qquad + 4\,\beta(\varepsilon)\int_0^T\int_\Omega g_\varepsilon\varphi\psi\,\mathrm{d}\lambda_\varepsilon^h\,\mathrm{d}t, \\
&\qquad \forall\,\varphi\in C_0^\infty(\mathbb{R}^2;\Gamma_\varepsilon^D),\ \forall\,\psi\in C_0^\infty(0,T),
\end{aligned} \tag{10.67}$$

where

$$\beta(\varepsilon) = \frac{(1-h(\varepsilon))}{(2-h(\varepsilon))} \cdot \xi(\varepsilon), \quad \text{and} \quad \xi(\varepsilon) = \frac{\varepsilon}{h(\varepsilon)}.$$

As follows from Proposition 9.23, the sequence of sets of test functions $\left\{C_0^\infty(\mathbb{R}^2; \Gamma_\varepsilon^D)\right\}_{\varepsilon>0}$ for the integral identity (10.67) converges in the sense of Kuratowski to $C_0^\infty(\mathbb{R}^2; \Gamma_0)$ as ε tends to 0. So, in the sequel, we suppose that $\varphi \in C_0^\infty(\mathbb{R}^2; \Gamma_0)$ in (10.67).

Let $\left\{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \in \widehat{\Sigma}_\varepsilon\right\}$ be any equibounded sequence of admissible solutions for the $\widehat{\mathbb{CP}}_\varepsilon$-problem. By Theorem 10.25, this sequence is relatively compact with respect to the w-convergence in the variable space $\mathbb{Z}_\varepsilon$. So, we may suppose that there exists a quadruple $(a, v, w, z) \in \mathbb{Y}$ such that $(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \overset{w}{\rightharpoonup} (a, v, w, z)$. The main question is: By what kind of relation can this quadruple be determined? We give the following result.

Theorem 10.28. *Suppose that $A_\varepsilon(x) = A(x/\varepsilon)$ in (10.67) is a Y-periodic measurable matrix satisfying conditions (9.21). Assume also that*

$$\lim_{\varepsilon\to 0} \xi(\varepsilon) = \xi^* < +\infty$$

and there exist functions $y^0 \in L^2(\Omega)$ and $g \in L^2(0,T; L^2(\Omega))$ such that

$$y_{\varepsilon,0} \rightharpoonup y_0 \quad \text{in} \quad H^1(\Omega) \quad \text{and} \quad g_\varepsilon \rightharpoonup g \quad \text{in} \quad L^2(0,T; L^2(\Omega, d\lambda_\varepsilon^h)). \tag{10.68}$$

Let

$$\begin{gathered}\left\{a_\varepsilon \in L^2(0,T; L^2(\Omega, \mathrm{d}\eta_\varepsilon^h))\right\}, \quad \left\{w_\varepsilon \in L^2(0,T; L^2(\Omega, \mathrm{d}\nu_\varepsilon^h))\right\}, \\ \left\{v_\varepsilon \in L^2(0,T; \left[H^1(\Gamma_0) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\right])\right\}\end{gathered} \tag{10.69}$$

be any bounded sequences of admissible controls for the $\widehat{\mathbb{CP}}_\varepsilon$-problems such that

$$a_\varepsilon \rightharpoonup a \quad \text{in} \quad L^2(0,T; L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)), \tag{10.70}$$

$$w_\varepsilon \rightharpoonup w \quad \text{in} \quad L^2(0,T; L^2(\Omega, \mathrm{d}\nu_\varepsilon^h)), \tag{10.71}$$

$$v_\varepsilon \to v \quad \text{in} \quad L^2(0,T; L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)). \tag{10.72}$$

Let $\left\{z_\varepsilon = z_\varepsilon(a_\varepsilon, v_\varepsilon, w_\varepsilon) \in L^2(0,T; H^1(\Omega, \mathrm{d}\eta_\varepsilon^h)) \cap L^2(0,T; L^2(\Omega, \mathrm{d}\lambda_\varepsilon^h))\right\}_{\varepsilon>0}$ be the corresponding solutions to the problem (10.65)–(10.67). Then

$$(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \overset{w}{\rightharpoonup} (a, v, w, z) \quad \text{as} \quad \varepsilon \to 0,$$

where z is a unique solution in $L^2(0,T; H^1(\Omega))$ of the initial-boundary value problem

$$z = v \ \text{a.e. on} \ (0,T) \times \Gamma_0, \ \|v\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0, \tag{10.73}$$

$$z(0,\cdot) = y_0 \quad \text{a.e. in} \quad \Omega, \tag{10.74}$$

$$-\int_0^T\int_\Omega z\varphi\,\psi'\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega (A^{hom}\nabla z\cdot\nabla\varphi)\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$+2d\,\xi^*\int_0^T\int_\Omega z\varphi\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^T\int_\Omega a\,\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t+2\,\xi^*\int_0^T\int_\Omega g\varphi\psi\,\mathrm{d}x\,\mathrm{d}t. \tag{10.75}$$

Here, the matrix A^{hom} is defined by the rule

$$A^{hom}\zeta=\int_Y A(y)(\zeta+\upsilon^0)\,\mathrm{d}\eta(y), \tag{10.76}$$

where $\upsilon^0\in L^2(\Omega,V_{pot})$ is the unique solution of the minimum problem (9.93).

Proof. First, we note that by the a priori estimate (10.41), the sequence of admissible solutions $\{(a_\varepsilon,v_\varepsilon,w_\varepsilon,z_\varepsilon)\in\widehat{\Sigma}_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $\mathbb{Z}_\varepsilon$. Hence, due to Theorem 10.25, we may suppose that there is a function $z\in L^2(0,T;H^1(\Omega))$ satisfying the conditions

$$\begin{cases} z_\varepsilon\to z \ \text{ in } \ L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)),\\ z_\varepsilon\rightharpoonup z \ \text{ in } \ L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)),\\ z_\varepsilon\to z \ \text{ in } \ L^6(0,T;L^6(\Omega,\mathrm{d}\eta_\varepsilon^h)),\\ \nabla z_\varepsilon\overset{2}{\rightharpoonup}\nabla z+r, \ \text{ where } \ r\in L^2(0,T;L^2(\Omega,V_{pot})). \end{cases} \tag{10.77}$$

On the other hand, in view of Theorem 10.22 and Remark 10.23, the limit element v in (10.72) is such that $v\in L^2(0,T;H^1(\Gamma_0))$ and $v_\varepsilon\overset{s_a}{\longrightarrow}v$. Moreover, taking the initial suppositions (10.68) and Lemma 10.13 into account, we can conclude

$$y_{\varepsilon,0}\to y_0 \ \text{ in } \ L^2(\Omega,d\eta_\varepsilon^h).$$

Passing to the limit in (10.65) and (10.66) as $\varepsilon\to 0$, we come to relations (10.73). Our aim is to prove that the integral equality (10.75) holds true.

For this, we consider the integral identity (10.67) with the test functions $\varphi(x)=\varepsilon\Psi(x)\omega(\varepsilon^{-1}x)$ and $\psi\in C_0^\infty(0,T)$, where $\omega\in C_{per}^\infty(Y)$ and $\Psi\in C_0^\infty(\mathbb{R}^2,\Gamma_\varepsilon^D)$. This yields

$$
\begin{aligned}
&- \varepsilon \int_0^T \int_\Omega z_\varepsilon \Psi \omega(\varepsilon^{-1}x)\, \psi' \, \mathrm{d}\eta_\varepsilon^h \, \mathrm{d}t \\
&+ \varepsilon \int_0^T \int_\Omega (A(x/\varepsilon)\nabla z_\varepsilon \cdot \nabla \Psi)\omega(\varepsilon^{-1}x)\psi \, \mathrm{d}\eta_\varepsilon^h \, \mathrm{d}t \\
&+ \int_0^T \int_\Omega (A(x/\varepsilon)\nabla z_\varepsilon \cdot \nabla \omega)\Psi\psi \, \mathrm{d}\eta_\varepsilon^h \, \mathrm{d}t + 4\varepsilon d\, \beta(\varepsilon) \int_0^T \int_\Omega z_\varepsilon \Psi \omega(\varepsilon^{-1}x)\psi \, \mathrm{d}\lambda_\varepsilon^h \, \mathrm{d}t \\
&= \varepsilon \int_0^T \int_\Omega a_\varepsilon \, \Psi \omega(\varepsilon^{-1}x)\, \psi \, \mathrm{d}\eta_\varepsilon^h \, \mathrm{d}t + 2\varepsilon\, \beta(\varepsilon) \int_0^T \int_\Omega w_\varepsilon \Psi \omega(\varepsilon^{-1}x)\psi \, \mathrm{d}\nu_\varepsilon^h \, \mathrm{d}t \\
&+ 4\varepsilon\, \beta(\varepsilon) \int_0^T \int_\Omega g_\varepsilon \Psi(x)\omega(\varepsilon^{-1}x)\psi \, \mathrm{d}\lambda_\varepsilon^h \, \mathrm{d}t. \qquad (10.78)
\end{aligned}
$$

Taking the mean value property which holds for every periodic Borel measure η_ε^h, λ_ε^h, and ν_ε^h into account, we pass to the limit in (10.78) as $\varepsilon \to 0$ and get

$$
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \left(A(\varepsilon^{-1}x)\nabla z_\varepsilon, \nabla \omega(\varepsilon^{-1}x) \right) \Psi\psi \, \mathrm{d}\eta_\varepsilon^h \, \mathrm{d}t = 0. \qquad (10.79)
$$

Then, in view of $(10.77)_4$ and the definition of the weak two-scale limit, we have

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Psi \left[\omega(\varepsilon^{-1}x) A(\varepsilon^{-1}x) \right] \nabla z_\varepsilon \, \mathrm{d}\eta_\varepsilon \psi \, \mathrm{d}t \\
&\qquad = \int_0^T \int_\Omega \int_Y \Psi(x)\omega(y)A(y) \left[\nabla z(t,x) + r(t,x,y) \right] \mathrm{d}\eta(y)\, dx\, \psi \, \mathrm{d}t.
\end{aligned}
$$

Thus,

$$
A(\varepsilon^{-1}x)\nabla z_\varepsilon \overset{2}{\rightharpoonup} A(y) \left[\nabla z + r(t,x,y) \right], \qquad (10.80)
$$

$$
\begin{aligned}
A(\varepsilon^{-1}x)\nabla z_\varepsilon(t,x) &\rightharpoonup \int_Y A(y) \left[\nabla z(t,x) + r(t,x,y) \right] \mathrm{d}\eta(y) \\
&\text{in } \; L^2(0,T; \left[L^2(\Omega, \mathrm{d}\eta_\varepsilon^h \right]^2), \qquad (10.81)
\end{aligned}
$$

that is, equality (10.79) can be rewritten in explicit form as

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \left(A(\varepsilon^{-1}x)\nabla z_\varepsilon, \nabla \omega(\varepsilon^{-1}x) \right) \Psi\psi \, \mathrm{d}\eta_\varepsilon^h \, \mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \Psi \left[\int_Y \left(A(y) \left[\nabla z(t,x) + r(t,x,y) \right], \nabla \omega \right) \mathrm{d}\eta(y) \right] \mathrm{d}x\, \psi \, \mathrm{d}t = 0.
\end{aligned}
$$

Since this equality holds true for every $\Psi \in C_0^\infty(\mathbb{R}^2, \Gamma_0)$ and every $\psi \in C_0^\infty(0,T)$, it follows that

$$\int_Y (A(\nabla z + r), \zeta)\ \mathrm{d}\eta = 0, \qquad \forall\, \zeta \in \left[C^\infty_{per}(Y)\right]^2. \tag{10.82}$$

However, following Zhikov [256], (10.82) can be viewed as Euler's equation for the minimum problem

$$\min_{\upsilon \in V_{pot}} \int_Y (\zeta + \upsilon, A(\zeta + \upsilon))\, \mathrm{d}\eta = \int_Y (\zeta + \upsilon^0, A(\zeta + \upsilon^0))\ \mathrm{d}\eta. \tag{10.83}$$

Since this problem has a unique solution $\upsilon^0 \in L^2(\Omega, V_{pot})$, it follows that υ^0 is the unique solution of (10.82) as well. Let us define the matrix A^{hom} as in (10.76). Then, having put $\upsilon = \upsilon^0$ in (10.81) and used formula (10.76), we immediately deduce that

$$A\left(\varepsilon^{-1}x\right)\nabla z_\varepsilon \rightharpoonup A^{hom}\nabla z \qquad \text{in}\ \ L^2(0,T;\left[L^2(\Omega, \mathrm{d}\eta^h_\varepsilon)\right]^2). \tag{10.84}$$

Our intention next is to pass to the limit in (10.67) as $\varepsilon \to 0$. For this, following Proposition 9.23, we choose the test functions in (10.67) as follows: $\varphi = \varphi_\varepsilon \in C^\infty_0(\mathbb{R}^2, \Gamma^D_\varepsilon)$ for every $\varepsilon > 0$, where $\varphi_\varepsilon \to \varphi_0$ in $C(\mathbb{R}^2)$, and φ_0 is an element of $C^1(\Omega, \Gamma_0)$. Then the passage to the limit in

$$\begin{aligned}
&-\int_0^T \int_\Omega z_\varepsilon \varphi_\varepsilon\, \psi'\, \mathrm{d}\eta^h_\varepsilon\, \mathrm{d}t + \int_0^T \int_\Omega (A(x/\varepsilon)\nabla z_\varepsilon \cdot \nabla \varphi_\varepsilon)\psi\, \mathrm{d}\eta^h_\varepsilon\, \mathrm{d}t \\
&\qquad + 4d\,\beta(\varepsilon) \int_0^T \int_\Omega z_\varepsilon \varphi_\varepsilon \psi\, \mathrm{d}\lambda^h_\varepsilon\, \mathrm{d}t \\
&\qquad = \int_0^T \int_\Omega a_\varepsilon\, \varphi_\varepsilon\, \psi\, \mathrm{d}\eta^h_\varepsilon\, \mathrm{d}t + 2\,\beta(\varepsilon) \int_0^T \int_\Omega w_\varepsilon \varphi_\varepsilon \psi\, \mathrm{d}\nu^h_\varepsilon\, \mathrm{d}t \\
&\qquad + 4\,\beta(\varepsilon) \int_0^T \int_\Omega g_\varepsilon \varphi_\varepsilon \psi\, \mathrm{d}\lambda^h_\varepsilon\, \mathrm{d}t
\end{aligned}$$

is justified by relations (10.70)–(10.72), (10.76), and (10.77),

$$\lim_{\varepsilon \to 0} \beta(\varepsilon) = (1/2) \lim_{\varepsilon \to 0} \xi(\varepsilon) = (1/2)\xi^*, \tag{10.85}$$

and Lemma 10.15. Hence, we obtain

$$\begin{aligned}
&-\int_0^T \int_\Omega z\varphi\, \psi'\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_\Omega (A^{hom}\nabla z \cdot \nabla \varphi)\psi\, \mathrm{d}x\, \mathrm{d}t \\
&\qquad + 2d\,\xi^* \int_0^T \int_\Omega z\varphi\psi\, \mathrm{d}x\, \mathrm{d}t \\
&\qquad = \int_0^T \int_\Omega a\, \varphi\, \psi\, \mathrm{d}x\, \mathrm{d}t \\
&\qquad + \xi^* \int_0^T \int_{\Gamma_0} w\varphi\psi\, \mathrm{d}x_1\, \mathrm{d}t + 2\,\xi^* \int_0^T \int_\Omega g\varphi\psi\, \mathrm{d}x\, \mathrm{d}t.
\end{aligned}$$

However, given $\varphi \in H^1(\Omega, \Gamma_0)$ have $\int_0^T \int_{\Gamma_0} w\varphi\psi \, \mathrm{d}x_1 \, \mathrm{d}t = 0$. As a result, we come to the required equality (10.75).

To conclude, we note that relations (10.73)–(10.75) can be interpreted as the variational formulation of the initial-boundary problem

$$\left.\begin{aligned} \partial_t z - \operatorname{div}\left(A^{hom}\nabla z\right) + 2d\xi^* z &= a + 2\xi^* g && \text{in } (0,T)\times\Omega, \\ \partial_{\nu_{A^{hom}}} z &= 0 && \text{on } (0,T)\times\partial\Omega\setminus\Gamma_0, \\ z &= v && \text{on } (0,T)\times\Gamma_0, \\ z(0,x) &= y_0 && \text{for a.a. } x\in\Omega, \end{aligned}\right\} \tag{10.86}$$

for which the following result is well known: For every $a, g \in L^2(0,T;L^2(\Omega))$, $y_0 \in H^1(\Omega)$, and $v \in L^2(0,T;H^1(\Gamma_0))$, there exists a unique solution of (10.86) (see Lions [168] and Michajlov [188]). This completes the proof. □

The following statement is a direct consequence of the well-known results of the theory of boundary value problems (see Lions [168]).

Corollary 10.29. *Let*

$$(a_1, v_1, z_1), (a_2, v_2, z_1) \in L^2((0,T)\times\Omega) \times L^2(0,T;H^1(\Gamma_0)) \times L^2(0,T;H^1(\Omega))$$

be any triplets satisfying relation (10.86). Then there exists a constant

$$\hat{C} > 0, \quad \hat{C} = \hat{C}(\Omega, \Gamma_0, v^0, g, \xi^*, \alpha_0)$$

such that

$$\|z_1 - z_2\|_{L^2(0,T;H^1(\Omega))} + \|z_1 - z_2\|_{L^6((0,T)\times\Omega)} \le \hat{C}\left[\|a_1 - a_2\|_{L^2((0,T)\times\Omega)} + \|v_1 - v_2\|_{L^2(0,T;H^1(\Gamma_0))}\right]. \tag{10.87}$$

10.7 The limit analysis of the C-extended optimal control problem

In this section, we show that for the sequence of constrained minimization problems (10.42), there exists a weak variational limit with respect to the w-convergence in the variable space $\mathbb{Z}_\varepsilon$ and that this limit problem can be recovered in an explicit form. We begin with the following result.

Theorem 10.30. *Let* $y_\varepsilon^T \in L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)$, $y_\varepsilon^* \in L^6(0,T;L^6(\Omega,\mathrm{d}\eta_\varepsilon^h))$, $y_{\varepsilon,0} \in H^1(\Omega)$, $g_\varepsilon \in L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h))$, *and* $f_\varepsilon \in L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h))$ *be given*

functions such that

$$\sup_{\varepsilon\to 0}\left(\int_\Omega \left(y_\varepsilon^T\right)^2 \,\mathrm{d}\eta_\varepsilon^h\right) < \infty, \quad y_\varepsilon^T \to y^T \quad in \quad L^2(\Omega, \mathrm{d}\eta_\varepsilon^h), \tag{10.88}$$

$$\left.\begin{aligned}\sup_{\varepsilon\to 0}\left(\int_0^T\int_\Omega (y_\varepsilon^*)^6 \,\mathrm{d}\eta_\varepsilon^h \mathrm{d}t\right) &< \infty,\\ y_\varepsilon^* \to y^* \quad in \quad L^6(0,T;L^6(\Omega,\mathrm{d}\eta_\varepsilon^h)),&\end{aligned}\right\} \tag{10.89}$$

$$\left.\begin{aligned}\sup_{\varepsilon\to 0}\left(\int_0^T\int_\Omega (f_\varepsilon)^2 \,\mathrm{d}\eta_\varepsilon^h \mathrm{d}t\right) &< \infty,\\ f_\varepsilon \to f \quad in \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta_\varepsilon^h)),&\end{aligned}\right\} \tag{10.90}$$

$$y_{\varepsilon,0} \rightharpoonup y_0 \quad in \quad H^1(\Omega), \tag{10.91}$$

$$\left.\begin{aligned}\sup_{\varepsilon\to 0}\left(\int_0^T\int_\Omega (g_\varepsilon)^2 \,\mathrm{d}\lambda_\varepsilon^h \mathrm{d}t\right) &< \infty,\\ g_\varepsilon \rightharpoonup g \quad in \quad L^2(0,T;L^2(\Omega,\mathrm{d}\lambda_\varepsilon^h)).&\end{aligned}\right\} \tag{10.92}$$

Assume that the correction factor $\kappa(\varepsilon)$ in the cost functional (10.39) is such that

$$\lim_{\varepsilon\to 0}\kappa(\varepsilon)h(\varepsilon) = \kappa^* < +\infty. \tag{10.93}$$

Then, for the sequence (10.42), there exists a unique weak variational limit in the sense of Definition 10.26 which has the representation

$$\left\langle \inf_{(a,v,w,z)\in\,\Sigma_0} J_0(a,v,w,z)\right\rangle, \tag{10.94}$$

where the cost functional J_0 and the set of admissible solutions Σ_0 are defined as follows:

$$\begin{aligned} J_0(a,v,w,z) = {} & 2\int_\Omega (z(T,x)-y^T)^2\,\mathrm{d}x + 2\int_0^T\int_\Omega (z-y^*)^6\,\mathrm{d}x\,\mathrm{d}t\\ & + \kappa^*\int_0^T\int_{\Gamma_0} v^2\,\mathrm{d}x_1\,\mathrm{d}t + 2\gamma^*\int_0^T\int_\Omega (a-z^3-f)^2\,\mathrm{d}x\,\mathrm{d}t\\ & + 2\int_0^T\int_{\Gamma_0} w^2\,\mathrm{d}x_1\,\mathrm{d}t, \end{aligned} \tag{10.95}$$

$(a,v,w,z)\in\Sigma_0$ *if and only if*

$$\Sigma_0 = \left\{ \begin{array}{c} z \in L^2(0,T;H^1(\Omega)), \quad a \in L^2((0,T)\times\Omega), \\ v \in L^2(0,T;H^1(\Gamma_0)), \quad w \in L^2(0,T;H^1(\Gamma_0)), \\ \|v\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0, \\ z = v \ \text{a.e. on} \ (0,T)\times\Gamma_0, \ z(0,\cdot) = y_0 \quad \text{a.e. on} \ \ \Omega; \\ -\displaystyle\int_0^T\int_\Omega z\varphi\,\psi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega (A^{hom}\nabla z\cdot\nabla\varphi)\psi\,\mathrm{d}x\,\mathrm{d}t \\ +2d\,\xi^*\displaystyle\int_0^T\int_\Omega z\varphi\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega a\,\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t \\ +2\,\xi^*\displaystyle\int_0^T\int_\Omega g\varphi\psi\,\mathrm{d}x\,\mathrm{d}t \quad \forall\,\varphi\in C^\infty(\Omega,\Gamma_0), \ \forall\,\psi\in C_0^\infty(0,T) \end{array} \right\}. \tag{10.96}$$

Proof. By analogy with Theorem 9.29, we divide the proof of this theorem into two steps for the verification of the correspondent items of Definition 10.26.

Step 1: *Statement (d) of Definition 8.24 is valid.*

Let $\{(a_k, v_k, w_k, z_k) \in \mathbb{Z}_{\varepsilon_k}\}_{\varepsilon>0}$ be a bounded sequence, w-convergent to a quadruple $(u, v, w, z) \in \mathbb{Y}$. Here, $\{\varepsilon_k\}$ is a subsequence of $\{\varepsilon > 0\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and $(a_k, v_k, w_k, z_k) \in \widehat{\Sigma}_{\varepsilon_k}$ for all $k \in \mathbb{N}$. Then due to Theorems 10.25 and 10.28, we have that the w-limit quadruple (u, v, w, z) belongs to the space

$$\mathbb{Y} = L^2((0,T)\times\Omega)\times L^2(0,T;H^1(\Gamma_0))\times L^2((0,T)\times\Gamma_0)\times L^2(0,T;H^1(\Omega))$$

and satisfies conditions (10.73)–(10.75). Thus, the first inclusion in (8.59) holds true.

We now verify inequality (8.59), where the limit cost functional J_0 has representation (10.95). Then, taking into account the property of lower semicontinuity of the weak convergence in variable spaces, we have

$$\begin{aligned} \lim_{k\to\infty} \Bigg[& (2-h(\varepsilon_k))\int_\Omega (z_{\varepsilon_k}(T,x) - y^T_{\varepsilon_k})^2\,\mathrm{d}\eta^h_{\varepsilon_k} \\ & + \kappa(\varepsilon_k)h(\varepsilon_k)\int_0^T\int_\Omega v^2_{\varepsilon_k}\,\mathrm{d}\mu^h_{\varepsilon_k}\,\mathrm{d}t + 2\,(1-h(\varepsilon_k))\int_0^T\int_\Omega w^2_{\varepsilon_k}\,\mathrm{d}\nu^h_{\varepsilon_k}\mathrm{d}t\Bigg] \\ & \ge 2\int_\Omega (z(T,x)-y^T)^2\,\mathrm{d}x + \kappa^*\int_0^T\int_{\Gamma_0} v^2\,\mathrm{d}x_1\,\mathrm{d}t \\ & + 2\int_0^T\int_{\Gamma_0} w^2\,\mathrm{d}x_1\mathrm{d}t. \end{aligned} \tag{10.97}$$

In order to pass to the limit in the rest of the terms of the cost functional (10.39), we make use of the strong convergence properties (10.60), (10.89), and (10.90), and the criterion of strong convergence in variable spaces. As a result, we get

$$\lim_{k\to\infty}(2-h(\varepsilon_k))\left[\int_0^T\int_\Omega (z_{\varepsilon_k}-y^*_{\varepsilon_k})^6\,\mathrm{d}\eta^h_{\varepsilon_k}\,\mathrm{d}t\right.$$
$$\left.+\gamma^*\int_0^T\int_\Omega (a_{\varepsilon_k}-z^3_{\varepsilon_k}-f_{\varepsilon_k})^2\,\mathrm{d}\eta^h_{\varepsilon_k}\,\mathrm{d}t\right]$$
$$\geq 2\int_0^T\int_\Omega (z-y^*)^6\,\mathrm{d}x\,\mathrm{d}t+2\gamma^*\int_0^T\int_\Omega (a-z^3-f)^2\,\mathrm{d}x\,\mathrm{d}t. \quad (10.98)$$

To conclude this step, it remains only to unite the relations (10.97)–(10.98) obtained above.

Step 2: *Statement (dd) of Definition 8.24 holds true.*

Let $(a,v,w,z)\in\Sigma_0$ be any admissible solution for the minimization problem (10.94). Let $\widetilde{a}\in L^2(0,T;H^1(\Omega))$ and $\widetilde{w}\in L^2(0,T;H^{3/2}(\Omega))$ be given functions and let $\widetilde{z}=z(\widetilde{a},v)$ be the corresponding solution of the problem (10.86). Then $\widetilde{w}|_{\Gamma_0}\in L^2(0,T;H^1(\Gamma_0))$ and $(\widetilde{a},v,\widetilde{w}|_{\Gamma_0},\widetilde{z})\in\Sigma_0$. Moreover, as follows from (10.95) and the a priori estimate (10.87), there are constants $D_1>0$ and $D_2>0$ such that

$$\left|J_0(a,v,w,z)-J_0(\widetilde{a},v,\widetilde{w}|_{\Gamma_0},\widetilde{z})\right|$$
$$\leq D_1\big(\|a-\widetilde{a}\|_{L^2((0,T)\times\Omega)}+\|w-\widetilde{w}\|_{L^2((0,T)\times\Gamma_0)}$$
$$+\|\widetilde{z}-z\|_{L^2(0,T;H^1(\Omega))}+\|\widetilde{z}-z\|_{L^6((0,T)\times\Omega)}\big),$$

$$\|\widetilde{z}-z\|_{L^2(0,T;H^1(\Omega))\cap L^6((0,T)\times\Omega)}$$
$$\leq D_2\left(\|a-\widetilde{a}\|_{L^2((0,T)\times\Omega)}+\|w-\widetilde{w}\|_{L^2((0,T)\times\Gamma_0)}\right).$$

Let $1>\delta>0$ be a given value. Using the density of the embedding $H^1(B)\hookrightarrow L^2(B)$, we can always choose the elements $\widetilde{a}\in L^2(0,T;H^1(\Omega))$ and $\widetilde{w}\in L^2(0,T;H^{3/2}(\Gamma_0))$ such that

$$\|a-\widetilde{a}\|_{L^2((0,T)\times\Omega)}+\|w-\widetilde{w}\|_{L^2((0,T)\times\Gamma_0)}<\varrho=\max\{D_1^{-1},D_2^{-1}\}\,\delta.$$

Due to the above estimates, we have

$$|J_0(a,v,w,z)-J_0(\widetilde{a},v,\widetilde{w},\widetilde{z})|\leq\delta. \quad (10.99)$$
$$\|\widetilde{z}-z\|_{L^2(0,T;H^1(\Omega))\cap L^6((0,T)\times\Omega)}\leq\delta, \quad (10.100)$$
$$\|(a,v,w,z)-(\widetilde{a},v,\widetilde{w},\widetilde{z})\|_{\mathbb{Y}}<\max\{1,D_1^{-1},D_2^{-1}\}\,\delta. \quad (10.101)$$

We now construct the δ-realizing sequence $\left\{(a^h_\varepsilon,v^h_\varepsilon,w^h_\varepsilon,z^h_\varepsilon)\in\mathbb{Z}_\varepsilon\right\}_{\varepsilon>0}$ as follows:

$$a^h_\varepsilon\to\widetilde{a}\ \text{ in } L^2(0,T;L^2(\Omega,\mathrm{d}\eta^h_\varepsilon)$$
$$\text{and}\quad \int_0^T\int_\Omega\left(a^h_\varepsilon\right)^2\,\mathrm{d}\eta^h_\varepsilon\,\mathrm{d}t\overset{\varepsilon\to0}{\longrightarrow}\int_0^T\int_\Omega a^2\,\mathrm{d}x\,\mathrm{d}t, \quad (10.102)$$

$$v_\varepsilon^h \rightharpoonup v \ \text{ in } L^2(0,T;L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)), \quad \|v_\varepsilon^h\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0$$
$$\text{and} \quad \int_0^T \int_\Omega \left(v_\varepsilon^h\right)^2 \mathrm{d}\mu_\varepsilon^h \,\mathrm{d}t \overset{\varepsilon\to 0}{\longrightarrow} \int_0^T \int_{\Gamma_0} v^2 \,\mathrm{d}x_1 \,\mathrm{d}t, \quad (10.103)$$

$$w_\varepsilon^h \rightharpoonup \widetilde{w} \ \text{ in } L^2(0,T;L^2(\Omega,\mathrm{d}\nu_\varepsilon^h))$$
$$\text{and} \quad \int_0^T \int_\Omega \left(w_\varepsilon^h\right)^2 \mathrm{d}\nu_\varepsilon^h \,\mathrm{d}t \overset{\varepsilon\to 0}{\longrightarrow} \int_0^T \int_{\Gamma_0} \widetilde{w}^2 \,\mathrm{d}x_1 \,\mathrm{d}t. \quad (10.104)$$

Let z_ε^h be the correspondent solutions of the initial-boundary value problem (10.65)–(10.67). Hence, $(a_\varepsilon^h, v_\varepsilon^h, w_\varepsilon^h, z_\varepsilon^h) \in \widehat{\Sigma}_\varepsilon$ for every $\varepsilon > 0$. It is clear that this sequence is equibounded in $\mathbb{Z}_\varepsilon$. Then applying Theorem 10.25, we conclude that the sequence $\left\{(a_\varepsilon^h, v_\varepsilon^h, w_\varepsilon^h, z_\varepsilon^h)\right\}$ is compact with respect to the w-convergence. Let $(\widetilde{a}, v, \widetilde{w}, \widetilde{z}^*)$ be its w-limit. Due to Theorem 10.28, we have $(\widetilde{a}, v, \widetilde{w}, \widetilde{z}^*) \in \Sigma_0$. Since the initial-boundary value problem (10.86) has a unique solution for every fixed $\widetilde{a}$ and v, it follows that $\widetilde{z} = \widetilde{z}^*$ and, hence, $(a_\varepsilon^h, v_\varepsilon^h, w_\varepsilon^h, z_\varepsilon^h) \overset{w}{\rightharpoonup} (\widetilde{a}, v, \widetilde{w}, \widetilde{z})$ as $\varepsilon \to 0$.

Thus, we have shown that the first part of property (dd) in Definition 8.24 holds true. It remains only to verify the validity of inequality (8.59). For this, we take into account properties (10.102)–(10.104) and Theorem 10.25. Then

$$\lim_{\varepsilon\to 0} \widehat{J}_\varepsilon(a_\varepsilon^h, v_\varepsilon^h, w_\varepsilon^h, z_\varepsilon^h) = J_0(\widetilde{a}, v, \widetilde{w}, \widetilde{z}).$$

Hence, applying inequality (10.99), we obtain the required inequality

$$J_0(a, v, w, z) \ge \lim_{\varepsilon\to 0} \widehat{J}_\varepsilon(a_\varepsilon^h, v_\varepsilon^h, w_\varepsilon^h, z_\varepsilon^h) - \delta.$$

□

Remark 10.31. As follows from (10.95) and (10.96), the Neumann boundary control $w \in L^2((0,T) \times \Gamma_0)$ has a "passive" influence on the limit object (10.94). At the same time, the limit cost functional (10.95) (in contrast to the limit problem (10.86)) is not indifferent to this control. Moreover, it can be represented in the form

$$\begin{aligned} J_0(a, v, w, z) &= 2\int_\Omega (z(T,x) - y^T)^2 \,\mathrm{d}x \\ &\quad + 2\int_0^T \int_\Omega (z - y^*)^6 \,\mathrm{d}x\mathrm{d}t + \kappa^* \int_0^T \int_{\Gamma_0} v^2 \,\mathrm{d}x_1\mathrm{d}t \\ &\quad + 2\gamma^* \int_0^T \int_\Omega (a - z^3 - f)^2 \,\mathrm{d}x\mathrm{d}t + \mathbf{C}(\mathbf{w}). \end{aligned} \quad (10.105)$$

Taking into account the variational properties of the problem (10.94) stated in Theorem 10.27, the constant $\mathbf{C}(\mathbf{w})$ should be defined as

$$\mathbf{C}(\mathbf{w}) = 2\int_0^T \int_{\Gamma_0} (w^0)^2 \, \mathrm{d}x_1 \, \mathrm{d}t,$$

where $w^0 \in L^2((0,T)\times\Gamma_0)$ is a weak limit in $L^2(0,T;L^2(\Omega,\mathrm{d}\nu_\varepsilon^h))$ of the optimal control $\widehat{w}_\varepsilon^0$ for the $\widehat{\mathbb{CP}}_\varepsilon$-problem (see (10.38)–(10.40)). On the other hand, it is clear that an optimal solution (a^0, v^0, w^0, z^0) to the problem (10.94)–(10.96) is always such that $w^0 = 0$. So, in the sequel, we will omit the constant $\mathbf{C}(\mathbf{w})$ in (10.105). Moreover, in view of the variational properties of the problem (10.94) the Neumann optimal control $\widehat{w}_\varepsilon^0$ for $\widehat{\mathbb{CP}}_\varepsilon$ converges to 0 weakly in $L^2(0,T;L^2(\Omega,\mathrm{d}\nu_\varepsilon^h))$.

Thus, since the constrained minimization problem (10.94) admits interpretation in the form of an OCP, it follows that for the C-extended OCP (10.38)–(10.40) (the so-called $\widehat{\mathbb{CP}}_\varepsilon$-problem) there exists a unique limit with respect to w-convergence as $\varepsilon \to 0$ and it can be represented in the form

$$\left.\begin{aligned} \partial_t z - \operatorname{div}\left(A^{hom}\nabla z\right) + 2d\xi^* z &= a + 2\xi^* g \quad \text{in } (0,T)\times\Omega, \\ \partial_{\nu_{A^{hom}}} z &= 0 \quad \text{on } (0,T)\times\partial\Omega\setminus\Gamma_0, \\ z &= v \quad \text{on } (0,T)\times\Gamma_0, \\ z(0,x) &= y_0 \quad \text{for a.a. } x\in\Omega, \end{aligned}\right\} \tag{10.106}$$

$$\begin{gathered} a \in L^2((0,T)\times\Omega), \quad v \in L^2(0,T;H^1(\Gamma_0)), \\ \|v\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0, \end{gathered} \tag{10.107}$$

$$\begin{aligned} J_0(a,v,z) = 2\int_\Omega (z(T,x) - y^T)^2 \, \mathrm{d}x \\ + 2\int_0^T \int_\Omega (z - y^*)^6 \, \mathrm{d}x \, \mathrm{d}t + \kappa^* \int_0^T \int_{\Gamma_0} v^2 \, \mathrm{d}x_1 \, \mathrm{d}t \\ + 2\gamma^* \int_0^T \int_\Omega (a - z^3 - f)^2 \, \mathrm{d}x \, \mathrm{d}t \longrightarrow \inf. \end{aligned} \tag{10.108}$$

This OCP will be called the $\mathbb{CP}^{hom}$-problem. In order to examine the variational properties for the $\mathbb{CP}^{hom}$-problem, we apply Theorem 10.27 and make use of the following result (for the proof see [172]).

Proposition 10.32. *The limit OCP (10.106)–(10.108) admits at least one solution.*

Then, combining the results of Theorem 10.27, Proposition 10.32, and Remark 10.31, we can give the conclusion concerning the variational properties of the $\mathbb{CP}^{hom}$-problem.

Theorem 10.33. *Let* $\left\{(a_\varepsilon^0, v_\varepsilon^0, w_\varepsilon^0, z_\varepsilon^0) \in \widehat{\Sigma}_\varepsilon\right\}_{\varepsilon>0}$ *be the optimal solutions of the C-extended problem $\widehat{\mathbb{CP}}_\varepsilon$. Then a subsequence*

$$\left\{(a^0_{\varepsilon_k}, v^0_{\varepsilon_k}, w^0_{\varepsilon_k}, z^0_{\varepsilon_k})\right\}_{k\in\mathbb{N}} \quad of \quad \left\{(a^0_\varepsilon, v^0_\varepsilon, w^0_\varepsilon, z^0_\varepsilon) \in \widehat{\Sigma}_\varepsilon\right\}_{\varepsilon>0}$$

and a triplet

$$(a^0, v^0, z^0) \in \mathbb{Y} = L^2((0,T)\times\Omega) \times L^2(0,T; H^1(\Gamma_0)) \times L^2(0,T; H^1(\Omega))$$

can be found such that

$$\begin{aligned}\lim_{\varepsilon\to 0} \widehat{J}_\varepsilon(a^0_\varepsilon, v^0_\varepsilon, w^0_\varepsilon, z^0_\varepsilon) &= \lim_{\varepsilon\to 0} \inf_{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon)\in\widehat{\Sigma}_\varepsilon} \widehat{J}_\varepsilon(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \\ &= \inf_{(a,\,v,z)\in\Sigma_0} J_0(a,\,v,z) = J_0(a^0, v^0, z^0), \end{aligned} \tag{10.109}$$

$$(a^0_{\varepsilon_k}, v^0_{\varepsilon_k}, w^0_{\varepsilon_k}, z^0_{\varepsilon_k}) \xrightarrow{w} (a^0, v^0, 0, z^0) \quad \textit{in the variable space} \quad \mathbb{Z}_\varepsilon. \tag{10.110}$$

10.8 Recovery of the limiting singular optimal control problem $\mathbb{P}^{hom}$

Our intention in this section is to show that the homogenized OCP with respect to the original one exists, preserves the fine variational properties, and can be recovered in an explicit form. For this, we will follow the diagram (10.28) and essentially use Lemmas 10.6–10.8 and Theorem 10.7.

Let us begin by considering a couple of new OCPs. For the first of them, we formulate the $\widehat{\mathbb{P}}^{hom}$-problem as follows:

$$\begin{aligned}\widehat{I}_0(q,u,y) &= 2\int_\Omega (y(T,x) - y^T)^2\,\mathrm{d}x \\ &\quad + 2\int_0^T\int_\Omega (y-y^*)^6\,\mathrm{d}x\,\mathrm{d}t + \kappa^*\int_0^T\int_{\Gamma_0} u^2\,\mathrm{d}x_1\,\mathrm{d}t \\ &\quad + 2\gamma^*\int_0^T\int_\Omega (q-f)^2\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \inf, \end{aligned} \tag{10.111}$$

$$\left.\begin{aligned} \partial_t y - \operatorname{div}\left(A^{hom}\nabla y\right) + 2d\xi^* y - y^3 &= q + 2\xi^* g \quad \text{in} \quad (0,T)\times\Omega, \\ \partial_{\nu_{A^{hom}}} y &= 0 \quad \text{on} \quad (0,T)\times\partial\Omega\setminus\Gamma_0, \\ y &= u \quad \text{on} \quad (0,T)\times\Gamma_0, \\ y(0,x) &= y_0 \quad \text{for a.a.} \quad x\in\Omega, \end{aligned}\right\} \tag{10.112}$$

$$\begin{aligned} q \in L^2((0,T)\times\Omega), \quad u \in L^2(0,T; H^1(\Gamma_0)), \\ \|u\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0. \end{aligned} \tag{10.113}$$

The second problem is called the $\mathbb{P}^{hom}$-problem and has the following description:

$$I_0(u,y) = 2\int_\Omega (y(T,x) - y^T)^2\,dx + 2\int_0^T\int_\Omega (y - y^*)^6\,dx\,dt + \kappa^*\int_0^T\int_{\Gamma_0} u^2\,dx_1\,dt \longrightarrow \inf, \tag{10.114}$$

$$\left.\begin{aligned} \partial_t y - \operatorname{div}\left(A^{hom}\nabla y\right) + 2d\xi^* y - y^3 &= f + 2\xi^* g \ \text{ in } \ (0,T)\times\Omega,\\ \partial_{\nu_{A^{hom}}} y &= 0 \ \text{ on } \ (0,T)\times\partial\Omega\setminus\Gamma_0,\\ y &= u \ \text{ on } \ (0,T)\times\Gamma_0,\\ y(0,x) &= y_0 \ \text{ for a.a. } \ x\in\Omega, \end{aligned}\right\} \tag{10.115}$$

$$u \in L^2(0,T;H^1(\Gamma_0)), \quad \|u\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0. \tag{10.116}$$

Here, the functions $y^T \in L^2(\Omega)$, $y^* \in L^6(0,T;L^6(\Omega))$, $y_0 \in H^1(\Omega)$, and $f \in L^2(0,T;L^2(\Omega))$ and the values κ^*, ξ^*, and γ^* are as in Theorems 10.28 and 10.30.

It is clear that $\widehat{\mathbb{P}}^{hom}$ can be interpreted as a virtual extension of the OCP $\mathbb{P}^{hom}$ with the virtual control function $q \in L^2(0,T;L^2(\Omega))$. On the other hand, $\mathbb{CP}^{hom}$ can be viewed as the C-extension for $\widehat{\mathbb{P}}^{hom}$ (see Lemma 10.6). However, the validity of these conclusions is essentially based on the regularity property of $\widehat{\mathbb{P}}^{hom}$ and $\mathbb{P}^{hom}$, namely the sets of admissible solutions for these problems have to be nonempty. In spite of the fact that it is still a challenging open problem in general, we have the following result.

Lemma 10.34. *Assume that conditions (10.88)–(10.93) are fulfilled. Assume also that the original OCP (10.5)–(10.11) is uniformly regular in the sense of Lemma 10.8. Then there are functions $y \in L^2(0,T;H^1(\Omega))$ and $u \in L^2(0,T;H^1(\Gamma_0))$ such that*

$$(f,u,y) \in \widehat{\Xi}_{hom}, \quad (u,y) \in \Xi_{hom}, \tag{10.117}$$

where $\widehat{\Xi}_{hom}$ and Ξ_{hom} denote the sets of admissible solutions for the $\widehat{\mathbb{P}}^{hom}$ and $\mathbb{P}^{hom}$-problems, respectively.

Proof. Since the original problem $\mathbb{P}_\varepsilon$ is uniformly regular, it follows that this problem is solvable for every $\varepsilon > 0$ (see Theorem 10.3). Let $\left\{(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon\right\}$ be a sequence of optimal solutions to the problem (10.5)–(10.11). Then, by Lemma 10.8, we have

$$(f_\varepsilon + (y_\varepsilon^0)^3, u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Sigma_\varepsilon \quad \text{ for every } \ \varepsilon > 0,$$

and, moreover, these quaternaries $(f_\varepsilon + (y_\varepsilon^0)^3, u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ are optimal solutions for the corresponding C-extended problems $\mathbb{CP}_\varepsilon$. Hence, due to Theorem 10.27, we conclude that the sequence of prototypes

$$\left\{(\widehat{f}_\varepsilon + (\widehat{y}_\varepsilon^0)^3, \widehat{u}_\varepsilon^0, \widehat{p}_\varepsilon^0, \widehat{y}_\varepsilon^0) \in \widehat{\Sigma}_\varepsilon\right\}_{\varepsilon>0}$$

is uniformly bounded in $\mathbb{Z}_\varepsilon$, and therefore, is relatively compact with respect to the w-convergence (see Theorem 10.25). Let $(q, u, 0, y) \in \mathbb{Y}$ be some w-cluster "quadruple" of it (see Theorem 10.33). Making use of Definition 10.24, the initial assumptions (10.88), and Theorem 10.25, we get

$$\begin{aligned}
\widehat{f}_{\varepsilon_k} + (\widehat{y}^0_{\varepsilon_k})^3 &\rightharpoonup q \quad \text{in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta^h_\varepsilon)),\\
\widehat{f}_{\varepsilon} &\to f \quad \text{in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta^h_\varepsilon)),\\
\widehat{y}^0_{\varepsilon_k} &\to y \quad \text{in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta^h_\varepsilon)),\\
\big(\widehat{y}^0_{\varepsilon_k}\big)^3 &\to y^3 \quad \text{in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta^h_\varepsilon)).
\end{aligned}$$

Thus, $(q, u, 0, y) = (f + y^3, u, 0, y)$, and due to Theorem 10.33, we obtain

$$\begin{aligned}
(f + y^3, u, 0, y) \in \Sigma_0 \quad &\Longrightarrow \quad (f, u, y) \in \widehat{\Xi}_{hom}\\
&\Longrightarrow \quad (u, y) \in \Xi_{hom}. \qquad (10.118)
\end{aligned}$$

This concludes the proof. □

Taking this assertion into account and using the well-known results of optimal control theory for distributed singular systems (see Lions [172]), we can give the following conclusion.

Corollary 10.35. *Under the assumptions of Lemma 10.34, there exist solutions to the OCPs* $\widehat{\mathbb{P}}^{hom}$ *and* $\mathbb{P}^{hom}$.

We are now in a position to establish the main result of this section concerning the variational properties of the $\mathbb{P}^{hom}$-problem.

Theorem 10.36. *Assume that the assumptions of Lemma 10.34 hold true. Let* $(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon$ *be an optimal solution to the problem (10.5)–(10.11) for every* $\varepsilon > 0$. *Then there is a sequence of indices* $\{\varepsilon_k\}$ *converging to* 0 *as* $k \to \infty$, *and functions* $y^0 \in L^2(0,T;H^1(\Omega))$ *and* $u^0 \in L^2(0,T;H^1(\Gamma_0))$ *such that*

$$\widehat{u}^0_{\varepsilon_k} \rightharpoonup u^0 \quad \textit{weakly in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\mu^h_{\varepsilon_k})), \qquad (10.119)$$

$$\widehat{p}^0_{\varepsilon_k} \rightharpoonup 0 \quad \textit{weakly in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\nu^h_{\varepsilon_k})); \qquad (10.120)$$

$$\widehat{y}^0_{\varepsilon_k} \longrightarrow y^0 \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta^h_{\varepsilon_k})), \qquad (10.121)$$

$$\big(\widehat{y}^0_{\varepsilon_k}\big)^3 \longrightarrow \big(y^0\big)^3 \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta^h_{\varepsilon_k})), \qquad (10.122)$$

$$\big(\nabla\widehat{y}^0_{\varepsilon_k} - \nabla y^0\big) \overset{2}{\rightharpoonup} r \in L^2(0,T;L^2(\Omega,V_{pot})), \qquad (10.123)$$

$$\lim_{\varepsilon\to 0} I_\varepsilon(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) = I_0(u^0, y^0), \qquad (10.124)$$

where the pair (u^0, y^0) *is an optimal solution to the nonlinear problem (10.114)–(10.116).*

Remark 10.37. We say that a sequence of triplets $\{(u_\varepsilon, p_\varepsilon, y_\varepsilon)\}$ converges weakly to (u, p, y) in the variable space

$$L^2(0,T;H^1(\Gamma_\varepsilon^D)) \times L^2(0,T;L^2(\Gamma_\varepsilon^N)) \times L^2(0,T;H^1(\Omega_\varepsilon))$$

if $(u,p,y) \in L^2(0,T;H^1(\Gamma_0)) \times L^2(0,T;L^2(\Gamma_0)) \times L^2(0,T;H^1(\Omega))$ and conditions similar to (10.119)–(10.123) are satisfied.

Proof. Using the same arguments as in the proof of Lemma 10.34, we can conclude that the sequence of quaternaries $\{(\widehat{f}_\varepsilon + (\widehat{y}_\varepsilon^0)^3, \widehat{u}_\varepsilon^0, \widehat{p}_\varepsilon^0, \widehat{y}_\varepsilon^0) \in \widehat{\Sigma}\}$ is relatively compact with respect to the w-convergence. Moreover, its every w-cluster quadruple can be represented in the form $(f + (y^0)^3, u^0, 0, y^0)$, where the limits elements u^0 and y^0 are determined by relations (10.119)–(10.123) (see Definition 10.24). Then by Theorem 10.25, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} \widehat{J}_\varepsilon(\widehat{f}_\varepsilon + (\widehat{y}_\varepsilon^0)^3, \widehat{u}_\varepsilon^0, \widehat{p}_\varepsilon^0, \widehat{y}_\varepsilon^0) &= \lim_{\varepsilon\to 0} \inf_{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon)\in \widehat{\Sigma}_\varepsilon} \widehat{J}_\varepsilon(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \\ &= \inf_{(a,v,z)\in\Sigma_0} J_0(a, v, z) \\ &= J_0(f + y^3, u, y). \end{aligned} \tag{10.125}$$

However, in view of the relations (10.9), (10.22), and (10.39), we immediately obtain

$$\widehat{J}_\varepsilon(\widehat{f}_\varepsilon + (\widehat{y}_\varepsilon^0)^3, \widehat{u}_\varepsilon^0, \widehat{p}_\varepsilon^0, \widehat{y}_\varepsilon^0) = J_\varepsilon(f_\varepsilon + (y_\varepsilon^0)^3, u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) = I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0),$$
$$J_0(f - y^3, u, y) = I_0(u, y).$$

Thus, summing up the above results, we conclude that the variational property (10.124) holds true for $(u, 0, y)$, and, hence, $(u, y) = (u^0, y^0)$ is an optimal solution to the limit problem (10.114)–(10.116). □

It is clear now that the problem (10.114)–(10.116) can be interpreted as a "limit" OCP for the original OCP (10.5)–(10.11) as ε tends to 0. In order to formalize this notion, we give the obvious conclusion, which immediately follows from Theorems 10.28, 10.30, 10.7, and 10.36.

Theorem 10.38. *Assume that the OCP (10.5)–(10.11) is uniformly regular (see Lemma 10.8) and assumptions (10.88)–(10.93) hold true. Then the OCP (10.114)–(10.116) is the variational limit of the original one as $\varepsilon \to 0$ in the following sense:*

(i) If a sequence of triplets

$$\{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in L^2(0,T;H^1(\Gamma_\varepsilon^D)) \times L^2(0,T;L^2(\Gamma_\varepsilon^N)) \times L^2(0,T;H^1(\Omega_\varepsilon))\}_{\varepsilon>0}$$

is weakly convergent to $(u, 0, y)$ in the space

$$L^2(0,T;H^1(\Gamma_\varepsilon^D)) \times L^2(0,T;L^2(\Gamma_\varepsilon^N)) \times L^2(0,T;H^1(\Omega_\varepsilon))$$

(see Remark 10.37) and there exists a subsequence of indices $\{\varepsilon_k\}$ *such that* $\varepsilon_k \to 0$ *as* $k \to \infty$ *and* $(u_{\varepsilon_k}, p_{\varepsilon_k}, y_{\varepsilon_k}) \in \widehat{\Xi}_{\varepsilon_k}$ *for all* $k \in \mathbb{N}$, *then*

$$\text{the pair } (u,y) \text{ is admissible for the } \mathbb{P}^{hom}\text{-problem}, \tag{10.126}$$

$$I_0(u,y) \le \liminf_{k\to\infty} I_{\varepsilon_k}(u_{\varepsilon_k}, p_{\varepsilon_k}, y_{\varepsilon_k}). \tag{10.127}$$

(ii) For every admissible pair (u,y) *and any value* $\delta > 0$, *there exist a* δ*-realizing sequence*

$$\big\{(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon) \in L^2(0,T;H^1(\Gamma_\varepsilon^D)) \times L^2(0,T;L^2(\Gamma_\varepsilon^N)) \times L^2(0,T;H^1(\Omega_\varepsilon))\big\}_{\varepsilon>0}$$

such that

$$(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi_\varepsilon \ \forall\, \varepsilon > 0, \quad (\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon) \overset{weakly}{\longrightarrow} (\widehat{u}, \widehat{p}, \widehat{y}), \tag{10.128}$$

$$\|(u,0,y) - (\widehat{u}, \widehat{p}, \widehat{y})\| \le \delta, \tag{10.129}$$

$$I_0(u,0,y) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon) - \delta. \tag{10.130}$$

10.9 On suboptimal controls for $\mathbb{P}_\varepsilon$-problems

In this section, we will derive the optimality conditions for the limit C-extended problem (10.106)–(10.108) from which an optimal pair for the $\mathbb{P}^{hom}$-problem may be determined. For this, we use the Lagrange multiplier principle. We obtain the weak form of optimal system equations that an optimal solution (u^0, y^0) and Lagrange multipliers must satisfy. This optimality system can serve as a basis for the construction of suboptimal solutions to the original problem on the periodic structure Ω_ε.

We are now in a position to apply the Lagrange principle to the OCP (10.106)–(10.108).

Theorem 10.39. *If a triplet*

$$(a^0, v^0, z^0) \in L^2(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Gamma_0)) \times L^2(0,T;H^1(\Omega))$$

is an optimal solution to the problem (10.106)–(10.108), then there exists a function

$$\Psi \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \tag{10.131}$$

such that the quadruple (a^0, v^0, z^0, Ψ) *satisfies the following optimality system:*

$$\left.\begin{aligned} \partial_t z^0 - \operatorname{div}\left(A^{hom}\nabla z^0\right) + 2d\xi^* z^0 &= a^0 + 2\xi^* g \quad \text{in } (0,T)\times\Omega,\\ \partial_{\nu_{A^{hom}}} z^0 &= 0 \quad \text{on } (0,T)\times\partial\Omega\setminus\Gamma_0,\\ z^0 &= v^0 \quad \text{on } (0,T)\times\Gamma_0,\\ z^0(0,x) &= y_0 \quad \text{for a.a. } x\in\Omega, \end{aligned}\right\} \tag{10.132}$$

$$\left.\begin{aligned} -\partial_t\Psi - \operatorname{div}\left(\left[A^{hom}\right]^t\nabla\Psi\right) + 2d\xi^*\Psi &= 12(z^0-y^*)^5\\ -12\gamma^*\left(a^0-\left(z^0\right)^3-f\right)\left(z^0\right)^2 &\quad \text{in } (0,T)\times\Omega,\\ \Psi &= 0 \quad \text{on } (0,T)\times\partial\Omega,\\ \Psi(T,x) = -4\left(z^0(T,x)-y^T\right) &\quad \text{for a.a. } x\in\Omega, \end{aligned}\right\} \tag{10.133}$$

$$a^0 = \left(z^0\right)^3 + f + \left[4\gamma^*\right]^{-1}\Psi \quad \text{a.e. in } (0,T)\times\Omega, \tag{10.134}$$

$$\left.\begin{aligned} \int_0^T\int_{\Gamma_0}\left(2\kappa^* v^0 - \partial_{\nu_{[A^{hom}]^t}}\Psi\right)(v-v^0)\,\mathrm{d}x_1\,\mathrm{d}t \ge 0\\ \forall v\in L^2(0,T;H^1(\Gamma_0)) \,:\, \|v\|_{L^2(0,T;H^1(\Gamma_0))}\le \mathbf{C}_0. \end{aligned}\right\} \tag{10.135}$$

Proof. Let (a^0,v^0,z^0) be an optimal solution to the problem (10.106)–(10.108). To apply the Lagrange multiplier principle, we set

$$\mathbb{Y} = \left\{z\in L^2(0,T;H^1(\Omega)), z'\in L^2(0,T;(H^1(\Omega))^*)\right\},$$

$\mathbb{U} = L^2(0,T;L^2(\Omega))\times L^2(0,T;H^1(\Gamma_0))$, $\mathbb{V} = L^2(0,T;(H^1(\Omega))^*)\times H^1(\Omega)$, and

$$F(a,v,z) = \left(z' - \operatorname{div}\left(A^{hom}\nabla z\right) + 2d\xi^* z - a - 2\xi^* g,\ z|_{t=0} - y_0\right).$$

Then $F'_z(a^0,v^0,z^0):\mathbb{Y}\to\mathbb{V}$ is an isomorphism. Hence, all assumptions of Theorem 9.35 are fulfilled.

We define the Lagrange function as follows:

$$\begin{aligned} \mathcal{L}(a,v,z,\Psi) &= 2\int_\Omega (z(T,x)-y^T)^2\,\mathrm{d}x + 2\int_0^T\int_\Omega (z-y^*)^6\,\mathrm{d}x\,\mathrm{d}t\\ &+ \kappa^*\int_0^T\int_{\Gamma_0} v^2\,\mathrm{d}x_1\,\mathrm{d}t + 2\gamma^*\int_0^T\int_\Omega (a-z^3-f)^2\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_0^T\int_\Omega z'\,\Psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega (A^{hom}\nabla z\cdot\nabla\Psi)\,\mathrm{d}x\,\mathrm{d}t\\ &+ 2d\,\xi^*\int_0^T\int_\Omega z\Psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega a\,\Psi\,\mathrm{d}x\,\mathrm{d}t\\ &- 2\,\xi^*\int_0^T\int_\Omega g\Psi\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for all $\Psi \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

Then, in accordance with Theorem 9.35, there exists a function $\Psi \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that relations (9.126)–(9.127) are valid. It is easy to see that in this case, relation (9.126) takes the form (10.132)–(10.133), whereas (9.127) can be written as (10.134)–(10.135). □

The next question we are going to consider in this section concerns the approximation of optimal solutions of the original problem $\mathbb{P}_\varepsilon$ for the values of the parameter ε small enough. It is clear that the numerical computation of optimal solutions to $\mathbb{P}_\varepsilon$-problems is very complicated due to the geometry of the original domain Ω_ε and the nonlinearity and large dimension of the corresponding optimality system. Hence, it is very difficult to keep an acceptable level of accuracy for the optimal solutions. Therefore, we will focus our attention on the possibility of defining approximate solutions which have to guarantee the closeness of the corresponding value of the cost functional (i.e., $I_\varepsilon(u_\varepsilon^{sub}, p_\varepsilon^{sub}, y_\varepsilon^{sub})$) to its minimal one for ε small enough. In view of this, we make use the following concept (for a comparison, see Definition 9.33).

Definition 10.40. *We say that a sequence of pairs*

$$\left\{(\widetilde{u}_\varepsilon^0, \widetilde{p}_\varepsilon^0) \in L^2(0,T;H^1(\Gamma_\varepsilon^D)) \times L^2(0,T;L^2(\Gamma_\varepsilon^N))\right\}_{\varepsilon>0}$$

is asymptotically suboptimal for the problem $\mathbb{P}_\varepsilon$ if for every $\delta > 0$, there is $\varepsilon_0 > 0$ such that

$$\left| \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) - I_\varepsilon(\widetilde{u}_\varepsilon^0, \widetilde{p}_\varepsilon^0, \widetilde{y}_\varepsilon) \right| < \delta, \quad \forall\, \varepsilon < \varepsilon_0. \tag{10.136}$$

Here by $\widetilde{y}_\varepsilon = \widetilde{y}_\varepsilon(\widetilde{u}_\varepsilon^0, \widetilde{p}_\varepsilon^0)$ we denote the corresponding solutions of the initial-boundary value problem (10.5).

Taking this concept into account, we get the following final result.

Theorem 10.41. *We suppose that the assumptions of Lemma 10.34 hold true. Let $u^0 \in L^2(0,T;H^1(\Gamma_0))$ be an optimal Dirichlet control for the $\mathbb{CP}^{hom}$-problem (10.106)–(10.108). Then the sequence of pairs*

$$\left\{(u^0|_{\Gamma_\varepsilon^D}, 0) \in L^2(0,T;H^1(\Gamma_\varepsilon^D)) \times L^2(0,T;L^2(\Gamma_\varepsilon^N))\right\}_{\varepsilon>0} \tag{10.137}$$

is asymptotically suboptimal for the original OCP $\mathbb{P}_\varepsilon$ if and only if the sequence $\left\{(u^0|_{\Gamma_\varepsilon^D}, 0)\right\}$ is a reproducing one for problem (10.5) (i.e., provided this problem is solvable on this sequence of controls).

Proof. We prove only the sufficient conditions of this assertion. Suppose that the sequence (10.137) is a reproducing one for the problem (10.5). Note that in view of recent results on the global steady-state boundary controllability of the blowing-up heat equations (see Coron and Trélat [75]), this supposition

has every reason to be true. Let us now consider the sequence of triplets $\{(u^0|_{\Gamma_\varepsilon^D}, 0, \widetilde{y}_\varepsilon)\}_{\varepsilon>0}$, where $\widetilde{y}_\varepsilon = y_\varepsilon(u^0|_{\Gamma_\varepsilon^D}, 0)$ are the corresponding solutions of the initial-boundary value problem (10.5).

It is easy to see that each of this triplet is admissible for the original control problem $\mathbb{P}_\varepsilon$. Then, by Lemma 10.8, we see that the quaternaries

$$(f_\varepsilon + (\widetilde{y}_\varepsilon)^3, u^0|_{\Gamma_\varepsilon^D}, 0, \widetilde{y}_\varepsilon)$$

are admissible for the C-extended problems $\mathbb{CP}_\varepsilon$ for every $\varepsilon > 0$. Hence, due to Theorem 10.27, we conclude that the sequence of prototypes

$$\left\{(\widehat{f}_\varepsilon + (\widehat{y}_\varepsilon)^3, \widehat{u}^{\,0}, 0, \widehat{y}_\varepsilon) \in \widehat{\Sigma}_\varepsilon\right\}_{\varepsilon>0}$$

is uniformly bounded in $\mathbb{Z}_\varepsilon$, and it means that this one is relatively compact with respect to the w-convergence (see Theorem 10.25). Here, $\widehat{u}^{\,0}$ is a prototype in $L^2(0,T;L^2(\Omega, \mathrm{d}\mu_\varepsilon^h))$ of the control function $u^0|_{\Gamma_\varepsilon^D}$.

Let $(q, u, 0, y) \in \mathbb{Y}$ be some w-cluster "quadruple" of this sequence (see Theorem 10.33). Applying Definition 10.24, the initial suppositions (10.88), Theorem 10.25, and the fact that $\widehat{u}^{\,0}|_{\Gamma_\varepsilon^D} = u^0|_{\Gamma_\varepsilon^D}$ for every $\varepsilon > 0$, we have

$$\begin{aligned}
\widehat{f}_{\varepsilon_k} + (\widehat{y}^0_{\varepsilon_k})^3 &\rightharpoonup q \ \text{ in } \ L^2(0,T;L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)),\\
\widehat{f}_\varepsilon &\to f \ \text{ in } \ L^2(0,T;L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)),\\
\widehat{y}_{\varepsilon_k} &\to y \ \text{ in } \ L^2(0,T;L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)),\\
(\widehat{y}_{\varepsilon_k})^3 &\to y^3 \ \text{ in } \ L^2(0,T;L^2(\Omega, \mathrm{d}\eta_\varepsilon^h)),\\
\widehat{u}^{\,0} &\rightharpoonup u \ \text{ in } \ L^2(0,T;L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)),\\
\widehat{u}^{\,0} &\xrightarrow{s_a} u^0.
\end{aligned}$$

Thus, $(q, u, 0, y) = (f + y^3, u^0, 0, y)$, and due to Theorem 10.33, we obtain

$$(f + y^3, u^0, 0, y) \in \Sigma_0 \quad \Longrightarrow \quad (f, u^0, y) \in \widehat{\Xi}_{hom} \quad \Longrightarrow \quad (u^0, y) \in \Xi_{hom}.$$

Hence, taking Theorem 10.30 into account, we have a more precise result: The pair $(u^0, y^0) \equiv (u^0, y) \in \Xi_{hom}$ is optimal for the limit problem $\mathbb{P}^{hom}$.

Let $\{(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be optimal solutions to the problems $\mathbb{P}_\varepsilon$. Then it is obvious that $\{(f_\varepsilon + (y_\varepsilon^0)^3, u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon\}_{\varepsilon>0}$ is the sequence of optimal solutions to $\mathbb{CP}_\varepsilon$. We now observe that

$$
\begin{aligned}
&\Big| \inf_{(u_\varepsilon,p_\varepsilon,y_\varepsilon)\in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon,p_\varepsilon,y_\varepsilon) - I_\varepsilon(u^0,0,\widetilde{y}_\varepsilon)\Big| \\
&= \big|I_\varepsilon(u^0_\varepsilon,p^0_\varepsilon,y^0_\varepsilon) - I_\varepsilon(u^0,0,\widetilde{y}_\varepsilon)\big| \\
&\le \big|I_\varepsilon(u^0_\varepsilon,p^0_\varepsilon,y^0_\varepsilon) - I_0(u^0,y^0)\big| + \big|I_0(u^0,y^0) - I_\varepsilon(u^0,0,\widetilde{y}_\varepsilon)\big| \\
&\le \big|I_\varepsilon(u^0_\varepsilon,p^0_\varepsilon,y^0_\varepsilon) - I_0(u^0,y^0)\big| \\
&+ \left|2\int_\Omega (y^0(T,x)-y^T)^2\,\mathrm{d}x - (2-h(\varepsilon))\int_\Omega (y^0(T,x)-y^T_\varepsilon)^2\,\mathrm{d}\eta^h_\varepsilon\right| \\
&+ \left|2\int_0^T\!\!\int_\Omega (y-y^*)^6\,\mathrm{d}x\mathrm{d}t - (2-h(\varepsilon))\int_0^T\!\!\int_\Omega (y^0_\varepsilon-y^*_\varepsilon)^6\,\mathrm{d}\eta^h_\varepsilon\mathrm{d}t\right| \\
&+ \left|\kappa^*\int_0^T\!\!\int_{\Gamma_0} u^2\,\mathrm{d}x_1\mathrm{d}t - \kappa(\varepsilon)h(\varepsilon)\int_0^T\!\!\int_\Omega \big(\widehat{u}^0\big)^2\,\mathrm{d}\mu^h_\varepsilon\,\mathrm{d}t\right| \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

To conclude the proof, we note that for a given $\delta > 0$ the following can always be found:

(i) $\varepsilon_1 > 0$ such that $J_1 < \delta/4$ for all $\varepsilon < \varepsilon_1$ by Theorem 10.36;
(ii) $\varepsilon_2 > 0$ such that $J_2 < \delta/4$ for all $\varepsilon < \varepsilon_2$ by Theorem 10.25;
(iii) $\varepsilon_3 > 0$ such that $J_3 < \delta/4$ for all $\varepsilon < \varepsilon_3$ by Theorem 10.25;
(iv) $\varepsilon_4 > 0$ such that $J_4 < \delta/4$ for all $\varepsilon < \varepsilon_4$ by Theorem 10.22 (see also Remark 10.23).

Thus, estimate (10.136) holds true for all $\varepsilon < \min\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}$. □

10.10 Optimal control problem for systems on thin lattice structures with blowup

This section deals with some generalizations of the blowing-up OCP considered above. First, we show that the 2D lattice structure Ω_ε can be taken in much more general form than was indicated in Fig. 9.1 – namely let us define the set Ω_ε as follows: $\Omega_\varepsilon = \Omega \cap \mathcal{F}^h_\varepsilon$, where $\mathcal{F}^h_\varepsilon = \varepsilon\mathcal{F}^h$ and $\mathcal{F}^h$ denotes a thin grid that is the h-thickening of some singular Y-periodic graph $\mathcal{F}$ in the plane. Thus, $\Omega_\varepsilon \subset \mathbb{R}^2$ can be viewed as a periodic open set with a reticulated-like structure contained in a fixed bounded open set $\Omega = (0, L_1) \times (0, L_2)$.

In contrast to (9.1)–(9.3), we define the sets Q_ε, Γ^D_ε, Γ^N_ε, S^ε_{int}, and S^ε_{ext} as follows:

$$
Q_\varepsilon = \big\{(x_1,x_2)\in\mathbb{R}^2 \;:\; x_1\in(0,L_1), x_2\in(0,\varepsilon)\big\} \subset \Omega, \tag{10.138}
$$

$$
\Gamma^D_\varepsilon \subset \partial\Omega\cap\mathcal{F}^h_\varepsilon \cap \{(x_1,x_2)\,:\,x_2=0\}, \quad \Gamma^N_\varepsilon \subset \partial\mathcal{F}^h_\varepsilon\cap Q_\varepsilon, \tag{10.139}
$$

$$
S^\varepsilon_{int} \subseteq [\Omega\setminus Q_\varepsilon]\cap\partial\mathcal{F}^h_\varepsilon, \qquad S^\varepsilon_{ext} = \partial\Omega_\varepsilon \setminus \big(\Gamma^D_\varepsilon \cup \Gamma^N_\varepsilon \cup S_{int}\big). \tag{10.140}
$$

As usual, we relate the parameters h and ε assuming that $h = h(\varepsilon) \to 0$ as $\varepsilon \to 0$. The case when $h = h(\varepsilon) \to 1$ as $\varepsilon \to 0$ is not considered in this chapter because it indicates periodically perforated domains rather than thin structures. Let us denote by $D(h)$ the area of the part of Y occupied by the graph $\mathcal{F}^h$. Let us define also the following values: $L^R(h)$ is the length of a boundary $\partial\mathcal{F}^h$ in Y where the Robin boundary conditions are given and $L^D(h)$ and $L^N(h)$ are the lengths of the parts of boundaries of the set $\mathcal{F}^h$ in $\overline{Y}$, where the Dirichlet and Neumann control zones are located, respectively.

In the domain Ω_ε, we consider the following nonlinear OCP $\mathbb{P}_\varepsilon$:

$$
\begin{aligned}
I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) = {} & \frac{1}{h}\int_{\Omega_\varepsilon} (y_\varepsilon(T,\cdot) - y_\varepsilon^T)^2 \,\mathrm{d}x \\
& + \frac{1}{h}\int_0^T \int_{\Omega_\varepsilon} (y_\varepsilon - y_\varepsilon^*)^6 \,\mathrm{d}x\,\mathrm{d}t + \kappa(\varepsilon)\int_0^T \int_{\Gamma_\varepsilon^D} u^2 \,\mathrm{d}x_1\,\mathrm{d}t \\
& + \int_0^T \int_{\Gamma_\varepsilon^N} p_\varepsilon^2 \,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t \longrightarrow \inf
\end{aligned}
\tag{10.141}
$$

subject to the constraints

$$
\left.
\begin{aligned}
\partial_t y_\varepsilon - \operatorname{div}(A(x/\varepsilon)\nabla y_\varepsilon) - y_\varepsilon^3 &= f_\varepsilon && \text{in } (0,T)\times\Omega_\varepsilon, \\
\partial_{\nu_A} y_\varepsilon &= 0 && \text{on } (0,T)\times S_{ext}^\varepsilon, \\
\partial_{\nu_A} y_\varepsilon &= \varepsilon^2(-d\, y_\varepsilon + g_\varepsilon) && \text{on } (0,T)\times S_{int}^\varepsilon, \\
y_\varepsilon &= u_\varepsilon && \text{on } (0,T)\times \Gamma_\varepsilon^D, \\
\partial_{\nu_A} y_\varepsilon &= \varepsilon p_\varepsilon && \text{on } (0,T)\times \Gamma_\varepsilon^N, \\
y_\varepsilon(0,x) &= y_{\varepsilon,0} && \text{for a.a. } x\in\Omega_\varepsilon,
\end{aligned}
\right\}
\tag{10.142}
$$

$$
p_\varepsilon \in L^2(0,T;L^2(\Gamma_\varepsilon^N)), \quad u_\varepsilon \in U_\varepsilon, \tag{10.143}
$$

$$
U_\varepsilon = \left\{ u|_{\Gamma_\varepsilon^D} \ : \ u \in L^2(0,T;\, H^1(\Gamma_0)), \quad \|u\|_{L^2(0,T;\, H^1(\Gamma_0))} \le \mathbf{C}_0 \right\}. \tag{10.144}
$$

Here, $p_\varepsilon \in L^2(0,T;L^2(\Gamma_\varepsilon^N))$ and $u_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon^D))$ are the boundary controls, $g_\varepsilon \in L^2(0,T;L^2(S_{int}^\varepsilon))$, $f_\varepsilon \in L^2(0,T;L^2(\Omega))$, $y_{\varepsilon,0} \in H^1(\Omega)$, $y_\varepsilon^* \in L^6((0,T)\times\Omega_\varepsilon)$, and $y_\varepsilon^T \in L^2(\Omega)$ are given functions, and $\kappa(\varepsilon)$ is a given value. As for the matrix A, we suppose that $A(x)$ is a Y-periodic measurable matrix such that

$$
\begin{gathered}
A(\cdot) \in L^\infty(Y, \mathbb{R}^{2\times 2}), \\
\alpha_0\|\xi\|^2 \le (A(x)\xi,\xi) \le \alpha_0^{-1}\|\xi\|^2 \quad \text{for a.a. } x\in Y
\end{gathered}
\tag{10.145}
$$

for some fixed constant $\alpha_0 > 0$ and every $\xi \in \mathbb{R}^2$.

Following the approach given above, our main intention is to construct the C-extended problem $\widehat{\mathbb{CP}}_\varepsilon$ to (10.141)–(10.144) and then study its asymptotic behavior as ε tends to 0. For this, we make relations (9.6)–(9.8), (10.30)–(10.34), and (9.17) more precise. Having introduced the singular scaling measures μ_ε^h, ν_ε^h, λ_ε^h, and η_ε^h as we did it earlier, we have

$$\int_{\Lambda_N^h} \varphi \,\mathrm{d}\mathcal{H}^1 = L^N(h) \int_{Y_1} \varphi \,\mathrm{d}\nu^h, \quad \int_{\Lambda_D^h} \varphi \,\mathrm{d}x_1 = L^D(h) \int_{Y_2} \varphi \,\mathrm{d}\mu^h, \tag{10.146}$$

$$\int_{\Lambda_R^h} \varphi \,\mathrm{d}\mathcal{H}^1 = L^R(h) \int_{Y_3} \varphi \,\mathrm{d}\lambda^h, \quad \int_{Y \cap \mathcal{F}^h} \varphi \,\mathrm{d}x = D(h) \int_{Y} \varphi \,\mathrm{d}\eta^h. \tag{10.147}$$

Then, after the transformations were applied in Sect. 10.3, formulas (10.30)–(10.34) take the form

$$\int_0^T \int_{S_{int}^\varepsilon} z_\varepsilon \varphi\psi \,\mathrm{d}\mathcal{H}^1 \,\mathrm{d}t = \varepsilon^{-1} L^R(h(\varepsilon)) \int_0^T \int_\Omega \widehat{z}_\varepsilon \varphi\psi \,\mathrm{d}\lambda_\varepsilon^h \,\mathrm{d}t,$$

$$\int_0^T \int_{\Gamma_\varepsilon^N} w_\varepsilon \varphi\psi \,\mathrm{d}\mathcal{H}^1 \,\mathrm{d}t = L^N(h(\varepsilon)) \int_0^T \int_\Omega \widehat{w}_\varepsilon \varphi\psi \,\mathrm{d}\nu_\varepsilon^h \,\mathrm{d}t,$$

$$\int_0^T \int_{\Gamma_\varepsilon^D} v^2 \,\mathrm{d}x_1 \,\mathrm{d}t = L^D(h(\varepsilon)) \int_0^T \int_\Omega \widehat{v}^2 \,\mathrm{d}\mu_\varepsilon^h \,\mathrm{d}t,$$

$$\int_0^T \int_{\Gamma_\varepsilon^N} w_\varepsilon^2 \,\mathrm{d}\mathcal{H}^1 \,\mathrm{d}t = L^N(h(\varepsilon)) \int_0^T \int_\Omega \widehat{w}_\varepsilon^2 \,\mathrm{d}\nu_\varepsilon^h \,\mathrm{d}t,$$

$$\int_0^T \int_{\Omega_\varepsilon} z_\varepsilon \varphi\, \psi' \,\mathrm{d}x \,\mathrm{d}t = D(h(\varepsilon)) \int_0^T \int_\Omega \widehat{z}_\varepsilon \varphi\, \psi' \,\mathrm{d}\eta_\varepsilon^h \,\mathrm{d}t,$$

$$\int_0^T \int_{\Omega_\varepsilon} a_\varepsilon\, \varphi\, \psi \,\mathrm{d}x \,\mathrm{d}t = D(h(\varepsilon)) \int_0^T \int_\Omega \widehat{a}_\varepsilon\, \varphi\, \psi \,\mathrm{d}\eta_\varepsilon^h \,\mathrm{d}t,$$

It means that we can formulate the $\widehat{\mathbb{CP}}_\varepsilon$-problem now as follows:

$$\inf \left\{ \widehat{J}_\varepsilon(\widehat{a}_\varepsilon, \widehat{v}_\varepsilon, \widehat{w}_\varepsilon, \widehat{z}_\varepsilon) \;:\; (\widehat{a}_\varepsilon, \widehat{v}_\varepsilon, \widehat{w}_\varepsilon, \widehat{z}_\varepsilon) \in \widehat{\Sigma}_\varepsilon \right\},$$

where

$$\begin{aligned}
\widehat{J}_\varepsilon(\widehat{a}_\varepsilon, \widehat{v}_\varepsilon, \widehat{w}_\varepsilon, \widehat{z}_\varepsilon) &= \frac{D(h(\varepsilon))}{h(\varepsilon)} \int_\Omega (\widehat{z}_\varepsilon(T,x) - y_\varepsilon^T)^2 \,\mathrm{d}\eta_\varepsilon^h \\
&+ \frac{D(h(\varepsilon))}{h(\varepsilon)} \int_0^T \int_\Omega (\widehat{z}_\varepsilon - y_\varepsilon^*)^6 \,\mathrm{d}\eta_\varepsilon^h \,\mathrm{d}t \\
&+ \kappa(\varepsilon) L^D(h(\varepsilon)) \int_0^T \int_\Omega \widehat{v}_\varepsilon^2 \,\mathrm{d}\mu_\varepsilon^h \,\mathrm{d}t + L^N(h(\varepsilon)) \int_0^T \int_\Omega \widehat{w}_\varepsilon^2 \,\mathrm{d}\nu_\varepsilon^h \,\mathrm{d}t \\
&+ \gamma^* \frac{D(h(\varepsilon))}{h(\varepsilon)} \int_0^T \int_\Omega (\widehat{a}_\varepsilon - \widehat{z}_\varepsilon^3 - f_\varepsilon)^2 \,\mathrm{d}\eta_\varepsilon^h \,\mathrm{d}t \longrightarrow \inf \text{ on } \widehat{\Sigma}_\varepsilon,
\end{aligned} \tag{10.148}$$

$$\widehat{\Sigma}_\varepsilon = \{(\widehat{a}_\varepsilon, \widehat{v}_\varepsilon, \widehat{w}_\varepsilon, \widehat{z}_\varepsilon) :$$

$$\left|\begin{array}{c} \widehat{z}_\varepsilon = \widehat{v}_\varepsilon \quad \mu_\varepsilon^h - \text{ a.e. on } (0,T)\times\Omega, \\ \widehat{z}_\varepsilon(0,\cdot) = y_{\varepsilon,0} \quad \eta_\varepsilon^h - \text{a.e. in } \Omega; \\ \|\widehat{v}_\varepsilon\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0, \\ \displaystyle\int_0^T \int_\Omega \Big(-\widehat{z}_\varepsilon \varphi\, \psi' + (A(x/\varepsilon)\nabla \widehat{z}_\varepsilon \cdot \nabla\varphi)\psi \Big)\, \mathrm{d}\eta_\varepsilon^h \mathrm{d}t \\ + d\,\dfrac{\varepsilon\, L^R(h(\varepsilon))}{D(h(\varepsilon))} \displaystyle\int_0^T \int_\Omega \widehat{z}_\varepsilon \varphi\psi\, \mathrm{d}\lambda_\varepsilon^h\, \mathrm{d}t = \int_0^T \int_\Omega \widehat{a}_\varepsilon\, \varphi\, \psi\, \mathrm{d}\eta_\varepsilon^h\, \mathrm{d}t \\ + \dfrac{\varepsilon\, L^R(h(\varepsilon))}{D(h(\varepsilon))} \displaystyle\int_0^T \int_\Omega \widehat{g}_\varepsilon \varphi\psi\, \mathrm{d}\lambda_\varepsilon^h\, \mathrm{d}t \\ + \dfrac{\varepsilon\, L^N(h(\varepsilon))}{D(h(\varepsilon))} \displaystyle\int_0^T \int_\Omega \widehat{w}_\varepsilon \varphi\psi\, \mathrm{d}\nu_\varepsilon^h\, \mathrm{d}t, \\ \forall\, \varphi \in C_0^\infty(\mathbb{R}^2; \Gamma_\varepsilon^D),\ \forall\, \psi \in C_0^\infty(0,T) \end{array}\right\}. \tag{10.149}$$

It is clear that for any thin grid which can be generated by the h-thickening of the corresponding singular graphs $\mathcal{F}$, we have

$$\lim_{\varepsilon\to 0} D(h(\varepsilon))/h(\varepsilon) = \beta_1 > 0 \ \text{ and } \ 0 \le \beta_N, \beta_R < +\infty,\ \beta_D = 0, \tag{10.150}$$

where

$$\begin{gathered} \beta_N = \lim_{\varepsilon\to 0} L^N(h(\varepsilon)),\ \beta_D = \lim_{\varepsilon\to 0} L^D(h(\varepsilon)), \\ \text{and } \ \beta_R = \lim_{\varepsilon\to 0} L^R(h(\varepsilon)). \end{gathered} \tag{10.151}$$

Under the assumptions of Theorem 10.38, the limiting OCP can be recovered in the following explicit form:

$$\begin{aligned} I_0(u,y) = \beta_1 \int_\Omega (y(T,x) - y^T)^2\, \mathrm{d}x + \beta_1 \int_0^T \int_\Omega (y - y^*)^6\, \mathrm{d}x\, \mathrm{d}t \\ + \lim_{\varepsilon\to 0} \Big[\kappa(\varepsilon) L^D(h(\varepsilon))\Big] \int_0^T \int_{\Gamma_0} u^2\, \mathrm{d}x_1\, \mathrm{d}t \longrightarrow \inf, \end{aligned} \tag{10.152}$$

subject to the constraints

$$\left|\begin{array}{c} \partial_t y - \operatorname{div}\big(A^{hom}\nabla y\big) + \xi^*\, d\, \dfrac{\beta_R}{\beta_1}\, y - y^3 \\ = f + \xi^* \dfrac{\beta_R}{\beta_1}\, g \ \text{ in } \ (0,T)\times\Omega, \\ \partial_{\nu_{A^{hom}}} y = 0 \ \text{ on } \ (0,T)\times\partial\Omega \setminus \Gamma_0, \\ y = u \ \text{ on } \ (0,T)\times\Gamma_0, \\ y(0,x) = y_0 \ \text{ a.e. } \ x \in \Omega, \end{array}\right\} \tag{10.153}$$

$$u \in L^2(0,T; H^1(\Gamma_0)),\ \|u\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C}_0. \tag{10.154}$$

Thus, the additional terms in the state equation (10.153) can appear only if the following conditions are valid: a "fattened" graph $\mathcal{F}^h$ is a thin grid in the above given sense, the value β_R is a strict positive constant, and the domain Ω_ε has a critical thickness, that is, $\xi^* = \lim_{\varepsilon\to 0} \varepsilon/h(\varepsilon) = \theta > 0$. It is interesting to note that for the domains Ω_ε which are sufficiently thick (i.e., when $\xi^* = \lim_{\varepsilon\to 0} \varepsilon/h(\varepsilon) = 0$) the limit state equation in (10.153) is similar to the original one in (10.142). However, it should be emphasized that in the case of sufficiently thin domains Ω_ε, the variational limit for the problem (10.141)–(10.144) does not exist! Nevertheless, the passage to the limit in the OCP in such domains is possible as $\varepsilon \to 0$ if only this problem instead of the Robin and Neumann conditions along the boundaries S^ε_{int} and Γ^N_ε, respectively, takes the homogeneous Neumann condition

$$\partial_{\nu_A} y_\varepsilon = 0 \quad \text{on} \quad \Gamma^N_\varepsilon \cup \Gamma^N_\varepsilon.$$

In this case, $d = 0$, $g_\varepsilon = 0$, and $p_\varepsilon = 0$ in (10.142). Then, as immediately follows from (10.153), the limiting form of such problem will be the same as in the previous case.

Our next observation deals with the structural "stability" of the limit OCP (10.152)–(10.154). Indeed, as follows from Proposition 9.23 and Theorem 10.28, the passage to the limit in $\mathbb{P}_\varepsilon$ containing boundary controls of two different types leads us to the OCP (10.152)–(10.154) with only one boundary control function: the Dirichlet control. Moreover, this conclusion holds true in spite of a "volume" of the boundary $\partial\Omega_\varepsilon$ occupied by the Dirichlet control zone. Thus, the Neumann boundary control in the limit problem can appear only if the domain Ω_ε has a critical thickness $\xi^* = \theta > 0$ and $L^D(h) = 0$ for every $h > 0$, that is, in the case when the Dirichlet boundary control is absent in the original statement. In particular, applying Theorems 10.28 and 10.30 provided the conditions $L^D(h) = 0$, (10.138)–(10.140), and (10.150) hold true, it can be easily shown that for the Neumann optimal boundary control problem

$$I_\varepsilon(p_\varepsilon, y_\varepsilon) = \frac{1}{h}\int_{\Omega_\varepsilon} (y_\varepsilon(T,\cdot) - y^T_\varepsilon)^2 \,\mathrm{d}x + \frac{1}{h}\int_0^T \int_{\Omega_\varepsilon} (y_\varepsilon - y^*_\varepsilon)^6 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Gamma^N_\varepsilon} p^2_\varepsilon \,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t \longrightarrow \inf, \tag{10.155}$$

$$\left.\begin{aligned} &\partial_t y_\varepsilon - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla y_\varepsilon\right) - y^3_\varepsilon = f_\varepsilon \quad \text{in } (0,T)\times\Omega_\varepsilon,\\ &\partial_{\nu_A} y_\varepsilon = 0 \quad \text{on } (0,T)\times\partial\Omega\setminus(S^\varepsilon_{int}\cup\Gamma^N_\varepsilon),\\ &\partial_{\nu_A} y_\varepsilon = \varepsilon^2(-d y_\varepsilon + g_\varepsilon) \quad \text{on } (0,T)\times S^\varepsilon_{int},\\ &\partial_{\nu_A} y_\varepsilon = \varepsilon p_\varepsilon \quad \text{on } (0,T)\times\Gamma^N_\varepsilon,\\ &y_\varepsilon(0,x) = y_{\varepsilon,0} \quad \text{for a.a. } x\in\Omega_\varepsilon, \end{aligned}\right\} \tag{10.156}$$

$$p_\varepsilon \in L^2(0,T;L^2(\Gamma^N_\varepsilon)), \tag{10.157}$$

there exists a limit which can be represented in the form

$$I_0(u,y) = \beta_1 \int_\Omega (y(T,x) - y^T)^2 \,\mathrm{d}x + \beta_1 \int_0^T \int_\Omega (y - y^*)^6 \,\mathrm{d}x\,\mathrm{d}t + \beta_N \int_0^T \int_{\Gamma_0} p^2 \,\mathrm{d}x_1\,\mathrm{d}t \longrightarrow \inf, \tag{10.158}$$

$$\left.\begin{aligned} \partial_t y - \operatorname{div}\left(A^{hom}\nabla y\right) + \xi^* d\,\frac{\beta_R}{\beta_1}\, y - y^3 \\ = f + \xi^* \frac{\beta_R}{\beta_1}\, g \quad \text{in} \quad (0,T)\times\Omega, \\ \partial_{\nu_{A^{hom}}} y = 0 \quad \text{on} \quad (0,T)\times\partial\Omega\setminus\Gamma_0, \\ \partial_{\nu_{A^{hom}}} y = \xi^* \frac{\beta_N}{\beta_1}\, p \quad \text{on} \quad (0,T)\times\Gamma_0, \\ y(0,x) = y_0 \quad \text{for a.a.} \quad x\in\Omega, \end{aligned}\right\} \tag{10.159}$$

$$p \in L^2(0,T;L^2(\Gamma_0)). \tag{10.160}$$

It is quite obvious now (see (10.158)–(10.160)) that for the problem (10.155)–(10.157) in a sufficiently thick domain Ω_ε (i.e., in the case when $\xi^* = 0$), we have no OCP in the limit as $\varepsilon \to 0$, but rather the initial-boundary value problem of the following structure:

$$\left.\begin{aligned} \partial_t y - \operatorname{div}\left(A^{hom}\nabla y\right) - y^3 = f \quad \text{in} \quad (0,T)\times\Omega, \\ \partial_{\nu_{A^{hom}}} y = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \\ y(0,x) = y_0 \quad \text{for a.a.} \quad x\in\Omega. \end{aligned}\right\} \tag{10.161}$$

It is interesting to note that in this case, in view of Theorem 10.36, the variational properties of the limit problem (10.161) can be formulated as follows:

Let $(p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ be an optimal pair for the Neumann boundary control problem (10.155)–(10.157). Then there is a sequence of indices $\{\varepsilon_k\}$ converging to 0 as $k\to\infty$, and functions $y^0 \in L^2(0,T;H^1(\Omega))$ and $p^0 \in L^2(0,T;L^2(\Gamma_0))$ such that

$$\widehat{p}^{\,0}_{\varepsilon_k} \longrightarrow p^0 \quad \textit{weakly in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\nu^h_{\varepsilon_k})), \tag{10.162}$$

$$\widehat{y}^{\,0}_{\varepsilon_k} \longrightarrow y^0 \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta^h_{\varepsilon_k})), \tag{10.163}$$

$$\left(\widehat{y}^{\,0}_{\varepsilon_k}\right)^3 \longrightarrow \left(y^0\right)^3 \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega,\mathrm{d}\eta^h_{\varepsilon_k})), \tag{10.164}$$

$$\left(\nabla\widehat{y}^{\,0}_{\varepsilon_k} - \nabla y^0\right) \overset{2}{\rightharpoonup} r \in L^2(0,T;L^2(\Omega,V_{pot})), \tag{10.165}$$

$$\lim_{\varepsilon\to 0} I_\varepsilon(p_\varepsilon^0, y_\varepsilon^0) = const \equiv \beta_1 \int_\Omega (y^0(T,x) - y^T)^2 \,\mathrm{d}x + \beta_1 \int_0^T \int_\Omega (y^0 - y^*)^6 \,\mathrm{d}x\,\mathrm{d}t + \beta_N \int_0^T \int_{\Gamma_0} (p^0)^2 \,\mathrm{d}x_1\,\mathrm{d}t, \tag{10.166}$$

where the function y^0 is a solution of the Neumann boundary value problem (10.161).

11

Asymptotic Analysis of Optimal Control Problems on Periodic Singular Graphs

Partial differential equations (PDEs) on graphs have frequently been considered in the literature, beginning with the works of Lumer [180] and A. Vol'pert [247]. We also mention Lagnese, Leugering, and Schmidt [164], Lagnese and Leugering [163], Pokornyi et al. [216], and von Below [248].

Applications of the theory of PDEs on multi-link structures are ubiquitous. For instance, we encounter problems on graphs, or, more generally, on networked domains, in structural control, civil engineering, infrastructures, mechatronics, biology, medicine, traffic flow, electrical and communication networks, and fluid flow through a network of pipes or canals. For more details, we refer the reader to a recent review [160] on these topics.

In this book, we will concentrate on the asymptotic analysis of optimal control problems for 1D elliptic equations on periodic graphs, as the period of the graph tends to 0. We focus on optimal control problems (OCPs) for linear elliptic equations with distributed and boundary controls. Using the variational approach, we show that the original problem on the periodic graph tends to a standard linear quadratic optimal control problem for a 2D homogenized system, and its solution can be used as a suboptimal control for the original problem.

Even though PDEs on networked domains and their homogenized substitutes are very important, only a few papers [56, 163, 165, 184, 246] deal with the homogenization problem on periodic networks. Typically, not only the process on the graph itself but also the optimization and control of processes on such networks are of great importance. On the one hand, as one goes down the scales of the periodic structures, the numerical computation of the solution of these problems is very costly due to the singular behavior for small scales and the complexity of large networks, whereas on the other hand, local computations on a regular locally 1D grid are much easier than 2D or 3D calculations. It may thus happen, as in sparse-grid computations, that an originally 2D or 3D problem is approximated by a semidiscretization on a suitable grid, whereas on the other hand, as in carbon-nanotube technology, photonic lattice devices and in infrastructural problems involving water

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_11,
© Springer Science+Business Media, LLC 2011

and gas networks, the vascular system, to mention just a few applications, a problem on a planar (or 3D) graph may be substituted by a homogenized one on a simple but higher-dimensional domain. Thus, the asymptotic analysis for OCPs on graphs appears to be a major tool in order investigate these questions. Moreover, in such an asymptotic analysis we may require both the optimal solution and the minimal value of the cost functional for the original problem to converge to the corresponding characteristics of a limit OCP, as ε tends to 0.

It should be stressed that the original problem (on the graph) and the homogenized one (in a 2D domain) live in different function spaces. Moreover, if the small parameter ε is changed, then all components of the original control problem, including the ε-periodic graph Ω_ε, the control constraint sets, the cost functional, and the set where we seek its infimum are changed as well. Let us observe also that the Lebesgue measure of the "material" included in the periodic graph Ω_ε is equal to 0 for every $\varepsilon > 0$, whereas there exists a set Ω which is filled up by this planar graph in the limit, as $\varepsilon \to 0$. In view of this, our approach is based on the description of OCPs for planar networks in terms of singular measures as proposed by Zhikov, Bouchitte, and Fragala in their recent works [26, 58, 255, 256]. Our emphasis is not on the homogenization of the underlying system of PDEs on graphs as such, but rather on the homogenization of the OCP with distributed controls and Neumann boundary controls on planar ε-periodic graphs as $\varepsilon \to 0$.

11.1 ε-Periodic graphlike structures in $\mathbb{R}^2$ and their description

In this section, we recall the main notions concerning ε-periodic graphlike structures in $\mathbb{R}^2$, the description of the geometry of periodic graph structures and their boundaries in terms of singular measures. It is important to note that we introduce two types of singular measures. One of them we use for the representation of a ε-periodic bounded graph, and the second one is a "boundary" measure for the description of boundary conditions on a graphlike domain.

Definition 11.1. *We say that the set $\square = [0,1)^2$ is the cell of periodicity for some planar graph $\mathcal{F}$ on $\mathbb{R}^2$ if $\square$ contains a "star" structure such that (see Fig. 7.6) the following hold:*

(i) All edges of this structure have a common point $M \in \operatorname{int}\square$.
(ii) Each edge is a line segment and all end points of these edges belong to the boundary of $\square$.
(iii) In the set of end points (vertices) there exist pairs $\underline{(M_i; M_k)}$ such that $x_1^{M_i} = x_1^{M_k}$ or $x_2^{M_i} = x_2^{M_k}$.

So, by analogy with Sect. 7.6, we admit the existence of isolated vertices.

As usual, we always suppose that the small parameter $\varepsilon \in E = (0, \varepsilon^0]$ varies in a strictly decreasing sequence of positive numbers which converges to 0.

Definition 11.2. *We say that $\mathcal{F}_\varepsilon$ is an ε-periodic graph on $\mathbb{R}^2$ if $\mathcal{F}_\varepsilon = \varepsilon\mathcal{F} = \{\varepsilon x : x \in \mathcal{F}\}$.*

Let $\square^S = \{(x_1, x_2) : x_1 \in [0,1),\ x_2 = 0\}$, $I^{ed} = \{I_j,\ j = 1, 2, \dots, K\}$ be the set of all edges on $\square$ and let $\mathcal{M} = \{M_i,\ i = 1, 2. \dots, L\}$ be the set of all vertices on $\square$ which belong to $\square^S$. Let Ω be an open bounded domain in $\mathbb{R}^2$ such that

$$\Omega = \{(x_1, x_2) : x_1 \in \Gamma_1, 0 < x_2 < \gamma(x_1)\}, \tag{11.1}$$

where $\Gamma_1 = (0, a)$, $\gamma \in C^1([0, a])$, and $0 < \gamma_0 = \inf_{x_1 \in [0,a]} \gamma(x_1)$. It is clear that in this case, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_2 = \partial\Omega \setminus \Gamma_1$.

Definition 11.3. *We say that Ω_ε has an ε-periodic graphlike structure (see Fig. 7.7) if $\Omega_\varepsilon = \Omega \cap \mathcal{F}_\varepsilon$.*

Following the approach of Sect. 7.6, we will describe the geometry of the set Ω_ε in terms of singular measures in $\mathbb{R}^2$. With this aim, for every segment $I_i \in I^{ed}, i = 1, 2, \dots, K$, we denote by μ_i its corresponding Lebesgue measure. Now, we define the $\square$-periodic Borel measure μ in $\mathbb{R}^2$ as

$$\mu = \sum_{i=1}^{K} g_i \cdot \mu_i, \tag{11.2}$$

where $g_1, g_2, \dots, g_K$ are non-negative weights such that $\int_\square d\mu = 1$. Thus, the support of the measure μ is the union of all edges $I_i \in I^{ed}$. Since the homothetic contraction of the plane with a factor of ε^{-1} takes the grid $\mathcal{F}$ to $\mathcal{F}_\varepsilon = \varepsilon\mathcal{F}$, we introduce a "scaling" $\varepsilon\square$-periodic measure μ_ε as

$$\mu_\varepsilon(B) = \varepsilon^2 \mu(\varepsilon^{-1} B) \text{ for every Borel set } B \subset \mathbb{R}^2. \tag{11.3}$$

As a result, $\int_{\varepsilon\square} \mathrm{d}\mu_\varepsilon = \varepsilon^2 \int_\square \mathrm{d}\mu = \varepsilon^2$. Then the measure μ_ε weakly converges to the Lebesgue measure on $\mathbb{R}^2$, that is,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \varphi \, \mathrm{d}\mu_\varepsilon = \int_{\mathbb{R}^2} \varphi \, \mathrm{d}x \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^2). \tag{11.4}$$

Let μ^S be the $\square^S$-periodic measure in $\mathbb{R}^1$ defined as

$$\mu^S = \sum_{j=1}^{L} \rho_j \delta_{M_j}, \tag{11.5}$$

where the non-negative weights ρ_j satisfy $\sum_{j=1}^{L} \rho_j = 1$ and δ_{M_j} are the Dirac measures located at the vertices M_j. It is the easy to see that μ^S is a Radon

measure, $\int_{\square^S} \mathrm{d}\mu^S = 1$. So, we may then define the corresponding scaling measure by

$$\mu_\varepsilon^S(B) = \varepsilon\mu^S(\varepsilon^{-1}B) \quad \text{for any Borel set} \quad B \subset \mathbb{R}^1. \tag{11.6}$$

Since $\int_{\varepsilon\square^S} \mathrm{d}\mu_\varepsilon^S = \varepsilon$, it follows that $\mathrm{d}\mu_\varepsilon^S$ converges weakly to $\mathrm{d}\ell$ as $\varepsilon \to 0$, where by $\mathrm{d}\ell$ we denote the linear Lebesgue measure on $\mathbb{R}^1$.

11.2 Statement of an optimal control problem on ε-periodic graphs

To formulate the OCP on singular structures such as the ε-periodic graph $\Omega_\varepsilon = \Omega \cap \varepsilon\mathcal{F}$, we introduce a certain associated Sobolev space $V(\Omega, \Gamma_2, \mathrm{d}\mu)$, where μ is the $\square$-periodic Borel non-negative measure on $\mathbb{R}^2$ that was defined in (11.2). Let us denote by $C^\infty(\Omega, \Gamma_2)$ the class of smooth functions $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi\big|_{\Gamma_2} = 0$. Here, Γ_2 is the second "part" of the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Let us define the spaces $L^2(\Omega, \mathrm{d}\mu_\varepsilon)$ and $L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S)$ in the usual way.

Definition 11.4. *We say that a function $y(x)$ belongs to the functional class $V(\Omega, \Gamma_2, \mathrm{d}\mu)$ if there exist a vector $z \in (L^2(\Omega, \mathrm{d}\mu))^2$ and a sequence $\{y_m \in C^\infty(\Omega, \Gamma_2)\}_{m\in\mathbb{N}}$ such that*

$$\lim_{m\to\infty} \int_\Omega (y_m - y)^2 \,\mathrm{d}\mu = 0, \qquad \lim_{m\to\infty} \int_\Omega \left|\nabla y_m - z\right|^2 \mathrm{d}\mu = 0. \tag{11.7}$$

In this case, we say that z is a gradient of y and denote it as ∇y (i.e., $z = \nabla y$).

Note that every function $y \in V(\Omega, \Gamma_2, \mathrm{d}\mu)$ may have many gradients $z = \nabla y$. Moreover, if we denote by $\Gamma(y)$ the set of gradients for a fixed function $y \in V(\Omega, \Gamma_2, \mathrm{d}\mu)$, then this set has the structure $\Gamma(y) = \nabla y + \Gamma(0)$ (see Sect. 6.4), where $\nabla y \in L^2(\Omega, \mathrm{d}\mu)^2$ is some fixed gradient and $\Gamma(0)$ is the set of gradients of 0, that is, by definition, $g \in \Gamma(0)$ if there exists a sequence $\{\varphi_m \in C^\infty(\overline{\Omega})\}$ such that

$$\int_\Omega \varphi_m^2 \,\mathrm{d}\mu \longrightarrow 0 \ \text{ as } \ m \to \infty, \quad \int_\Omega \left|\nabla\varphi_m - g\right|^2 \mathrm{d}\mu \longrightarrow 0 \ \text{ as } \ m \to \infty.$$

From this, it follows immediately that the set of gradients $\Gamma(y)$ is a closed subspace of $L^2(\Omega, \mathrm{d}\mu)^2$. Let us recall also that any gradient ∇y can be represented as a sum of two orthogonal terms: $\nabla y = \nabla^t y + g$, where $g \in \Gamma(0)$ and $\nabla^t y \perp \Gamma(0)$. Hence, the first term is also a gradient of y, and this gradient is minimal in the following sense:

$$\int_\Omega |\nabla^t y|^2 \,\mathrm{d}\mu = \min_{\nabla y \in \Gamma(y)} \int_\Omega |\nabla y|^2 \,\mathrm{d}\mu.$$

The crucial fact is that the space $\Gamma(0)$ admits a pointwise description; namely there is a μ-measurable subspace $T(x) \subset \mathbb{R}^2$ such that

$$\Gamma(0) = \left\{ g \in L^2(\Omega, \mathrm{d}\mu)^2 \ : \ g(x) \in T^{\perp}(x) \right\}.$$

So, the "minimal" gradient is determined by the tangential condition $\nabla y(x) \in T(x)$ μ-a.e.

The following result can be viewed as the necessary and sufficient conditions for the inclusion $y \in V(\Omega, \Gamma_2, \mathrm{d}\mu)$, where μ is defined in (11.3).

Proposition 11.5. *Let $\mathcal{F}$ be a $\square$-periodic unbounded graph on $\mathbb{R}^2$, let μ be the $\square$-periodic Borel measure in $\mathbb{R}^2$ defined by (11.3), and let y be any function of $V(\Omega, \Gamma_2, \mathrm{d}\mu)$. Then the following hold:*

(i) $y\big|_{I_i} \in H^1(I_i)$ for any edge $I_i \in \Omega \cap \mathcal{F}$, where $H^1(I_i)$ is the 1D Sobolev space on the segment I_i.
(ii) The values of $y\big|_{I_i}$ coincide at the vertices of the graph.

The validity of Proposition 11.5 immediately follows from Lemma 3 in [58].

Remark 11.6. Due to the classical Sobolev embedding theorem, we have the obvious consequence of the previous statement: If $y \in V(\Omega, \Gamma_2, \mathrm{d}\mu)$, then its restriction on the set $\Omega \cap \mathcal{F}$ is a continuous function.

Let $\{A_\varepsilon(x) \ : \ A_\varepsilon(\cdot) \in L^\infty(\Omega, \mathbb{R}^{2\times 2}; \mathrm{d}\mu_\varepsilon)\}_{\varepsilon \in \mathbb{N}}$ be a family of matrices such that

$$\alpha_0 \parallel \xi \parallel^2 \leq (A_\varepsilon(x)\xi, \xi) \leq \alpha_0^{-1} \parallel \xi \parallel^2 \qquad \text{for } \mu_\varepsilon\text{-a.e. } x \in \Omega, \tag{11.8}$$

where $\alpha_0 > 0$ is some constant which is independent of ε.

We define the OCP on the ε-periodic graphlike domain $\Omega_\varepsilon = \Omega \cap \varepsilon\mathcal{F}$ as follows: Find a "boundary" control $h_\varepsilon^0 \in L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S)$, a "distributed" control $u_\varepsilon^0 \in L^2(\Omega, \mathrm{d}\mu_\varepsilon)$, and a corresponding state $y_\varepsilon^0 \in V(\Omega, \Gamma_2, \mathrm{d}\mu_\varepsilon)$ such that the cost functional

$$I_\varepsilon(y_\varepsilon, u_\varepsilon, h_\varepsilon) = k_1 \int_\Omega (y_\varepsilon - z_d)^2 \, \mathrm{d}\mu_\varepsilon + k_2 \int_\Omega u_\varepsilon^2 \, \mathrm{d}\mu_\varepsilon + k_3 \int_{\Gamma_1} h_\varepsilon^2 \, \mathrm{d}\mu_\varepsilon^S \tag{11.9}$$

is minimized subject to the following constraints:

$$y_\varepsilon \in V(\Omega, \Gamma_2, \mathrm{d}\mu_\varepsilon), \ u_\varepsilon \in L^2(\Omega, \mathrm{d}\mu_\varepsilon), \ h_\varepsilon \in L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S), \tag{11.10}$$

$$\begin{aligned} &\int_\Omega (A_\varepsilon(x)\nabla y_\varepsilon, \nabla\varphi) \, \mathrm{d}\mu_\varepsilon + \int_\Omega \alpha \, y_\varepsilon \, \varphi \, \mathrm{d}\mu_\varepsilon \\ &\qquad = \int_\Omega u_\varepsilon \, \varphi \, \mathrm{d}\mu_\varepsilon + \int_{\Gamma_1} h_\varepsilon \, \varphi \, \mathrm{d}\mu_\varepsilon^S, \quad \forall \, \varphi \in C^\infty(\Omega, \Gamma_2) \ (\alpha > 0), \end{aligned} \tag{11.11}$$

$$|u_\varepsilon| \le c_u \quad \mu_\varepsilon\text{-a.e. in } \Omega, \qquad |h_\varepsilon| \le c_h \quad \mu_\varepsilon^S\text{-a.e. on } \Gamma_1. \tag{11.12}$$

Here, c_u and c_h are some positive constants, $z_d \in C^0(\overline{\Omega})$ is a given function, μ_ε and μ_ε^S are the "scaling" measures defined by (11.3) and (11.6), respectively, and $A_\varepsilon(x) \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ is a μ_ε-measurable symmetric matrix satisfying inequality (11.8), $k_1, k_2, k_3 > 0$ are given constants. Note that by the vector-function ∇y_ε in (11.11), we mean only that $\nabla y_\varepsilon \in \Gamma(y_\varepsilon)$ is some gradient of y_ε in the sense of Definition 11.4 and ∇y_ε satisfies, with y_ε, the integral identity (11.11).

We emphasize also that the notion "boundary" and "distributed" controls should be understood with respect to the corresponding measures and the integral identity (11.11). For example, the inclusion $h \in L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S)$ implies that this function is uniquely defined by the respective set of values at the points $K_\varepsilon = \Gamma_1 \cap (\cup_{n\in\mathbb{Z}}(\varepsilon\mathcal{M} + n\varepsilon))$. Here, $\mathcal{M} = \{M_i, i = 1, 2, \dots, L\}$ is the set of all vertices on $\square$ which belong to $\square^S$. Note that by definition of the μ^S we have $\mu_\varepsilon^S(\Gamma_1 \backslash K_\varepsilon) = 0$.

First, we show that for every $u_\varepsilon \in L^2(\Omega, \mathrm{d}\mu_\varepsilon)$, $h_\varepsilon \in L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S)$, and $\varepsilon \in E$, there exists a unique pair $(y_\varepsilon, \nabla y_\varepsilon)$ that satisfies identity (11.11).

Lemma 11.7. *Under the standing assumption with respect to the measures μ and μ^S and with respect to the matrix A_ε, there exists a unique function $y_\varepsilon \in V(\Omega, \Gamma_2, \mathrm{d}\mu_\varepsilon)$ and a unique gradient of this function $\nabla y_\varepsilon \in \Gamma(y_\varepsilon)$ that satisfy (11.11).*

Proof. Since the set $C^\infty(\Omega, \Gamma_2)$ is dense in the class $V(\Omega, \Gamma_2, \mathrm{d}\mu_\varepsilon)$, the left-hand side of (11.11) induces a new scalar product on $L^2(\Omega, \mathrm{d}\mu_\varepsilon) \times L^2(\Omega, \mathrm{d}\mu_\varepsilon)^2$, and the corresponding norm is equivalent to the usual norm in this space. Thus, the existence and uniqueness of the solution regarded as the pair $(y_\varepsilon, \nabla y_\varepsilon)$ is an easy consequence of the Lax–Milgram lemma. However, the uniqueness is twofold here: There exists a unique function y_ε in the Sobolev space $V(\Omega, \Gamma_2, \mathrm{d}\mu_\varepsilon)$ such that only one of its gradients satisfies the identity (11.11). The uniqueness and existence of such a gradient was proved in [256]. It is interesting to note that the gradient ∇y_ε in this identity is defined only by matrix A_ε alone and it is not related to the equation itself (for details, we refer to Zhikov [256]). □

Remark 11.8. It is easy to see that the solution of (11.11) satisfies the following estimate:

$$\| y_\varepsilon \|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon)} + \| \nabla y_\varepsilon \|_{(L^2(\Omega, \mathrm{d}\mu_\varepsilon))^2} \le \alpha^{-1} \left(\| u_\varepsilon \|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon)} + \| h_\varepsilon \|_{L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S)} \right) \tag{11.13}$$

for every $\varepsilon \in E, u_\varepsilon \in L^2(\Omega, \mathrm{d}\mu_\varepsilon)$, and $h_\varepsilon \in L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S)$, where the constant α is independent of ε. Indeed, if we take $\varphi = y_\varepsilon$ as a test function in (11.11) and use Young's inequality, we immediately obtain relation (11.13).

Definition 11.9. *The triplet* $(y,u,h) \in \mathbb{Z}_\varepsilon(\Omega,\Gamma_1)$ *is called admissible if* (y,u,h) *satisfies the restrictions (11.10)–(11.12). We denote by* Ξ_ε *the set of all admissible triplets for the OCP (11.9)–(11.12).*

Remark 11.10. In view of (11.12) and estimate (11.13), we see that the sequence of sets $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ is uniformly bounded in the following sense: There is a constant $C>0$ such that

$$\sup_{\varepsilon\in E}\sup_{(y_\varepsilon,u_\varepsilon,h_\varepsilon)\in\Xi_\varepsilon}\big[\|\,y_\varepsilon\,\|_{L^2(\Omega,\mathrm{d}\mu_\varepsilon)}+\|\,u_\varepsilon\,\|_{L^2(\Omega,\mathrm{d}\mu_\varepsilon)} + \|\,h_\varepsilon\,\|_{L^2(\Gamma_1,\mathrm{d}\mu_\varepsilon^S)}+\|\,\nabla y_\varepsilon\,\|_{(L^2(\Omega,\mathrm{d}\mu_\varepsilon))^2}\big]\le C. \quad (11.14)$$

Moreover, each of the sets Ξ_ε is convex and closed with respect to the weak convergence in the space $L^2(\Omega,\mathrm{d}\mu_\varepsilon)\times L^2(\Omega,\mathrm{d}\mu_\varepsilon)\times L^2(\Gamma_1,\mathrm{d}\mu_\varepsilon^S)$. Indeed, let $\{u_k\}_{k\in\mathbb{N}}\subset L^2(\Omega,\mathrm{d}\mu_\varepsilon)$ and $\{h_k\}_{k\in\mathbb{N}}\subset L^2(\Gamma_1,\mathrm{d}\mu_\varepsilon^S)$ be any bounded sequences such that

$$u_k\to u \text{ weakly in } L^2(\Omega,\mathrm{d}\mu_\varepsilon),\quad h_k\to h \text{ weakly in } L^2(\Gamma_1,\mathrm{d}\mu_\varepsilon^S). \quad (11.15)$$

We define the sequence $\{y_k\in H^1(\Omega,\mathrm{d}\mu_\varepsilon)\}$ as the corresponding solutions of the problem (11.11) with $u=u_k$ and $h=h_k$. Then by inequality (11.13), there exists a constant $C>0$ such that

$$\|\,y_k\,\|_{L^2(\Omega,\mathrm{d}\mu_\varepsilon)}\le C,\qquad \|\,\nabla y_k\,\|_{(L^2(\Omega,\mathrm{d}\mu_\varepsilon))^2}\le C,$$

where, for every $k\in\mathbb{N}$, ∇y_k is the unique gradient of y_k satisfying the integral identity (11.11) for suitable u_k and h_k.

Hence, we may assume that there is a function $y\in L^2(\Omega,\mathrm{d}\mu_\varepsilon)$ and a vector $p\in L^2(\Omega,\mathrm{d}\mu_\varepsilon)^2$ such that

$$y_k\to y \text{ weakly in } L^2(\Omega,\mathrm{d}\mu_\varepsilon),\quad \nabla y_k\to p \text{ weakly in } L^2(\Omega,\mathrm{d}\mu_\varepsilon)^2. \quad (11.16)$$

Therefore, using (11.15) and (11.16), we can pass to the limit as $k\to\infty$ in

$$\int_\Omega\left[(A_\varepsilon\nabla y_k,\nabla\varphi)+\alpha y_k\varphi\right]\,\mathrm{d}\mu_\varepsilon=\int_\Omega u_k\varphi\,\mathrm{d}\mu_\varepsilon+\int_\Omega h_k\varphi\,\mathrm{d}\mu_\varepsilon^S$$

and obtain

$$\int_\Omega\left[(A_\varepsilon p,\nabla\varphi)+\alpha y\varphi\right]\,\mathrm{d}\mu_\varepsilon=\int_\Omega u\varphi\,\mathrm{d}\mu_\varepsilon+\int_{\Gamma_1}h\varphi\,\mathrm{d}\mu_\varepsilon^S\quad\forall\,\varphi\in C^\infty(\Omega,\Gamma_2).$$

This proves that $p\in\Gamma_\varepsilon(y)$, that is, $p=\nabla y$ is some gradient of the function $y\in H^1(\Omega,\mathrm{d}\mu_\varepsilon)$. As a result, we have obtained that for every $\varepsilon\in E$, the set Ξ_ε is sequentially closed with respect to the weak convergence.

Taking this remark into account and also the fact that the norm in $L^2(\Omega,\mathrm{d}\mu_\varepsilon)\times L^2(\Omega,\mathrm{d}\mu_\varepsilon)\times L^2(\Gamma_1,\mathrm{d}\mu_\varepsilon^S)$ is sequentially weakly lower semicontinuous, we come to the following conclusion (due to Tonelli's direct method).

Lemma 11.11. *For every $\varepsilon \in E$, the OCP (11.9)–(11.12) has a unique solution, that is, there exists a unique triplet $(y_\varepsilon^0, u_\varepsilon^0, h_\varepsilon^0) \in \Xi_\varepsilon$ such that*

$$I_\varepsilon(y_\varepsilon^0, u_\varepsilon^0, h_\varepsilon^0) = \inf_{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(y_\varepsilon, u_\varepsilon, h_\varepsilon).$$

Note that the uniqueness of this solution is a consequence of the convexity property for Ξ_ε and strict convexity of the cost functional I_ε.

As was mentioned earlier, the main question of this chapter is to study the asymptotic behavior of the OCP (11.9)–(11.12) as $\varepsilon \to 0$. With this aim, we represent this problem for varying values of $\varepsilon \in E$ in the form of the sequence of corresponding constrained minimization problems

$$\left\{ \Big\langle \inf_{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(y_\varepsilon, u_\varepsilon, h_\varepsilon) \Big\rangle;\ \varepsilon \to 0 \right\}, \tag{11.17}$$

where the cost functional $I_\varepsilon : \Xi_\varepsilon \to \mathbb{R}$ and the sets of admissible triplets are defined in (11.9) and (11.10)–(11.12), respectively. Then the definition of an appropriate limiting OCP to the family (11.9)–(11.12) as $\varepsilon \to 0$ can be reduced to the analysis of the limit properties of the sequences (11.17). This will be done through the concept of variational convergence of constrained minimization problems (see Sect. 8.4).

Definition 11.12. *We say that the OCP (11.9)–(11.12) has a limiting problem as ε tends to 0, if the following hold:*

(i) For the sequence of the corresponding constrained minimization problems (11.17), there exists a variational limit as $\varepsilon \to 0$ in the sense of Definition 8.22, that is,

$$\Big\langle \inf_{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(y_\varepsilon, u_\varepsilon, h_\varepsilon) \Big\rangle \overset{as\ \ \epsilon \to 0}{\longrightarrow} \Big\langle \inf_{(y,u,h) \in \Xi_0} I_0(y, u, h) \Big\rangle. \tag{11.18}$$

(ii) The minimization problem (11.18) can be recovered in the form of some OCP.

11.3 Convergence formalism in the variable spaces associated with ε-periodic graphs

In this section, we focus our attention on the convergence formalism in variable spaces associated with boundary OCPs on ε-periodic graphs. For the definitions and main properties of the weak and two-scale weak convergence in the variable spaces $L^2(\Omega, \mathrm{d}\mu_\varepsilon)$ and $L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S)$, we refer to Sects. 6.2 and 6.3. Note that the "scaling" measure μ_ε^S is singular with respect to μ_ε and associated with a "boundary condition" on the set $\partial(\Omega \cap \mathcal{F}_\varepsilon)$. Moreover, at the heart of the two-scale convergence in $L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S)$, there is the following mean value property of periodic functions (see Theorem 6.21 for the proof).

Proposition 11.13. *Assume that η is any μ^S-measurable $\square^S$-periodic function on $\mathbb{R}^1$. Then for any $a \in C_0^\infty(\mathbb{R}^1)$, the following equality holds true:*

$$\lim_{\varepsilon\to 0}\int_{\Gamma_1} a(l)\eta(\varepsilon^{-1}l)\,\mathrm{d}\mu_\varepsilon^S = \langle\eta\rangle^S\int_{\Gamma_1} a(l)\,\mathrm{d}l,$$

where $\langle\eta\rangle^S = \int_{\square^S}\eta(\xi)\,\mathrm{d}\mu^S(\xi)$ is the mean value of η on the cell $\square^S$.

We define the sets of potential and solenoidal vectors on the cell of periodicity $\square$ following Sect. 6.4. Let $C_{per}^\infty = C_{per}^\infty(\square)$ be the space of infinitely differentiable periodic functions and let $L_{per}^2(\square,\mathrm{d}\mu)^2$ be the space of μ-measurable periodic functions $\mathbf{f} = [f_1, f_2]$ such that

$$\int_\Omega (f_1^2(x) + f_2^2(x))\,\mathrm{d}\mu < \infty.$$

We say that a vector function $\mathbf{g}$ belongs to the space V_{pot} of potential vectors if there exists a sequence $\{\varphi_m \in C_{per}^\infty\}$ such that

$$\nabla\varphi_m \to \mathbf{g} \text{ in } L_{per}^2(\square,\mathrm{d}\mu)^2, \quad \text{that is,} \quad \int_\square \left|\nabla\varphi_m - \mathbf{g}\right|^2\mathrm{d}\mu \to 0 \text{ as } m\to\infty.$$

We also say that a vector function $\mathbf{b}$ belongs to the space V_{sol} of solenoidal vectors if $\mathbf{b}$ is orthogonal to all potential vectors (i.e., $\int_\square(\mathbf{b},\mathbf{g})_{\mathbb{R}^2}\,\mathrm{d}\mu = 0$ for every $\mathbf{g}\in V_{pot}$). In view of this, we always have the decomposition

$$L^2(\square,\mathrm{d}\mu)^2 = V_{pot}\oplus V_{sol}. \tag{11.19}$$

Let us denote by $L^2(\Omega,\square)$ the space $L^2(\Omega\times\square,\ \mathrm{d}x\times\mathrm{d}\mu)$, that is, $y(x,z)\in L^2(\Omega;\square)$ if y is $\mathrm{d}x\times\mathrm{d}\mu$-measurable on $\Omega\times\square$ and

$$\int_\Omega\int_\square y^2(x,z)\,\mathrm{d}x\,\mathrm{d}\mu(z) < \infty.$$

Then from (11.19), we immediately have that

$$L^2(\Omega,\square)^2 \equiv L^2\left(\Omega, L^2(\square)^2\right) = L^2(\Omega,V_{pot})\oplus L^2(\Omega,V_{sol}). \tag{11.20}$$

Here, by $L^2(\Omega,V_{pot})$ and $L^2(\Omega,V_{sol})$ we understand the following spaces:

(i) $L^2(\Omega,V_{pot})$ is the closure in $L^2(\Omega,\square)$ of the linear span of the vectors $f(x)\nabla_z\varphi(z)$, where $f\in C_0^\infty(\Omega)$ and $\varphi\in C_{per}^\infty$.
(ii) $L^2(\Omega,V_{sol})$ is the closure in $L^2(\Omega,\square)$ of the linear span of the vectors $f(x)\mathbf{b}(z)$, where $f\in C_0^\infty(\Omega)$ and $\mathbf{b}\in V_{sol}$.

In what follows, we need the following properties of the $\square$-periodic Borel measure μ (see Sect. 6.4).

Theorem 11.14. *Let μ be the non-negative periodic Borel measure in $\mathbb{R}^2$ which is defined in (11.3). Then μ is nondegenerate and ergodic, that is,*

(i) (nondegeneratedness) For every vector $\eta \in \mathbb{R}^2$*, there exists a vector* $\mathbf{b} \in V_{pot}$ *such that* $\int_\square \mathbf{b}(z)\,\mathrm{d}\mu = \eta$*;*
(ii) (ergodicity) Every function $y \in H^1(\square, \mathrm{d}\mu)$ *with gradient* 0 *is constant* μ*-everywhere.*

Remark 11.15. The validity of this theorem can also be illustrated by the following observation. In view of (11.3), we have $\mu(\Omega \backslash \mathcal{F}) = 0$. Therefore, any functions taking the same values on the graph $\Omega \cap \mathcal{F}$ coincide as elements of $L^2(\Omega, \mathrm{d}\mu)$. Thus, every element of the space $H^1(\Omega, \mathrm{d}\mu)$ is uniquely defined by the respective element of the space $\cup H^1(I_j)$, where $\{I_j\}$ is the set of all edges on the graph $\Omega \cap \mathcal{F}$.

Let $\{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon \in E}$ be any sequence of admissible triplets. For every fixed ε, each of these triplets $(y_\varepsilon, u_\varepsilon, h_\varepsilon)$ belongs to the corresponding functional space

$$\mathbb{Z}_\varepsilon(\Omega, \Gamma_1) \equiv V(\Omega, \Gamma_2, \mathrm{d}\mu_\varepsilon) \times L^2(\Omega, \mathrm{d}\mu_\varepsilon) \times L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S)$$

depending on the small parameter ε.

Definition 11.16. *We say that the sequence of triplets* $\{(y_\varepsilon, u_\varepsilon, h_\varepsilon)\}_{\varepsilon \in E}$ *is weakly convergent in the variable space* $\mathbb{Z}_\varepsilon(\Omega, \Gamma_1)$*, or, in short, is w-convergent, if there are functions*

$$y \in H^1(\Omega),\ u \in L^2(\Omega),\ h \in L^2(\Gamma_1),\ \mathbf{p} \in L^2(\Omega, \square)^2,$$

and there exists a sequence of gradients $\{\nabla y_\varepsilon \in L^2(\Omega,\, \mathrm{d}\mu_\varepsilon)^2\}_{\varepsilon \in E}$ *such that*

$$u_\varepsilon \rightharpoonup u \quad in \quad L^2(\Omega, \mathrm{d}\mu_\varepsilon), \quad h_\varepsilon \rightharpoonup h \quad in \quad L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S), \tag{11.21}$$

$$y_\varepsilon \overset{2}{\rightharpoonup} y(x) \quad in \quad L^2(\Omega, \mathrm{d}\mu_\varepsilon), \quad \nabla y_\varepsilon \overset{2}{\rightharpoonup} \mathbf{p}(x, z) \quad in \quad L^2(\Omega, \mathrm{d}\mu_\varepsilon)^2, \tag{11.22}$$

$$\mathbf{p}(x, y) - \nabla y(x) \in L^2(\Omega, V_{pot}). \tag{11.23}$$

Remark 11.17. Note that in Definition 11.16, a two-scale limit y has to be independent of the second variable z. Moreover, as follows from (11.23), we do not conjecture that for the family of H^1-functions $\{y_\varepsilon \in V(\Omega, \Gamma_2, \mathrm{d}\mu_\varepsilon)\}$, there exists a sequence of gradients $\{\nabla y_\varepsilon\}_{\varepsilon \in E}$ such that $\nabla y_\varepsilon \to \nabla y$ μ-weakly or two-scale weakly.

As we will see later, conditions (11.21)–(11.23) will be sufficient in order to identify the limit OCP on the graph. In the following theorem, we establish sufficient conditions of the relative w-compactness of uniformly bounded sequences.

Theorem 11.18. *Let* $\{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \mathbb{Z}_\varepsilon(\Omega, \Gamma_1)\}_{\varepsilon \in E}$ *be a sequence for which the following conditions hold:*

(i) There exists a constant $C > 0$ *such that*

$$\sup_\varepsilon \left\{ \| y_\varepsilon \|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon)}, \| u_\varepsilon \|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon)}, \| h_\varepsilon \|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon^S)} \right\} \le C. \tag{11.24}$$

(ii) There exists a bounded sequence of gradients $\{\nabla y_\varepsilon \in L^2(\Omega, \mathrm{d}\mu_\varepsilon)^2\}$*, that is,*

$$\limsup_{\varepsilon\to 0} \int_\Omega |\nabla y_\varepsilon|^2 \, \mathrm{d}\mu_\varepsilon < +\infty.$$

Then the sequence $\{(y_\varepsilon, u_\varepsilon, h_\varepsilon)\}_{\varepsilon\in E}$ *is relatively compact with respect to the w-convergence.*

Proof. First, we note that in view of the properties of the weak (two-scale) convergence in the variable spaces, it may be supposed without loss of generality that

$$\left.\begin{aligned} y_\varepsilon &\overset{2}{\rightharpoonup} y(x,z) \in L^2(\Omega, \square), \\ h_\varepsilon &\rightharpoonup h(l) = \int_{\square^S} h(l,\xi)\, \mathrm{d}\mu^S(\xi) \quad \text{in} \quad L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S), \\ \nabla y_\varepsilon &\overset{2}{\rightharpoonup} \mathbf{p}\,(x,z) \in L^2(\Omega, \square), \\ u_\varepsilon &\rightharpoonup u(x) = \int_\square u(x,z)\, \mathrm{d}\mu(z) \quad \text{in} \quad L^2(\Omega, \mathrm{d}\mu_\varepsilon). \end{aligned}\right\} \tag{11.25}$$

Let $\mathbf{b} \in L^2(\square, \mathrm{d}\mu)^2$ be a vector-valued function for which there is an element $a \in L^2(\square, \mathrm{d}\mu)$ such that

$$-\int_\square a(z)\, \varphi(z)\, \mathrm{d}\mu = \int_\square (\mathbf{b}(z), \nabla\varphi(z))\, \mathrm{d}\mu, \qquad \forall\, \varphi \in C^\infty_{per}(\square). \tag{11.26}$$

Taking $\varphi \in C_0^\infty(\Omega)$ as a test function and using the equality $\nabla(\varphi y_\varepsilon) = \varphi\nabla y_\varepsilon + y_\varepsilon \nabla\varphi$, we have

$$\begin{aligned} \varepsilon \int_\Omega \left(\nabla y_\varepsilon(x)\varphi(x), \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\mu_\varepsilon &= \varepsilon \int_\Omega \left(\nabla(\varphi y_\varepsilon), \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\mu_\varepsilon \\ &\quad - \varepsilon \int_\Omega y_\varepsilon(x) \left(\nabla\varphi(x), \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\mu_\varepsilon \end{aligned} \tag{11.27}$$

and

$$-\int_\Omega a(\varepsilon^{-1}x)\varphi(x)\, \mathrm{d}\mu_\varepsilon = \varepsilon \int_\Omega (\mathbf{b}(\varepsilon^{-1}x), \nabla\varphi)\, \mathrm{d}\mu_\varepsilon. \tag{11.28}$$

Therefore, applying (11.28) to (11.27), we obtain

$$\begin{aligned} \varepsilon \int_\Omega \varphi(x) \left(\nabla y_\varepsilon(x), \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\mu_\varepsilon &= -\int_\Omega a(\varepsilon^{-1}x) y_\varepsilon(x)\varphi(x)\, \mathrm{d}\mu_\varepsilon \\ &\quad - \varepsilon \int_\Omega y_\varepsilon(x) \left(\nabla\varphi(x), \mathbf{b}(\varepsilon^{-1}x)\right)\, \mathrm{d}\mu_\varepsilon. \end{aligned} \tag{11.29}$$

Taking (11.25) and supposition (ii) into account, one gets

$$\lim_{\varepsilon\to 0}\varepsilon \parallel \nabla y_\varepsilon \parallel_{(L^2(\Omega,\mathrm{d}\mu_\varepsilon))^2}=0, \quad \lim_{\varepsilon\to 0}\varepsilon\int_\Omega \varphi(x)\left(\nabla y_\varepsilon(x),\mathbf{b}(\varepsilon^{-1}x)\right)\,\mathrm{d}\mu_\varepsilon=0,$$

and by definition of the weak two-scale convergence, we have

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega y_\varepsilon(x)\left(\nabla\varphi(x),\mathbf{b}(\varepsilon^{-1}x)\right)\,\mathrm{d}\mu_\varepsilon=0.$$

Therefore, passing to the limit in (11.29), we conclude

$$\lim_{\varepsilon\to 0}\int_\Omega a(\varepsilon^{-1}x)y_\varepsilon(x)\varphi\,\mathrm{d}\mu_\varepsilon=\int_\Omega\int_\square a(z)y(x,z)\,\mathrm{d}\mu(z)\varphi(x)\,\mathrm{d}x=0. \quad (11.30)$$

Due to the approximation lemma [257] and the ergodicity property of the measure μ (see Theorem 11.14), the set of all functions $a\in L^2(\square,\mathrm{d}\mu)$ satisfying condition (11.26), where $\mathbf{b}$ is a vector from $(L^2(\square,\mathrm{d}\mu))^2$, is dense in the subspace of functions in $L^2(\square,\mathrm{d}\mu)$ with mean value 0. Thus, from (11.30), we immediately conclude that $y(x,z)=y(x)$, that is, the weak two-scale limit $y(x,z)$ in (11.25) is independent of z.

Let us show now that condition (11.23) is satisfied. For this, we consider equality (11.27) with any vector $\mathbf{b}\in V_{sol}$ (i.e., $\mathbf{b}\perp V_{pot}$). This yields

$$\int_\Omega \varphi(x)\left(\nabla y_\varepsilon(x),\mathbf{b}(\varepsilon^{-1}x)\right)\,\mathrm{d}\mu_\varepsilon = -\int_\Omega y_\varepsilon(x)\left(\nabla\varphi,\mathbf{b}(\varepsilon^{-1}x)\right)\,\mathrm{d}\mu_\varepsilon,\quad \forall\,\varphi\in C_0^\infty(\Omega). \quad (11.31)$$

Passing in (11.31) to the limit (in the sense of two-scale convergence) and using (11.25), we obtain

$$\int_\Omega\left[\int_\square\Big(p(x,y),\mathbf{b}(z)\Big)\,\mathrm{d}\mu(z)\right]\varphi(x)\,\mathrm{d}x = -\int_\Omega\left(y(x)\int_\square\mathbf{b}(z)\,\mathrm{d}\mu(z),\nabla\varphi(x)\right)\,\mathrm{d}x. \quad (11.32)$$

Note that since $p\in L^2(\Omega,\square)$, it follows that the function

$$v(x)=\int_\square(p(x,z),\mathbf{b}(z))\,\mathrm{d}\mu(x)$$

belongs to $L^2(\Omega)$. Let us set $\Theta(x)=y(x)\int_\square\mathbf{b}(z)\,\mathrm{d}\mu(z)$. Then rewriting (11.32) in the form

$$\int_\Omega v\varphi\,\mathrm{d}x=-\int_\Omega(\Theta,\nabla\varphi)\,\mathrm{d}x$$

and integrating by parts the expression on the right-hand side, we conclude that

$$v(x) = \left(\nabla y(x), \int_{\square} \mathbf{b}(z)\, \mathrm{d}\mu(z)\right).$$

Now, we may use the nondegeneracy property of the measure μ (Theorem 11.14). As a result, for $\eta = \int_{\square} \mathbf{b}(z)\, \mathrm{d}\mu(z)$, there exists a vector $\mathbf{q} \in V_{pot}$ ($V_{pot} \subset L^2(\square)$) such that $\eta = \int_{\square} \mathbf{q}(z)\, \mathrm{d}\mu(z)$. Therefore,

$$v(x) = \left(\nabla y(x), \int_{\square} \mathbf{q}(z)\, \mathrm{d}\mu(z)\right),$$

and we can draw the following conclusion: Since $(\nabla y, \int_{\square} \mathbf{q}\ \mathrm{d}\mu) \in L^2(\Omega)$ in the sense of distributions and $\mathbf{q} \in V_{pot}$, it follows that $\nabla y \in L^2(\Omega)^2$ in the sense of distributions as well. Hence,

$$y \in L^2(\Omega), \nabla y \in L^2(\Omega)^2 \Rightarrow y \in H^1(\Omega).$$

In view of this, (11.32) can be rewritten in the form

$$\int_{\Omega} \varphi(x) \left(\int_{\square} (p(x,z), \mathbf{b}(z))\ \mathrm{d}\mu(z)\right)\ \mathrm{d}x = \int_{\Omega} \varphi(x) \left(\nabla y, \int_{\square} b(z)\, \mathrm{d}\mu(z)\right)\ \mathrm{d}x,$$

that is,

$$\int_{\Omega} \int_{\square} (\mathbf{p}(x,z) - \nabla y(x), \mathbf{b}(z))\ \mathrm{d}\mu(z)\varphi(x)\ \mathrm{d}x = 0$$

for every $\varphi \in C_0^{\infty}(\Omega)$ and $\mathbf{b} \in V_{sol}$. Since the linear span of the vector-valued functions $\varphi(x)b(z)$ is dense in $L^2(\Omega, V_{sol})$ and the orthogonal decomposition (11.20) holds, it leads to the inclusion $p(x,z) - \nabla y(x) \in L^2(\Omega, V_{pot})$. This concludes the proof. □

11.4 Variational convergence of constrained minimization problems on varying graphs

The main object in this section is the sequence of constrained minimization problems

$$\left\{\left\langle \inf_{(y,u,h)\in \Xi_{\varepsilon}} I_{\varepsilon}(y,u,h)\right\rangle,\ \ \varepsilon \to 0\right\}, \tag{11.33}$$

where $I_{\varepsilon} : \Xi_{\varepsilon} \to \mathbb{R}$ and $\Xi_{\varepsilon} \subset \mathbb{Z}_{\varepsilon}(\Omega, \Gamma_1)$, $\forall\, \varepsilon \in E$.

To begin, we show that the space $L^2(\Omega)$ possesses a weak approximation property in the sense of Definition 8.23.

Lemma 11.19. *The weak approximation property for $L^2(\Omega)$ with respect to the family of Borel measures $\{\mu_\varepsilon\}_{\varepsilon\in E}$ holds true, that is,*

$$\|u-u^*\|_{L^2(\Omega)}\le\delta \quad \text{and} \quad u_\varepsilon \rightharpoonup u^* \quad \text{in} \quad L^2(\Omega,\mathrm{d}\mu_\varepsilon) \tag{11.34}$$

(in this case, the sequence $\{u_\varepsilon\in L^2(\Omega,\mathrm{d}\mu_\varepsilon)\}_{\varepsilon\in E}$ is called δ-realizing).

Proof. Let u be any element of $L^2(\Omega)$. Since the inclusion $C_0^\infty(\mathbb{R}^2)\subset L^2(\Omega)$ is dense with respect to the strong topology for $L^2(\Omega)$, it follows that for a given value $\delta>0$, there is an element $u^*\in C_0^\infty(\mathbb{R}^2)$ such that $\|u-u^*\|_{L^2(\Omega)}\le\delta$. Let us construct the δ-realizing sequence as follows: $u_\varepsilon=u^*$ for every $\varepsilon>0$. Then due to the weak convergence of the measures $\mathrm{d}\mu_\varepsilon\rightharpoonup\mathrm{d}x$, we have

$$\lim_{\varepsilon\to0}\int_\Omega u_\varepsilon\,\varphi\psi\,\mathrm{d}\mu_\varepsilon=\int_\Omega u^*\,\varphi\psi\,\mathrm{d}x,\ \forall\,\varphi\in C_0(\Omega), \tag{11.35}$$

$$\lim_{\varepsilon\to0}\int_\Omega (u_\varepsilon)^2\,\mathrm{d}\mu_\varepsilon=\int_\Omega (u^*)^2\,\mathrm{d}x. \tag{11.36}$$

Hence, by the criterium of strong convergence in $L^2(\Omega,\mathrm{d}\mu_\varepsilon)$, we obtain $u_\varepsilon\to u^*$ in $L^2(\Gamma_1,\mathrm{d}\mu_\varepsilon^S)$. □

It is clear that the same assertion holds for the space $L^2(\Gamma_1)$, that is, this space possesses the weak approximation property with respect to the family of measures $\{\mu_\varepsilon^S\}$. Taking into account these results, we give the following notion of set convergence in the space with variable measure (for comparison, we refer to Definition 7.58).

Definition 11.20. *We say that a set $\Xi_0\subset\mathbb{Y}_0=H^0(\Omega)\times L^2(\Omega)\times L^2(\Gamma_1)$ is the sequential two-scale limit, or K-limit, of the sequence*

$$\{\Xi_\varepsilon\subset\mathbb{Z}_\varepsilon(\Omega,\Gamma_1)\equiv V(\Omega,\Gamma_2,\mathrm{d}\mu_\varepsilon)\times L^2(\Omega,\mathrm{d}\mu_\varepsilon)\times L^2(\Gamma_1,\mathrm{d}\mu_\varepsilon^S)\}_{\varepsilon\in E} \tag{11.37}$$

if the following conditions are satisfied:

(i) For every triplet $(y,u,h)\in\Xi_0$ and any value $\delta>0$, there exist a constant $\varepsilon^0\in E$ and a δ-realizing sequence $\{(y_\varepsilon,u_\varepsilon,h_\varepsilon)\}_{\varepsilon\in E}$ such that

$$(y_\varepsilon,u_\varepsilon,h_\varepsilon)\in\Xi_\varepsilon,\ \forall\,\varepsilon\le\varepsilon^0,\quad (y_\varepsilon,u_\varepsilon,h_\varepsilon)\xrightarrow{w}(\widehat{y},\widehat{u},\widehat{h}),$$
$$\|(y,u,h)-(\widehat{y},\widehat{u},\widehat{h})\|_{\mathbb{Y}_0}\le\delta.$$

(ii) If $\{\Xi_{\varepsilon_k}\}$ is a subsequence of $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ and $\{(y_k,u_k,h_k)\}$ is a sequence w-converging to (y,u,h) such that $(y_k,u_k,h_k)\in\Xi_{\varepsilon_k}$ for every $k\in\mathbb{N}$, then $(y,u,h)\in\Xi_0$.

Remark 11.21. Note also that if $\mathrm{d}\mu_\varepsilon=\mathrm{d}x$, $\delta=0$, and $\mathrm{d}\mu_\varepsilon^S=\mathrm{d}l$ in (11.37), then the notion of the K-limit set coincides with the well-known notion of the sequential topological limit in the sense of Kuratowski with respect to the product of weak topologies in $H^1(\Omega)$, $L^2(\Omega)$, and $L^2(\Gamma_1)$, respectively (see Definition 7.21).

Now, we are in the position to clarify the convergence property (11.18). Following the concept of variational convergence of constrained minimization problem in variable spaces (see Definition 8.24), we make use of the following notion.

Definition 11.22. *We say that a minimization problem*

$$\left\langle \inf_{(y,u,h)\in \Xi_0} I_0(y,u,h) \right\rangle \tag{11.38}$$

is the variational limit of the sequence (11.33) with respect to w-convergence if $\Xi_0 \subset \mathbb{Y}_0$ is a nonempty K-limit of the sets $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ and the conditions (d)–(dd) of Definition 8.24 hold true with

$$\mathbb{X}_\varepsilon = V(\Omega,\Gamma_2,\mathrm{d}\mu_\varepsilon)\times L^2(\Omega,\mathrm{d}\mu_\varepsilon)\times L^2(\Gamma_1,\mathrm{d}\mu_\varepsilon^S),$$
$$\mathbb{X} = H^0(\Omega)\times L^2(\Omega)\times L^2(\Gamma_1).$$

As a result, Theorem 8.25 (see also Corollary 8.37 of Theorem 8.35) implies the following properties of variational limit in the sense of Definition 11.22.

Theorem 11.23. *Assume that the constrained minimization problem (11.38) is the variational limit of the sequence (11.33) in the sense of Definition 11.20 and this problem has a unique solution (y^0,u^0,h^0). For every $\varepsilon\in E$, let $(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0)\in\Xi_\varepsilon$ be an optimal solution of the corresponding problem (11.9)–(11.12). If the sequence $\{(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0)\}_{\varepsilon\in E}$ is relatively w-compact, then*

$$(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0) \xrightarrow{w} (y^0,u^0,h^0), \tag{11.39}$$

$$\begin{aligned}\inf_{(y,u,h)\in\Xi_0} I_0(y,u,h) &= I_0\left(y^0,u^0,h^0\right)\\ &= \lim_{\varepsilon\to 0} I_\varepsilon(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0)\\ &= \lim_{\varepsilon\to 0}\inf_{(y_\varepsilon,u_\varepsilon,h_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(y_\varepsilon,u_\varepsilon,h_\varepsilon).\end{aligned} \tag{11.40}$$

11.5 Asymptotic analysis of optimal control problems on ε-periodic graphs

We begin with the so-called "convergence property of gradients of arbitrary solutions." This property can be viewed as a natural requirement on the homogenized matrix to the family of matrices A_ε in (11.11). Let $\{A_\varepsilon(x)\in\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)\}_{\varepsilon\in\mathbb{N}}$ be a family of square μ_ε-measurable matrices satisfying inequality (11.8) for every $\varepsilon\in E$.

Definition 11.24. *(Convergence of gradients of arbitrary solutions). We say that a matrix $A^{hom}(x)\in\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)$ is the homogenized matrix with respect to the family $\{A_\varepsilon(x)\}$ as ε tends to 0, if the following hold:*

(a) $A^{hom} \in [L^\infty(\Omega)]^{2\times 2}$, $A^{hom}(x)$ *is coercive.*
(aa) $A_\varepsilon \nabla y_\varepsilon \rightharpoonup A^{hom}\nabla y_0$ *in* $L^2(\Omega, \mathrm{d}\mu_\varepsilon)^2$ *for every sequence* $\{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \mathbb{Z}_\varepsilon(\Omega, \Gamma_1)\}_{\varepsilon\in E}$ *such that* $(y_\varepsilon, \nabla y_\varepsilon)$ *satisfy relation (11.11) and*

$$(y_\varepsilon, u_\varepsilon, h_\varepsilon) \overset{w}{\to} (y_0, u_0, h_0).$$

In order to find a liming problem to the family (11.9)–(11.12), we introduce two auxiliary sets

$$\begin{aligned}\mathrm{Graph}(\mathbb{P}_\varepsilon) = \Big\{ (y,u,h) \in \mathbb{Z}_\varepsilon(\Omega,\Gamma_1) : \int_\Omega [(A_\varepsilon(x)\nabla y, \nabla\varphi) + \alpha y\varphi]\ \mathrm{d}\mu_\varepsilon \\ = \int_\Omega u\varphi\ \mathrm{d}\mu_\varepsilon + \int_{\Gamma_1} h\varphi\ \mathrm{d}\mu_\varepsilon^S, \qquad \forall\ \varphi \in C^\infty(\Omega,\Gamma_2)\Big\}, \quad (11.41)\end{aligned}$$

$$\begin{aligned}\mathrm{Graph}(\mathbb{P}^{hom}) = \Big\{ (y,u,h) \in \mathbb{Z}_0(\Omega,\Gamma_1) \equiv H^1(\Omega,\Gamma_2)\times L^2(\Omega)\times L^2(\Gamma_1) : \\ \int_\Omega \big[(A^{hom}\nabla y, \nabla\varphi) + \alpha y\varphi\big]\ \mathrm{d}x \\ = \int_\Omega u\varphi\ \mathrm{d}x + \int_{\Gamma_1} h\varphi\ \mathrm{d}l,\ \ \forall\ C^\infty(\Omega,\Gamma_2)\Big\},\end{aligned}$$

and make use of the following lemma.

Lemma 11.25. *Assume that for the family* $\{A_\varepsilon \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)\}$ *there exists a homogenized matrix in the sense of Definition 11.24. Then the sequence of sets* $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ *is K-convergent to the set* Ξ_0*, where*

$$\begin{aligned}\Xi_0 = \Big\{ (y,u,h) \in \mathbb{Z}_0(\Omega,\Gamma_1)\ :\ |u| \le c_u\ \text{a.e. in}\ \Omega,\ |h| \le c_h\ \text{a.e. on}\ \Gamma_1, \\ \int_\Omega \big[(A^{hom}\nabla y, \nabla\varphi) + \alpha y\varphi\big]\ \mathrm{d}x \\ = \int_\Omega u\varphi\ \mathrm{d}x + \int_{\Gamma_1} h\varphi\ \mathrm{d}l,\ \ \forall\ C^\infty(\Omega,\Gamma_2)\Big\}. \quad (11.42)\end{aligned}$$

Proof. Let (y^0, u^0, h^0) be any triplet of Ξ_0 and let us fix a value $\delta > 0$. Since the space of smooth functions $C^\infty(\Omega)$ is dense in $L^2(\Omega)$, it follows that for a given value $\delta > 0$, there is an element $u^* \in C_0^\infty(\mathbb{R}^2)$ such that $\|u^0 - u^*\|_{L^2(\Omega)} \le \delta$ and $|u^*| \le c_u$ in Ω. Let us construct a δ-realizing sequence as follows: $u_\varepsilon = u^*$ for every $\varepsilon > 0$. Then due to the weak convergence of the measures $\mathrm{d}\mu_\varepsilon \rightharpoonup \mathrm{d}x$, we have

$$\int_\Omega \varphi u_\varepsilon\ \mathrm{d}\mu_\varepsilon \to \int_\Omega \varphi u^*\ \mathrm{d}x \quad \text{for every} \quad \varphi \in C_0^\infty(\mathbb{R}^2),$$

that is, $\{u_\varepsilon\}$ is a δ-realizing sequence of admissible controls for u^0. By analogy, we may choose the sequence $\{h_\varepsilon \in C^\infty(\Gamma_1)\}$ such that

$$h_\varepsilon = h^*,\ \forall\, \varepsilon, \quad |h_\varepsilon| \le c_h\ \mu^S\text{-a.e. on } \Gamma_1,$$
$$h_\varepsilon \in L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon), \quad h_\varepsilon \rightharpoonup h^* \text{ in } L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S),$$

where $\|h^0 - h^*\|_{L^2(\Gamma_1)} \le \delta$ and $|h^*| \le c_h$ on Γ_1.

Let $\{y_\varepsilon \in V(\Omega, \Gamma_2, \mathrm{d}\mu_\varepsilon)\}$ be the sequence of solution of the boundary value problem (11.10)–(11.11) with the corresponding control functions $u = u_\varepsilon$ and $h = h_\varepsilon$. Using estimate (11.13), we find that

$$\| y_\varepsilon \|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon)} \le 2\alpha^{-1}C\ ,\ \| \nabla y_\varepsilon \|_{(L^2(\Omega, \mathrm{d}\mu_\varepsilon))^2} \le 2\alpha^{-1}C.$$

Then, thanks to Remark 11.8 and Theorem 11.18, we may suppose that the sequence of the triplets $\{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in E}$ is w-convergent. Let (y^*, u^*, h^*) be its w-limit. Our aim is to prove that $(y^*, u^*, h^*) \in \mathrm{Graph}(\mathbb{P}^{hom})$. In view of our assumptions and Definition 11.24, we have

$$u_\varepsilon \rightharpoonup u^* \text{ in } L^2(\Omega, \mathrm{d}\mu_\varepsilon), \quad h_\varepsilon \rightharpoonup h^* \text{ in } L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S),$$
$$y_\varepsilon \overset{2}{\rightharpoonup} y^*(x), \quad y_\varepsilon \rightharpoonup y^* \text{ in } L^2(\Omega, \mathrm{d}\mu_\varepsilon),$$
$$A_\varepsilon(\cdot)\nabla y_\varepsilon \rightharpoonup A^{hom}\nabla y^* \text{ in } L^2(\Omega, \mathrm{d}\mu_\varepsilon)^2.$$

Therefore, passing to the limit as $\varepsilon \to 0$ in the integral identity

$$\int_\Omega \left[(A_\varepsilon(x)\nabla y_\varepsilon, \nabla\varphi) + \alpha y_\varepsilon\varphi\right] \mathrm{d}\mu_\varepsilon = \int_\Omega u_\varepsilon\varphi\, \mathrm{d}\mu_\varepsilon + \int_{\Gamma_1} h_\varepsilon\varphi\, \mathrm{d}\mu_\varepsilon^S\ ,\ \forall\ \varphi \in C^\infty(\Omega, \Gamma_2),$$

we obtain

$$\int_\Omega \left[\left(A^{hom}\nabla y^*, \nabla\varphi\right) + \alpha y^*\varphi\right] \mathrm{d}x = \int_\Omega u^*\varphi\, \mathrm{d}x + \int_{\Gamma_1} h^*\varphi\, \mathrm{d}l, \quad \forall\, \varphi \in C^\infty(\Omega, \Gamma_2). \quad (11.43)$$

Hence, we have $(y^*, u^*, h^*) \in \mathrm{Graph}(\mathbb{P}^{hom})$. In view of the coerciveness property of the homogenized matrix A^{hom}, we have the standard a priori estimate

$$\|y^0 - y^*\|_{H^1(\Omega)} \le C\left[\|u^0 - u^*\|_{L^2(\Omega)} + \|h^0 - h^*\|_{L^2(\Gamma_1)}\right] \le 2\delta C.$$

Thus,

$$\|(y^0, u^0, h^0) - (y^*, u^*, h^*)\|_{\mathbb{Y}_0} \le 2\max\{C, 1\}\delta,$$

that is, $\{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in E}$ is a δ-realizing sequence with the required properties. Therefore, the following inclusion $K - \lim(\Xi_\varepsilon) \supseteq \Xi_0$ is established.

In order to obtain the inverse inclusion, we consider a w-convergent sequence $\{(y_k, u_k, h_k)\}_{k\in\mathbb{N}}$ with the following properties:

(a) $(y_k, u_k, h_k) \in \Xi_{\varepsilon_k}$ for every $k \in \mathbb{N}$, where $\{\varepsilon_k\}$ is some subsequence of indices converging to 0 as k tends to ∞.
(aa) $(y_k, u_k, h_k) \to (y^*, u^*, h^*)$ in the w-sense.

Then, proceeding as in the previous part of this lemma, we can show that

$$\nabla y_k \overset{2}{\rightharpoonup} \nabla y^* + \mathbf{v}(x,z), \quad \mathbf{v} \in L^2(\Omega, V_{pot}),$$
$$A_{\varepsilon_k}(\cdot)\nabla y_k \to A^{hom}\nabla y^* \text{ in } L^2(\Omega, \mathrm{d}\mu_\varepsilon)^2.$$

Therefore, passing to the limit in the integral identity

$$\int_\Omega [(A_{\varepsilon_k}(x)\nabla y_k, \nabla\varphi) + \alpha y_k\varphi]\, \mathrm{d}\mu_{\varepsilon_k} = \int_\Omega u_k\varphi\, \mathrm{d}\mu_{\varepsilon_k} + \int_{\Gamma_1} h_k\varphi\, \mathrm{d}\mu^S_{\varepsilon_k}, \quad \forall\, \varphi \in C^\infty(\Omega, \Gamma_2),$$

leads us to relation (11.43). Hence,

$$(y^*, u^*, h^*) \in \mathrm{Graph}(\mathbb{P}^{hom}). \tag{11.44}$$

Hence, it remains to show that the limit control functions $u^* \in L^2(\Omega)$ and $h^* \in L^2(\Gamma_1)$ satisfy the corresponding constraints

$$|u^*| \le c_u \text{ a.e. in } \Omega, \quad |h^*| \le c_h \text{ a.e. on } \Gamma_1. \tag{11.45}$$

Indeed, for every $k \in \mathbb{N}$ and every positive function $\varphi \in C^\infty(\Omega, \Gamma_2)$, we have

$$\begin{cases} \displaystyle\int_\Omega \varphi(c_u - u_k)\, \mathrm{d}\mu_{\varepsilon_k} \ge 0 \text{ and } \int_\Omega \varphi(u_k + c_u)\, \mathrm{d}\mu_{\varepsilon_k} \ge 0, \\ \displaystyle\int_{\Gamma_1} \varphi(c_h - h_k)\, \mathrm{d}\mu^S_{\varepsilon_k} \ge 0 \text{ and } \int_{\Gamma_1} \varphi(h_k + c_h)\, \mathrm{d}\mu^S_{\varepsilon_k} \ge 0. \end{cases} \tag{11.46}$$

Using the facts that $\mathrm{d}\mu_\varepsilon \rightharpoonup \mathrm{d}x$ and $\mathrm{d}\mu^S_\varepsilon \rightharpoonup \mathrm{d}l$ and passing the limit in (11.46) as $k \to \infty$, one gets

$$\left.\begin{array}{l} \displaystyle\int_\Omega \varphi(c_u - u^*)\, \mathrm{d}x \ge 0 \text{ and } \int_\Omega \varphi(u^* + c_u)\, \mathrm{d}x \ge 0, \\ \displaystyle\int_{\Gamma_1} \varphi(c_h - h^*)\, \mathrm{d}l \ge 0 \text{ and } \int_{\Gamma_1} \varphi(h^* + c_h)\, \mathrm{d}l \ge 0. \end{array}\right\} \tag{11.47}$$

Since φ in (11.47) is an arbitrary positive function, it follows that inequalities (11.45) hold true. As a result, combining (11.44) and (11.45), we deduce $(y^*, u^*, h^*) \in \Xi_0$. Thus, we have obtained the required inclusion $K - \lim \Xi_\varepsilon \subseteq \Xi_0$, which concludes the proof. □

Corollary 11.26. *Suppose that the matrix $A_\varepsilon(x)$ is defined as $A_\varepsilon(x) = A(\varepsilon^{-1}x)$, where $A(z)$ is a $\square$-periodic μ-measurable matrix satisfying condition (11.8). Then the limiting matrix A^{hom} can be defined as*

$$A^{hom}\xi = \int_{\square} A(z)\left(\xi + v^0\right)\, \mathrm{d}\mu(z), \tag{11.48}$$

where $v^0 \in V_{pot}$ is the solution of the problem

$$\min_{v \in V_{pot}} \int_{\square} (\xi + v, A(\xi + v))\, \mathrm{d}\mu = \int_{\square} (\xi + v^0, A(\xi + v^0))\, \mathrm{d}\mu. \tag{11.49}$$

Proof. Let

$$\{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon \in E}$$

be a w-convergent sequence and let (y^*, u^*, h^*) be its w-limit. Then

$$\nabla y_\varepsilon \overset{2}{\rightharpoonup} \nabla y^* + \mathbf{v}(x, z), \tag{11.50}$$

where $\mathbf{v} \in L^2(\Omega, V_{pot})$. We emphasize that the gradients in (11.50) are uniquely defined (Lemma 11.7). For this, we consider the integral identity (11.11) with the test function $\varphi(x) = \varepsilon\Psi(x)\omega(\varepsilon^{-1}x)$, where $\Psi \in C^\infty(\Omega, \Gamma_2)$ and $\omega \in C^\infty_{per}(\square)$. This yields

$$\begin{aligned}\int_\Omega (A(\varepsilon^{-1}x)\nabla y_\varepsilon, \nabla_z \omega)\Psi\, \mathrm{d}\mu_\varepsilon &\\ + \varepsilon \int_\Omega (A(\varepsilon^{-1}x)\nabla y_\varepsilon, \nabla\Psi)\omega\, \mathrm{d}\mu_\varepsilon + \varepsilon \int_\Omega \alpha y_\varepsilon \Psi\omega\, \mathrm{d}\mu_\varepsilon &\\ = \varepsilon \int_\Omega u_\varepsilon \Psi\omega\, \mathrm{d}\mu_\varepsilon + \varepsilon \int_{\Gamma_1} h_\varepsilon \Psi\omega\, \mathrm{d}\mu^S_\varepsilon.&\end{aligned}$$

It is easy to see that after passing to the limit as $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega \left(A(\varepsilon^{-1}x)\nabla y_\varepsilon, \nabla_z \omega(\varepsilon^{-1})\right)\Psi(x)\, \mathrm{d}\mu_\varepsilon = 0. \tag{11.51}$$

In view of (11.50) and the definition of the weak two-scale limit, we have

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_\Omega \Psi(x)\left[\omega(\varepsilon^{-1}x)A(\varepsilon^{-1}x)\right]\nabla y_\varepsilon(x)\, \mathrm{d}\mu_\varepsilon &\\ = \int_\Omega \int_{\square} \Psi(x)\omega(z)A(z)\left[\nabla y^*(x) + \mathbf{v}(x, z)\right]\, \mathrm{d}\mu(z)\, \mathrm{d}x.&\end{aligned}$$

Therefore,

$$\begin{aligned}&A(\varepsilon^{-1}x)\nabla y_\varepsilon(x) \overset{2}{\rightharpoonup} A(z)\left[\nabla y^*(x) + \mathbf{v}(x, z)\right],\\ &A(\varepsilon^{-1}x)\nabla y_\varepsilon(x) \rightharpoonup \int_{\square} A(z)\left[\nabla y^*(x) + \mathbf{v}(x, z)\right]\, \mathrm{d}\mu(z)\\ &\qquad\qquad \text{in } L^2(\Omega, \mathrm{d}\mu_\varepsilon)^2.\end{aligned} \tag{11.52}$$

As a result, we can rewrite (11.51) in the explicit form

$$\lim_{\varepsilon\to 0}\int_\Omega \left(A(\varepsilon^{-1}x)\nabla y_\varepsilon, \nabla_z\omega\right)\Psi\,\mathrm{d}\mu_\varepsilon = \int_\Omega\int_\square \left(A(z)\left[\nabla y^* + \mathbf{v}(x,z)\right], \nabla_z\omega\right)\,\mathrm{d}\mu(z)\Psi(x)\,\mathrm{d}x = 0.$$

Since this equality holds true for every $\Psi \in C^\infty(\Omega, \Gamma_2)$, this means that

$$\int_\square \left(A(\nabla y^* + \mathbf{v}), p\right)\,\mathrm{d}\mu = 0, \qquad \forall\, p \in C^\infty_{per}(\square)^2. \tag{11.53}$$

However, as follows from (11.49), equality (11.53) can be viewed as Euler's equation for the minimum problem (11.49). Since this problem has a unique solution $\mathbf{v}^0$, it follows that $\mathbf{v}^0$ is the unique solution of (11.53) as well. Thus, putting $\mathbf{v} = \mathbf{v}^0$ in (11.52) and using formula (11.48), we immediately deduce that

$$A\left(\varepsilon^{-1}x\right)\nabla y_\varepsilon \rightharpoonup A^{hom}\nabla y^* \qquad \text{in } L^2(\Omega, \mathrm{d}\mu_\varepsilon)^2.$$

□

Lemma 11.27. *Under the assumptions of Lemma 11.25, the limit functional $I_0 : \Xi_0 \to \mathbb{R}$ in (11.18) has the representation*

$$I_0(y,u,h) = k_1\int_\Omega (y - z_d)^2\,\mathrm{d}x + k_2\int_\Omega u^2\,\mathrm{d}x + k_3\int_{\Gamma_1} h^2\,\mathrm{d}l. \tag{11.54}$$

Proof. To prove representation (11.54), we have to verify conditions (ii)–(iii) of Definition 11.20. Let (y,u,h) be any triplet of Ξ_0 and let $\{(y_k,u_k,h_k)\}_{k\in\mathbb{N}}$ be a w-convergent sequence such that

$$(y_k,u_k,h_k) \xrightarrow{w} (y,u,h), \quad (y_k,u_k,h_k) \in \Xi_{\varepsilon_k} \text{ for every } k \in \mathbb{N}, \tag{11.55}$$

where $\{\varepsilon_k\}$ is a subsequence of E converging to 0. Then, the following inequalities hold (see Sect. 6.6):

$$\begin{gathered}\liminf_{k\to\infty}\int_\Omega u_k^2\,\mathrm{d}\mu_{\varepsilon_k} \ge \int_\Omega u^2\,\mathrm{d}x, \quad \liminf_{k\to\infty}\int_{\Gamma_1} h_k^2\,\mathrm{d}\mu^S_{\varepsilon_k} \ge \int_\Omega h^2\,\mathrm{d}l,\\ \liminf_{k\to\infty}\int_\Omega y_k^2\,\mathrm{d}\mu_{\varepsilon_k} \ge \int_\Omega y^2\,\mathrm{d}x.\end{gathered} \tag{11.56}$$

First, we note that the property of the weak compactness (11.4) for the sequence of measures $\{\mu_\varepsilon\}$ holds automatically in a wider class of test functions, when $\varphi \in C_0(\mathbb{R}^2)$. Thus,

$$\lim_{k\to\infty}\int_\Omega (z_d)^2\,\mathrm{d}\mu_\varepsilon = \int_\Omega (z_d)^2\,\mathrm{d}x,$$

where $z_d \in C(\overline{\Omega})$ by the standing assumptions.

Therefore, using this, (11.56), and the definition of the weak two-scale convergence, it follows that

$$\liminf_{k\to\infty}\int_\Omega (y_k - z_d)^2\,\mathrm{d}\mu_{\varepsilon_k} = \liminf_{k\to\infty}\int_\Omega y_k^2\,\mathrm{d}\mu_{\varepsilon_k} - 2\int_\Omega yz_d\,\mathrm{d}x + \int_\Omega (z_d)^2\,\mathrm{d}x \geq \int_\Omega (y - z_d)^2\,\mathrm{d}x. \tag{11.57}$$

Thus, summing up (11.56) and (11.57), we get

$$\liminf_{k\to\infty} I_{\varepsilon_k}(y_k, u_k, h_k) \geq I_0(y, u, h),$$

that is, property (d) of Definition 8.24 is valid.

We now verify the correctness of the reverse inequality (see (8.61)). Let (y, u, h) be any triplet of the limit set Ξ_0 and let $\delta > 0$ be a fixed small value. We construct the δ-realizing sequence $\{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in E}$ such that $u_\varepsilon = \widetilde{u}$, $h_\varepsilon = \widetilde{h}$, and $y_\varepsilon = y_\varepsilon(\widetilde{u}, \widetilde{h})$ is the corresponding solution of the boundary value problem (11.10)–(11.11) with $u = \widetilde{u}$ and $h = \widetilde{h}$. Here, $\widetilde{u}$ and $\widetilde{h}$ are the functions with the following properties:

(i) $\widetilde{u} \in C^\infty(\Omega)$, $|\widetilde{u}| \leq c_u$ on Ω, and $\|\widetilde{u} - u\|_{L^2(\Omega)} \leq \delta$.
(ii) $\widetilde{h} \in C^\infty(\Gamma_1)$, $|\widetilde{h}| \leq c_h$ on Γ_1, and $\|\widetilde{h} - h\|_{L^2(\Gamma_1)} \leq \delta$.

It is clear that, in this case, we have (see (11.35))

$$u_\varepsilon \to \widetilde{u} \ \text{ in } \ L^2(\Omega, \mathrm{d}\mu_\varepsilon), \quad h_\varepsilon \to \widetilde{h} \ \text{ in } \ L^2(\Gamma_1, \mathrm{d}\mu_\varepsilon^S).$$

Moreover, as follows from Lemma 11.25, $y_\varepsilon \overset{2}{\rightharpoonup} \widetilde{y}$, where $\widetilde{y}$ is the unique solution of the problem (11.10)–(11.12) with $u = \widetilde{u}$ and $h = \widetilde{h}$. Hence, $(y_\varepsilon, u_\varepsilon, h_\varepsilon) \overset{w}{\longrightarrow} (\widetilde{y}, \widetilde{u}, \widetilde{h})$. In view of the coerciveness property of the homogenized matrix A^{hom}, we have the standard a priori estimate

$$\|y - \widetilde{y}\|_{H^1(\Omega)} \leq C\left[\|u - \widetilde{u}\|_{L^2(\Omega)} + \|h - \widetilde{h}\|_{L^2(\Gamma_1)}\right] \leq 2\delta C. \tag{11.58}$$

Thus, $\|(y, u, h) - (\widetilde{y}, \widetilde{u}, \widetilde{h})\|_{\mathbb{Y}} \leq 2\max\{C, 1\}\delta$, that is, $\{(y_\varepsilon, u_\varepsilon, h_\varepsilon)\}_{\varepsilon\in E}$ is the δ-realizing sequence with the required properties.

By the initial construction, we have

$$\limsup_{\varepsilon\to 0}\int_\Omega \widetilde{u}_\varepsilon^2\,\mathrm{d}\mu_\varepsilon = \int_\Omega u^2\,\mathrm{d}x, \qquad \limsup_{\varepsilon\to 0}\int_{\Gamma_1} \widetilde{h}_\varepsilon^2\,\mathrm{d}\mu_\varepsilon^S = \int_{\Gamma_1} h^2\,\mathrm{d}l. \tag{11.59}$$

In order to obtain the convergence

$$\limsup_{\varepsilon\to 0}\int_\Omega (y_\varepsilon - z_d)^2\,\mathrm{d}\mu_\varepsilon = \int_\Omega (\widetilde{y} - z_d)^2\,\mathrm{d}x, \tag{11.60}$$

we make use the idea of Cioranescu, Murat, and Zhikov (see [65, 256]). For this, we introduce the following auxiliary problem: Find $p_\varepsilon \in V(\Omega, \Gamma_2, \mathrm{d}\mu_\varepsilon)$ such that

$$\int_\Omega \left[(A_\varepsilon \nabla p_\varepsilon, \nabla \Psi) + \alpha p_\varepsilon \Psi\right] \mathrm{d}\mu_\varepsilon = \int_\Omega y_\varepsilon \Psi \,\mathrm{d}\mu_\varepsilon, \ \forall\, \Psi \in C^\infty(\Omega, \Gamma_2). \quad (11.61)$$

Note that the linear span of the test functions $\Psi \in C^\infty(\Omega, \Gamma_2)$ is dense in $L^2(\Omega)$; hence, we may take p_ε as a test function in (11.11) with $u = u_\varepsilon$ and $h = h_\varepsilon$ and y_ε as a test function in (11.61). Then, using the symmetry property of the matrix A_ε, we get

$$\int_\Omega y_\varepsilon^2 \,\mathrm{d}\mu_\varepsilon = \int_\Omega u_\varepsilon p_\varepsilon \,\mathrm{d}\mu_\varepsilon + \int_{\Gamma_1} h_\varepsilon p_\varepsilon \,\mathrm{d}\mu_\varepsilon^S, \qquad \forall\, \varepsilon \in E. \quad (11.62)$$

Since $u_\varepsilon = \widetilde{u} \in C^\infty(\Omega)$, $h_\varepsilon = \widetilde{h} \in C^\infty(\Gamma_1)$, and $p_\varepsilon \rightharpoonup p$ in $L^2(\Omega, \mathrm{d}\mu_\varepsilon)$, it follows that

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon p_\varepsilon \,\mathrm{d}\mu_\varepsilon = \lim_{\varepsilon\to 0} \int_\Omega \widetilde{u} p_\varepsilon \,\mathrm{d}\mu_\varepsilon = \int_\Omega \widetilde{u}\, p \,\mathrm{d}x,$$
$$\lim_{\varepsilon\to 0} \int_{\Gamma_1} h_\varepsilon p_\varepsilon \,\mathrm{d}\mu_\varepsilon^S = \int_{\Gamma_1} \widetilde{h}\, p \,\mathrm{d}l,$$

where $p \in H^1(\Omega)$ is the solution of a limit problem to (11.61). Note that since $y_\varepsilon \overset{2}{\rightharpoonup} \widetilde{y}(x)$, it follows that $p_\varepsilon \overset{2}{\rightharpoonup} p(x)$, where

$$\int_\Omega \left[(A^{hom}(x)\nabla p, \nabla \Psi) + \alpha p \Psi\right] \mathrm{d}x = \int_\Omega \widetilde{y}\Psi \,\mathrm{d}x, \quad \forall\, \Psi \in C^\infty(\Omega, \Gamma_2).$$

Hence, returning to (11.62), we have

$$\lim_{\varepsilon\to 0} \int_\Omega y_\varepsilon^2 \,\mathrm{d}\mu_\varepsilon = \lim_{\varepsilon\to 0} \left[\int_\Omega u_\varepsilon p_\varepsilon \,\mathrm{d}\mu_\varepsilon + \int_{\Gamma_1} h_\varepsilon p_\varepsilon \,\mathrm{d}\mu_\varepsilon^S\right]$$
$$= \int_\Omega \widetilde{u}\, p \,\mathrm{d}x + \int_{\Gamma_1} \widetilde{h}\, p \,\mathrm{d}l = \int_\Omega \widetilde{y}^2 \,\mathrm{d}x,$$

that is, $y_\varepsilon \to \widetilde{y}$ strongly in $L^2(\Omega, \mathrm{d}\mu_\varepsilon)$. Using this fact, we immediately obtain

$$\lim_{\varepsilon\to 0} \int_\Omega (y_\varepsilon - z_d)^2 \,\mathrm{d}\mu_\varepsilon = \int_\Omega (\widetilde{y} - z_d)^2 \,\mathrm{d}x. \quad (11.63)$$

As a result, combining (11.59) and (11.63), we deduce

$$\limsup_{\varepsilon\to 0} I_\varepsilon(y_\varepsilon, u_\varepsilon, h_\varepsilon) = I_0(\widetilde{y}, \widetilde{u}, \widetilde{h}).$$

Since

$$\left|I_0(\widetilde{y}, \widetilde{u}, \widetilde{h}) - I_0(y, u, h)\right| \le \widehat{C}\delta$$

with some constant $\widehat{C}$ independent of δ, this concludes the proof. □

Remark 11.28. Note that the result of the Lemma 11.25 remains correct without assuming A_ε to be symmetric, if we assume that the homogenized matrix in Definition 11.24 satisfies the condition $A_\varepsilon^T(x)\nabla y_\varepsilon \rightharpoonup (A^{hom})^T\nabla y_0$ in $L^2(\Omega, \mathrm{d}\mu_\varepsilon)^2$ for every $\{y_\varepsilon\}$ such that

$$\Big\{(y,u,h)\in \mathbb{Z}_\varepsilon(\Omega,\Gamma_1)\ :\ \int_\Omega \left[(A_\varepsilon^T(x)\nabla y,\nabla\varphi)+\alpha y\varphi\right]\ \mathrm{d}\mu_\varepsilon$$
$$=\int_\Omega u\varphi\ \mathrm{d}\mu_\varepsilon+\int_{\Gamma_1} h\varphi\ \mathrm{d}\mu_\varepsilon^S,\ \forall\ \varphi\in C^\infty(\Omega,\Gamma_2)\Big\}$$
$$\ni (y_\varepsilon,u_\varepsilon,h_\varepsilon)\to(y_0,u_0,h_0)\ \text{in the sense of } w\text{-convergence.}$$

Furthermore, we emphasize that this property holds automatically for a μ-measurable symmetric periodic matrix $A_\varepsilon(x)=A(\varepsilon^{-1}x)$ satisfying the conditions of ellipticity and boundedness (11.8).

Now, we are in a position to prove the main result of this section.

Theorem 11.29. *Under the supposition of Lemma 11.11 for the family of problems (11.9)–(11.12), there exists a unique limiting OCP which has the representation*

$$-\mathrm{div}(A^{hom}(x)\nabla y)+\alpha y=u\quad in\ \ \Omega, \tag{11.64}$$
$$y=0\quad on\ \ \Gamma_2\qquad \partial y/\partial\nu_{A^{hom}}=h\quad on\ \ \Gamma_1, \tag{11.65}$$
$$|u|\le c_u\quad a.e.\ on\ \ \Omega,\qquad |h|\le c_h\quad a.e.\ on\ \ \Gamma_1, \tag{11.66}$$
$$I_0(y,u,h)=k_1\int_\Omega (y-z_d)^2\,\mathrm{d}x+k_2\int_\Omega u^2\,\mathrm{d}x+k_3\int_{\Gamma_1}h^2\,\mathrm{d}l\to\inf, \tag{11.67}$$

where

$$\frac{\partial y}{\partial\nu_{A^{hom}}}=\sum_{i,j=1}^{2}a_{ij}^{hom}(x)\frac{\partial y}{\partial x_{ij}}\cos(n,x_i),$$

$\cos(n,x_i)$ *is ith direction cosine of* n*, and* n *is the normal of* Γ_1 *exterior to* Ω*.*

Moreover, the sequence of optimal solutions $\{(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0)\in\Xi_\varepsilon\}$ *for the original problems (11.9)–(11.12) and corresponding minimal values of the cost functional (11.9) satisfy the following variational properties:*

$$\lim_{\varepsilon\to0}\inf_{(y,u,h)\in\Xi_\varepsilon} I_\varepsilon(y,u,h)=I_0(y^0,u^0,h^0)=\inf_{(y,u,h)\in\Xi_0} I_0(y,u,h), \tag{11.68}$$

$$y_\varepsilon^0\overset{2}{\to}y^0,\quad u_\varepsilon^0\to u^0\ \ in\ \ L^2(\Omega,\mathrm{d}\mu_\varepsilon),\quad h_\varepsilon^0\to h^0\ \ in\ \ L^2(\Gamma_1,\mathrm{d}\mu_\varepsilon^S), \tag{11.69}$$

where (y^0,u^0,h^0) *is the unique solution of the limiting problem (11.64)–(11.67).*

Proof. As immediately follows from Lemmas 11.11 and 11.25 for the sequence of constrained minimization problems (11.17), there exists a variational limit

(11.18) the main components of which can be recovered in the form (11.42) and (11.54), respectively. Moreover, from (11.42), we have the implication

$$(y,u,h)\in \Xi_0 \Rightarrow (y,u,h)\in \mathrm{Graph}(\mathbb{P}^{hom}).$$

It is clear now that the limit problem (11.18) can be written in the form of the OCP (11.64)–(11.67). Therefore, in view of Definition 11.12, the problem (11.64)–(11.67) is the homogenized optimal control problem for the original family (11.9)–(11.12).

Applying now Theorem 11.23, we come to the following variational properties of the limit problem (11.64)–(11.67): Let $\{(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0)\}$ be the sequence of optimal triplets for the problems (11.9)–(11.12), then

$$\begin{aligned}\lim_{\varepsilon\to 0} I_\varepsilon(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0) &= \lim_{\varepsilon\to 0}\inf_{(y_\varepsilon,u_\varepsilon,h_\varepsilon)\in \Xi_\varepsilon} I_\varepsilon(y_\varepsilon,u_\varepsilon,h_\varepsilon)\\ &= \inf_{(y,u,h)\in\Xi_0} I_0(y,u,h) = I_0(y^0,u^0,h^0),\end{aligned}$$

$$(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0) \overset{w}{\longrightarrow} (y^0,u^0,h^0). \tag{11.70}$$

Hence, $I_\varepsilon(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0)\to I_0(y^0,u^0,h^0)$. Rewriting this in the explicit form, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\left[k_1\int_\Omega (y_\varepsilon^0-z_d)^2\,\mathrm{d}\mu_\varepsilon + k_2\int_\Omega (u_\varepsilon^0)^2\,\mathrm{d}\mu_\varepsilon + k_3\int_{\Gamma_1}(h_\varepsilon^0)^2\,\mathrm{d}\mu_\varepsilon^S\right]\\ = k_1\int_\Omega (y^0-z_d)^2\,\mathrm{d}x + k_2\int_\Omega (u^0)^2\,\mathrm{d}x + k_3\int_{\Gamma_1}(h^0)^2\,\mathrm{d}l.\end{aligned}$$

Therefore, the validity of this equality for every $k_i\ge 0$ implies

$$\lim_{\varepsilon\to 0}\int_\Omega (y_\varepsilon^0)^2\,\mathrm{d}\mu_\varepsilon = \int_\Omega (y^0)^2\,\mathrm{d}x,\quad \lim_{\varepsilon\to 0}\int_\Omega (u_\varepsilon^0)^2\,\mathrm{d}\mu_\varepsilon = \int_\Omega (u^0)^2\,\mathrm{d}x,$$
$$\lim_{\varepsilon\to 0}\int_{\Gamma_1}(h_\varepsilon^0)^2\,\mathrm{d}\mu_\varepsilon^S = \int_\Omega (h^0)^2\,\mathrm{d}l.$$

Combining these properties with (11.70) and using Lemma 6.19, we obtain the required assertions (11.68) and (11.69). □

11.6 Modeling of suboptimal controls

The main question we are going to consider here concerns the approximation of the optimal solution of the original problem (11.9)–(11.12) for ε small enough. We define suboptimal controls $(u_\varepsilon^{sub},h_\varepsilon^{sub})$ which approximate the optimal value of the original problem (11.9)–(11.12). To do so, we introduce the following concept (for comparison, see Definition 4.5).

Definition 11.30. *We say that a sequence of admissible controls*

$$\left\{(u_\varepsilon^{sub}, h_\varepsilon^{sub})\right\}_{\varepsilon>0}$$

is asymptotically suboptimal to the problem (11.9)–(11.12) if there exists a constant $\widehat{C} > 0$ independent of ε such that for every δ $(0 < \delta < \delta_0)$ there is $\varepsilon_0 > 0$ satisfying

$$\left| \inf_{(y,u,h)\in \Xi_\varepsilon} I_\varepsilon(y,u,h) - I_\varepsilon(y_\varepsilon^{sub}, u_\varepsilon^{sub}, h_\varepsilon^{sub}) \right| < \widehat{C}\delta, \quad \forall\, \varepsilon < \varepsilon_0, \tag{11.71}$$

where $y_\varepsilon^{sub} = y_\varepsilon^{sub}(u_\varepsilon^{sub}, h_\varepsilon^{sub})$ denotes the corresponding solutions of the boundary value problem (11.10)–(11.11).

Theorem 11.29 leads to the following final result.

Theorem 11.31. *Let $u^0 \in L^2(\Omega)$ and $h^0 \in L^2(\Gamma_1)$ be the optimal controls for the homogenized problem (11.64)–(11.67). Then any δ-realizing sequence to the pair (u^0, h^0) is asymptotically suboptimal for the original OCP (11.9)–(11.12).*

Proof. Let (y^0, u^0, h^0) be the unique solution of the homogenized problem (11.64)–(11.67). For a given $\delta > 0$, we construct the δ-realizing sequence $\{(y_\varepsilon, u_\varepsilon, h_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in E}$ in the usual way, that is, $u_\varepsilon = \widetilde{u}$, $h_\varepsilon = \widetilde{h}$, and $y_\varepsilon = y_\varepsilon(\widetilde{u}, \widetilde{h})$ is the corresponding solution of the boundary value problem (11.10)–(11.11) with $u = \widetilde{u}$ and $h = \widetilde{h}$. Here,

$$\left.\begin{array}{l} \widetilde{u} \in C^\infty(\Omega),\ |\widetilde{u}| \le c_u \ \text{ on } \ \Omega,\ \|\widetilde{u} - u^0\|_{L^2(\Omega)} \le \delta, \\ \widetilde{h} \in C^\infty(\Gamma_1),\ |\widetilde{h}| \le c_h \ \text{ on } \ \Gamma_1,\ \|\widetilde{h} - h^0\|_{L^2(\Gamma_1)} \le \delta. \end{array}\right\} \tag{11.72}$$

It is clear that (see Lemma 11.27)

$$\begin{gathered} \limsup_{\varepsilon\to 0} \int_\Omega (y_\varepsilon - z_d)^2 \,\mathrm{d}\mu_\varepsilon = \int_\Omega (\widetilde{y} - z_d)^2 \,\mathrm{d}x, \\ \|y^0 - \widetilde{y}\|_{H^1(\Omega)} \le 2\delta C \ \text{(see (11.58))}, \end{gathered} \tag{11.73}$$

$$\begin{aligned} \limsup_{\varepsilon\to 0} \int_\Omega \widetilde{u}_\varepsilon^2 \,\mathrm{d}\mu_\varepsilon &= \int_\Omega (u^0)^2 \,\mathrm{d}x, \\ \limsup_{\varepsilon\to 0} \int_{\Gamma_1} \widetilde{h}_\varepsilon^2 \,\mathrm{d}\mu_\varepsilon^S &= \int_{\Gamma_1} (h^0)^2 \,\mathrm{d}l, \end{aligned} \tag{11.74}$$

where $\widetilde{y}$ is the unique solution of the problem (11.64)–(11.65) with $u = \widetilde{u}$ and $h = \widetilde{h}$. In addition, in view of these relations, we have the following obvious estimates:

$$k_2 \left| \int_\Omega (u^0)^2 \,\mathrm{d}x - \int_\Omega (\widetilde{u})^2 \,\mathrm{d}x \right| \le k_2 \left(\delta_0 + 2\|u^0\|_{L^2(\Omega)}\right) \delta = C_2\delta, \tag{11.75}$$

$$k_3 \left| \int_\Omega (h^0)^2 \,\mathrm{d}x - \int_\Omega (\widetilde{h})^2 \,\mathrm{d}x \right| \le k_3 \left(\delta_0 + 2\|h^0\|_{L^2(\Gamma_1)}\right) \delta = C_3\delta, \tag{11.76}$$

$$k_1 \left| \int_\Omega (y^0 - z_d)^2 \,\mathrm{d}x - \int_\Omega (\widetilde{y} - z_d)^2 \,\mathrm{d}x \right| \leq 4k_1 \left(\delta_0 C + \|y^0 - z_d\|_{L^2(\Omega)} \right) C\delta = C_1 \delta. \quad (11.77)$$

We now observe that

$$\begin{aligned}
&\left| \inf_{(y,u,h)\in\Xi_\varepsilon} I_\varepsilon(y,u,h) - I_\varepsilon(y_\varepsilon,u_\varepsilon,h_\varepsilon) \right| \\
&\quad = \left| I_\varepsilon(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0) - I_\varepsilon(y_\varepsilon,\widetilde{u},\widetilde{h}) \right| \\
&\quad \leq \left| I_\varepsilon(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0) - I_0(y^0,u^0,h^0) \right| + \left| I_0(\widetilde{y},\widetilde{u},\widetilde{h}) - I_\varepsilon(y_\varepsilon,\widetilde{u},\widetilde{h}) \right| \\
&\quad + \left| I_0(y^0,u^0,h^0) - I_0(\widetilde{y},\widetilde{u},\widetilde{h}) \right| \\
&\quad \leq \left| I_\varepsilon(y_\varepsilon^0,u_\varepsilon^0,h_\varepsilon^0) - I_0(y^0,u^0,h^0) \right| \\
&\quad + \left| I_0(\widetilde{y},\widetilde{u},\widetilde{h}) - I_\varepsilon(y_\varepsilon,\widetilde{u},\widetilde{h}) \right| \\
&\quad + k_1 \left| \int_\Omega (y^0 - z_d)^2 \,\mathrm{d}x - \int_\Omega (\widetilde{y} - z_d)^2 \,\mathrm{d}x \right| \\
&\quad + k_2 \left| \int_\Omega (u^0)^2 \,\mathrm{d}x - \int_\Omega (\widetilde{u})^2 \,\mathrm{d}x \right| + k_3 \left| \int_{\Gamma_1} (h^0)^2 \,\mathrm{d}l - \int_{\Gamma_1} (\widetilde{h})^2 \,\mathrm{d}l \right| \\
&\quad = J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

To conclude the proof, we note that for a given $\delta > 0$, one can always find (i) $\varepsilon_1 > 0$ such that $J_1 < \delta/2$ for all $\varepsilon < \varepsilon_1$ by Theorem 11.29 and (ii) $\varepsilon_2 > 0$ such that $J_2 < \delta/2$ for all $\varepsilon < \varepsilon_2$ by Lemma 11.27. Additionally, $J_3 < C_1\delta$, $J_4 < C_2\delta$, and $J_5 < C_3\delta$ by estimates (11.75)–(11.77). As a result, we have

$$\left| \inf_{(y,u,h)\in\Xi_\varepsilon} I_\varepsilon(y,u,h) - I_\varepsilon(y_\varepsilon,u_\varepsilon,h_\varepsilon) \right| \leq \left(1 + \sum_{i=1}^{3} C_i \right) \delta, \ \forall\, \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}.$$

Thus, we have obtained the required estimate (11.71). □

11.7 An example of an optimal control problem on ε-periodic square grid

On the domain Ω that was defined in (11.1), we consider the ε-periodic square grid $\varepsilon\mathcal{F}$ with the cell of periodicity $\varepsilon\square$. Here, the set $\square = [0,1)^2$ contains the "cross"-structure such as indicated in Fig. 11.1(a).

Following the notation of Sect. 11.6, we say that Ω_ε has ε-periodic grid-like structure if $\Omega_\varepsilon = \Omega \cap \varepsilon\mathcal{F}$. We also set $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{x \in \bar{\Omega} | x_2 = 0,\ 0 < x_1 < a\}$.

Now, we recall the standard notations on graphs (see Lagnese and Leugering [163]). Let $V^\varepsilon = \{v_J : J \in \mathcal{J}_\varepsilon\}$ be the set of vertices of our ε-periodic

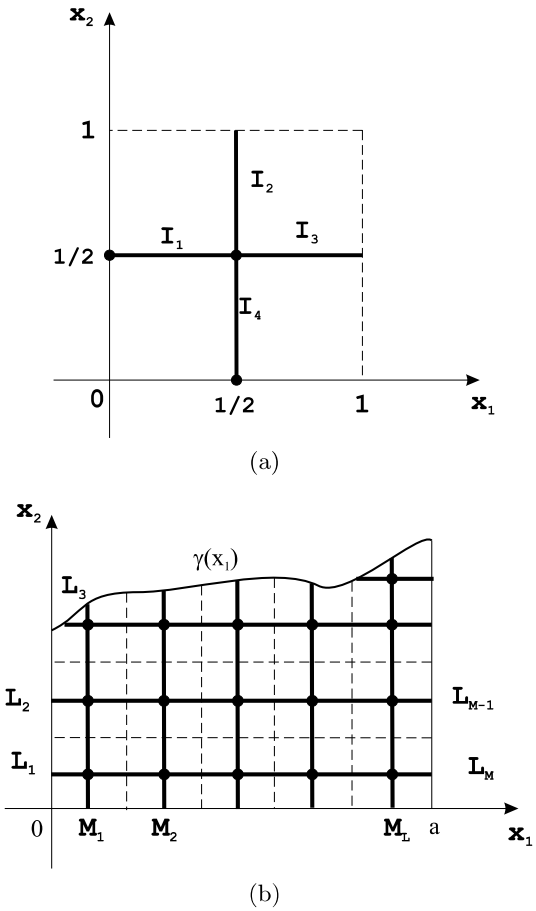


Fig. 11.1. (a) The cell of grid periodicity; (b) periodic grid on Ω

graph (grid) Ω_ε and let $E^\varepsilon = \{e_i : i \in \dot{\mathcal{I}}_\varepsilon\}$ be the index set of corresponding edges. Here, by $\mathcal{J}_\varepsilon$ and $\dot{\mathcal{I}}_\varepsilon$, we denote the index sets for vertices and edges, respectively. For a given vertex v_J, we consider the set of edges that are incident at v_J. The corresponding set of edge indices is denoted by

$$\dot{\mathcal{I}}_J = \{i \in \dot{\mathcal{I}}_\varepsilon : e_i \text{ is incident at } v_J\}.$$

The cardinality of $\dot{\mathcal{I}}_J$ is the edge degree at v_J, that is, $d_J = |\dot{\mathcal{I}}_J|$. It is easy to see that $d_J \leq 4$ in our case.

Note that as follows from Fig. 11.1(b), every edge e_i on the graph Ω_ε can be parameterized by $x \in [0, l_i]$, where $l_i \leq \varepsilon/2$ denotes the length of the edge e_i. With every edge e_i we will associate a so-called "state function"

$$y_i : [0, l_i] \to \mathbb{R}^1, \ i \in \dot{\mathcal{I}}_\varepsilon.$$

Since we are going to consider an OCP on a bounded periodic graph, it is necessary to specify the boundary and the so-called transmission conditions at the vertices V^ε of Ω_ε. To this end, we subdivide the set of vertices (nodes) as follows:

$$V^\varepsilon = V_S^\varepsilon \cup V_M^\varepsilon,$$

where V_S^ε denotes the set of simple modes such that $d_J = 1$ and V_M^ε signifies the set of multiple nodes where $4 \geq d_J > 1$. The set of simple nodes V_S^ε will be divided as

$$V^\varepsilon = V_{\Gamma_1}^\varepsilon \cup V_{\Gamma_2}^\varepsilon,$$

where $V_{\Gamma_1}^\varepsilon$ represents the set of simple nodes belonging to the Γ_1-boundary and $V_{\Gamma_2}^\varepsilon$ signifies those simple nodes which belong to the Γ_2-boundary. It is easy to see that, in the case of ε-periodic grid on Ω, there is no simple node lying in the interior of the domain Ω. Further, we will look at the set $V_{\Gamma_1}^\varepsilon$ as the set of control-active Neumann nodes and at $V_{\Gamma_2}^\varepsilon$ as the set of nodes with zero Dirichlet conditions.

On all the edges e_i, we consider the differential operator L_i of the form

$$L_i y_i = -R_i y_i'' + \alpha y_i,$$

where $R_i \geq \alpha > 0$. Moreover, using the ε-periodic structure of Ω_ε, we will always suppose that for every ε-cell $\varepsilon\square_j$, we have (see Fig. 11.3(a))

$$R_i = \beta,\ R_{i+1} = \gamma, \quad R_{i+2} = \beta, \quad R_{i+3} = \gamma, \tag{11.78}$$

where $\alpha^{-1} \geq \beta, \gamma \geq \alpha > 0$.

We now define the classes of admissible controls U^ε and H^ε where

$$U^\varepsilon = \left\{u : \Omega_\varepsilon \to \mathbb{R}^1 \ : \ u\big|_{l_i} \in L^2(0, l_i); \quad |u(x)| \leq c_u \right. \\ \left. \text{for almost every } \ x \in \Omega_\varepsilon\right\}, \tag{11.79}$$

$$H^\varepsilon = \Big\{h = (h_1, h_2, \ldots, h_{L_\varepsilon}) \in \mathbb{R}^{L_\varepsilon} \ : \ L_\varepsilon = |V_{\Gamma_1}^\varepsilon|, \\ \sum_{K=1}^{L_\varepsilon} h_K^2 < \infty, \quad |h_K| \leq c_h\Big\}. \tag{11.80}$$

Here, c_u and c_h are some positive constant; by $|V_{\Gamma_1}^\varepsilon|$ we denote the number of all simple nodes belonging to Γ_1.

Let k_1, k_2, and k_3 ($k_i > 0$) be penalty terms and let z_d, u_d, and h_d be given functions of $C(\overline{\Omega})$. We consider the following OCP on the grid Ω_ε:

$$-R_i y_i'' + \alpha y_i = u_i, \ \ x \in (0, l_i), \ \ i \in \dot{\mathcal{I}}_\varepsilon, \tag{11.81}$$

$$y_i(v_J) = 0, \ \ i \in \dot{\mathcal{I}}_J^\varepsilon, \ \ v_J \in V_{\Gamma_2}^\varepsilon, \tag{11.82}$$

$$\sum_{i \in \mathcal{I}_J^\varepsilon} R_i y_i'(v_J) = 0, \ \ v_J \in V_M^\varepsilon, \tag{11.83}$$

$$y_i(v_J) = h_{k(J)}, \qquad \forall\, v_J \in V_{\Gamma_1}^\varepsilon, \ \ i \in \mathcal{I}_J^\varepsilon; \ \ k(J) \in \{1, 2, \ldots, L_\varepsilon\}, \tag{11.84}$$

$$u \in U^\varepsilon, \quad h \in H^\varepsilon, \tag{11.85}$$

$$I_\varepsilon(y,u,h) = \sum_{i\in\dot{I}_\varepsilon} k_1 \int_0^{l_i} (y_i - z_d\big|_{e_i})^2 \,\mathrm{d}x + k_2 \sum_{i\in\dot{\mathcal{I}}_\varepsilon} \int_0^{l_i} (u_i - u_d\big|_{e_i})^2 \,\mathrm{d}x$$
$$+ k_3 \sum_{K=1}^{L_\varepsilon} (h_K - h_d(v_J))^2 \to \inf. \tag{11.86}$$

Using the properties of the sets U^ε and H^ε and invoking the standard arguments, it is easy to prove that for every $\varepsilon \in E$, the problem (11.81)–(11.86) admits a unique optimal triplet $(y^0_\varepsilon, u^0_\varepsilon, h^0_\varepsilon)$ which can be characterized by some adjoint system.

Our aim is to study the asymptotic behavior of this problem as ε tends to 0. For this, we reformulate the problem (11.81)–(11.86) in terms of some variational control problem defined on spaces with singular measures. We introduce the $\square$-periodic Borel measure μ in $\mathbb{R}^2$ as follows:

$$\mu = \frac{1}{2}(\mu_1 + \mu_2 + \mu_3 + \mu_4),$$

where μ_i are the 1D Lebesgue measures on the corresponding line segments (edges) I_i (see Fig. 11.1).

Also, we define the $\square^S$-periodic Radon measure μ^S on $\mathbb{R}^1$ as $\mu^S = \delta_{(1/2,0)}$, where $\delta_{(1/2,0)}$ is the Dirac measure at the point $(1/2, 0) \in \square^S$. It is easy to see that

$$\int_\square \mathrm{d}\mu = 1 \quad \text{and} \quad \int_{\square^S} \mathrm{d}\mu^S = 1.$$

Therefore, we may define the "scaling" measures

$$\mu_\varepsilon(B) = \varepsilon^2 \mu(\varepsilon^{-1}B), \qquad \mu^S_\varepsilon(B_1) = \varepsilon \mu^S(\varepsilon^{-1}B_1),$$

where B and B_1 are corresponding Borel sets in $\mathbb{R}^2$ and $\mathbb{R}^1$, respectively. Obviously, each of these measures μ_ε and μ^S_ε converges weakly to the corresponding Lebesgue measure: $\mathrm{d}\mu_\varepsilon \rightharpoonup \mathrm{d}x$ and $\mathrm{d}\mu^S_\varepsilon \rightharpoonup \mathrm{d}l$. Here, $\mathrm{d}x$ and $\mathrm{d}l$ are the Lebesgue measures on $\mathbb{R}^2$ and $\mathbb{R}^1$, respectively.

Now, we define the matrix $A_\varepsilon(x) = A(\varepsilon^{-1}x)$ as

$$A(z) = \begin{bmatrix} a_{11}(z) & 0 \\ 0 & a_{22}(z) \end{bmatrix},$$

where $a_{11}(z) = \beta$ and $a_{22}(z) = \gamma$. It is easy to see that the matrix so defined is symmetric, μ-measurable, and satisfying property (11.8).

As a result, the original OCP can be presented in the form

$$\int_\Omega \left[(A(\varepsilon^{-1}x)\nabla y, \nabla\varphi) + \alpha y\varphi\right] \mathrm{d}\mu_\varepsilon$$
$$= \int_\Omega u\varphi \,\mathrm{d}\mu_\varepsilon + \int_{\Gamma_1} h\varphi \,\mathrm{d}\mu^S_\varepsilon, \ \forall\, \varphi \in C^\infty(\Omega, \Gamma_2), \tag{11.87}$$

$$y \in V(\Omega, \Gamma_2, \mathrm{d}\mu_\varepsilon), \tag{11.88}$$

$$|u| \le c_u \quad \mu_\varepsilon\text{-a.e. in } \Omega, \quad |h| \le c_h \quad \mu_\varepsilon^S\text{-a.e. on } \Gamma_1, \tag{11.89}$$

$$I_\varepsilon(y,u,h) = k_1 \int_\Omega (y - z_d)^2 \,\mathrm{d}\mu_\varepsilon + k_2 \int_\Omega (u - u_d)^2 \,\mathrm{d}\mu_\varepsilon + k_3 \int_{\Gamma_1} (h - h_d)^2 \,\mathrm{d}\mu_\varepsilon^S \to \inf. \tag{11.90}$$

The validity of this representation immediately follows from Proposition 11.5 and Remark 11.15. Then, due to Theorem 11.29, for the control problem (11.87)–(11.90) there exists a limit problem which can be recovered in the form

$$-\mathrm{div}(A^{hom}\nabla y) + \alpha y = u \text{ in } \Omega, \tag{11.91}$$

$$y = 0 \text{ on } \Gamma_2, \quad \partial y/\partial \nu_{A^{hom}} = h \text{ on } \Gamma_1, \tag{11.92}$$

$$|u| \le c_u \text{ a.e. on } \Omega, \quad |h| \le c_h \text{ a.e. on } \Gamma_1, \tag{11.93}$$

$$I_0(y,u,h) = k_1 \int_\Omega (y - z_d)^2 \,\mathrm{d}x + k_2 \int_\Omega (u - u_d)^2 \,dx + k_3 \int_{\Gamma_1} (h - h_d)^2 \,\mathrm{d}l \longrightarrow \inf. \tag{11.94}$$

The identification of the matrix A^{hom} can be done either by definition, that is, as the solution of the minimum problem

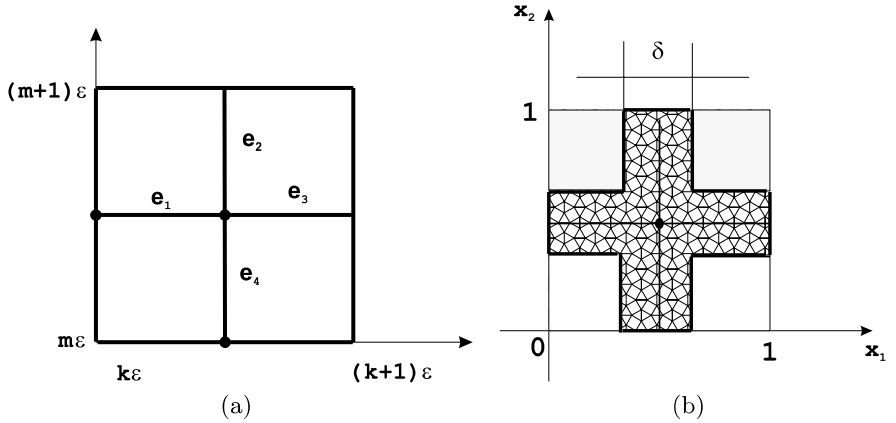
$$(A^{hom}\xi, \xi) = \min_{p \in V_{pot}} (A(\xi + p), \xi + p)$$

(see [256]) or using a more classical method: First, we homogenize the problem on the grid with nonzero thickness in the usual way (see [66, 261]) and then pass to the limit as thickness goes to 0. As for the second approach, it was shown in [58] that in this case the corresponding diagram of homogenization (see Fig. 11.2) is commutative.

$$\begin{array}{ccc} A_{\varepsilon,\delta} & \xrightarrow{\delta\to 0} & A_\varepsilon^{sing} \\ {\scriptstyle \varepsilon\to 0}\downarrow & & \downarrow{\scriptstyle \varepsilon\to 0} \\ A_\delta^{hom} & \xrightarrow{\delta\to 0} & A^{hom} \end{array}$$

Fig. 11.2. Homogenization diagram

Here, δ denotes a small parameter which characterizes the fixed "thickness" of graphs. The corresponding structures will be called δ-grids $\mathcal{F}^\delta$.



(a) (b)

Fig. 11.3. (a) $\varepsilon\square_J$-cell; (b) periodicity cell for $\mathcal{F}^\delta$

We will follow the second approach. Therefore, we define the ε-periodic δ-grids $\mathcal{F}^\delta_\varepsilon$ by setting $\mathcal{F}^\delta_\varepsilon = \varepsilon^{-1}\mathcal{F}^\delta$; the cell of periodicity $\square_\delta$ for $\mathcal{F}^\delta$ has the form that Fig. 11.3(b) shows.

Let $\mu^\varepsilon_\delta(B) = \varepsilon^2\mu_\delta(\varepsilon^{-1}\,B)$ and let μ_δ be the measure on $\mathcal{F}^\delta$ that can be defined as the probability measure on Ω supported by the cross-bar in $\square_\delta$ and is uniformly distributed on it. The weak limit of this measure μ_δ as $\delta \to 0$ is the singular measure μ which was defined earlier. Assume that the matrix $A_{\varepsilon,\delta} = A(\varepsilon^{-1}x)$ (i.e., $A_{\varepsilon,\delta} = \mathrm{diag}(\beta, \gamma)$.)

Then using the standard technique of homogenization (see [66, 209]), we get $A_{\varepsilon,\delta} \overset{\varepsilon\to 0}{\longrightarrow} A^{hom}_\delta$, where

$$A^{hom}_\delta = \begin{bmatrix} b^\delta_{11} & 0 \\ 0 & b^\delta_{22} \end{bmatrix}, \quad b^\delta_{11} = \frac{1}{(2-\delta)}\left[\beta + o(\delta^{1/2})\right],$$

$$b^\delta_{22} = \frac{1}{(2-\delta)}\left[\gamma + o(\delta^{1/2})\right].$$

Furthermore, due to the results of Cioranescu and Saint Jean Paulin ([66]), we may take the limit $A^{hom}_\delta \overset{\delta\to 0}{\longrightarrow} A^{hom}$. As a result, we obtain (for details, we refer to [22, 66, 207, 209]) $A^{hom} = \mathrm{diag}(\frac{1}{2}\beta, \frac{1}{2}\gamma)$.

Thus, the homogenized state equation for the limit problem (11.91)–(11.94) has the form

$$-\beta\frac{\partial^2 y}{\partial x_1^2} - \gamma\frac{\partial^2 y}{\partial x_2^2} + 2\alpha y = 2u \quad \text{in } \Omega,\ y = 0 \ \text{ on } \ \Gamma_2, \quad \gamma\frac{\partial y}{\partial x_2} = 2h \ \text{ on } \ \Gamma_1.$$

Furthermore, in view of the result of Theorem 11.29, the homogenized OCP (11.91)–(11.94) satisfies the variational properties (11.68)–(11.69). Note also that other approaches for the identification of homogenized operators

$-\mathrm{div}(A^{hom}\nabla y)$ have been discussed in the literature (see, for instance, Mazja and Slutskij [184]).

In the case when

$$a = 2,\ \alpha = \beta = \gamma = 1,\ \Omega = \left\{(x_1, x_2)\ :\ x_1 \in (0,2),\ 0 < x_2 < \sqrt{2x_1 - x_1^2}\right\},$$

$$h_d = 0,\ z_d = \exp(A),\ u_d = \exp(A)\left(1 - 2A^2 - 6A^3 - 2A^4\right),$$

$$A = (x_1^2 + x_2^2 - 2x_1)^{-1},$$

we have that

$$u^0 = \exp(A)\left(1 - 2A^2 - 6A^3 - 2A^4\right) \quad \text{and} \quad h^0 = 0$$

are the optimal controls to the homogenized problem (11.91)–(11.94). Hence, in view of Theorem 11.29, the restriction of these functions on the sets Ω_ε and $V_{\Gamma_1}^\varepsilon$, respectively, can be taken as suboptimal controls to the original problem on an ε-periodic square grid (11.81)–(11.85).

12

Suboptimal Boundary Control of Elliptic Equations in Domains with Small Holes

Our prime interest in this chapter concerns the construction of suboptimal solutions to a class of boundary optimal control problems (OCPs) in ε-periodically perforated domains with small holes. We suppose that the support of controls is contained in the set of boundaries of the holes. This set is divided into two parts: On one part, the controls are of Dirichlet type; on the other one, the controls are of Neumann type. Using the ideas of the Γ-convergence theory and the concept of the variational convergence of constrained minimization problems, we show that the limit problem, as ε tends to 0, can be recovered in an explicit analytical form. However, in contrast to the case of thin periodic structures (see Chap. 9), the control through the boundary of small holes leads us in the limit to an OCP with drastically different properties and structures. In particular, we show that in this case, a "strange term" appears both in the limit equation and the cost functional, and that this term depends on the geometry of the holes. Moreover, the characteristic feature of the obtained limiting control problem is the fact that it contains two independent distributed control functions which can be used as suboptimal controls to the original one.

12.1 Statement of the problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open domain and let ε be a small positive parameter. In order to define a perforated domain Ω_ε, we introduce the following sets: $Y = [-1/2, +1/2)^n$; Q and K are compact subsets of Y such that $0 \in \operatorname{int} K \cap \partial Q$ (see Fig. 12.1),

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_12,
© Springer Science+Business Media, LLC 2011

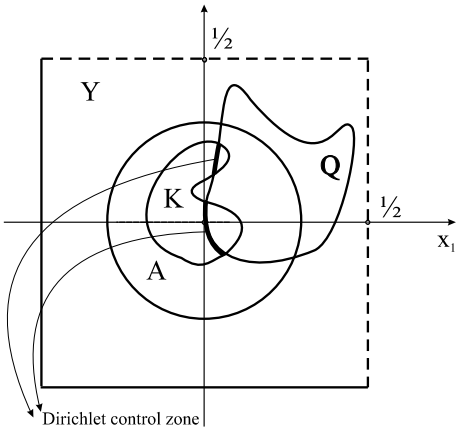


Fig. 12.1. The perforation scheme

$$\Theta_\varepsilon = \{\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n \ : \ (\varepsilon Y + \varepsilon\,\mathbf{k}) \cap \Omega \neq \emptyset\}\,; \tag{12.1}$$

$$Y_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \{\varepsilon(Y + \mathbf{k})\}\,; \qquad T_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \left\{\varepsilon^{n/(n-1)}Q + \varepsilon\,\mathbf{k}\right\}; \tag{12.2}$$

$$S_\varepsilon = \begin{cases} \varepsilon^{n/(n-2)}K \cap \partial(\varepsilon^{n/(n-1)}Q), & n \geq 3, \\ \exp(-1/\varepsilon^2)K \cap \partial(\varepsilon^{n/(n-1)}Q), & n = 2; \end{cases} \tag{12.3}$$

$$\Gamma_\varepsilon^D = \left[\bigcup_{\mathbf{k}\in\Theta_\varepsilon} \{S_\varepsilon + \varepsilon\,\mathbf{k}\}\right] \cap \overline{\Omega}, \qquad \Gamma_\varepsilon^N = \left[\partial T_\varepsilon \setminus \Gamma_\varepsilon^D\right] \cap \overline{\Omega}. \tag{12.4}$$

Then, we set $\Omega_\varepsilon = \Omega \setminus T_\varepsilon$. Notice that (in contrast to the case of thin periodic structures) the size of the holes $Q_\varepsilon + \varepsilon\,\mathbf{k}$ tends to 0 as $\varepsilon \to 0$ and their boundaries Γ_ε^D and Γ_ε^N are not proportional to the size of the periodicity cell εY. In Ω_ε, we consider the following boundary value problem:

$$\left.\begin{aligned} -\Delta\, y_\varepsilon + y_\varepsilon &= f_\varepsilon && \text{in } \Omega_\varepsilon, \\ \partial_\nu y_\varepsilon &= -k_0 y_\varepsilon + p_\varepsilon && \text{on } \Gamma_\varepsilon^N, \\ y_\varepsilon &= u_\varepsilon && \text{on } \Gamma_\varepsilon^D, \\ y_\varepsilon &= 0 && \text{on } \Sigma_\varepsilon = \partial\Omega \cap \partial\Omega_\varepsilon, \end{aligned}\right\} \tag{12.5}$$

where $f_\varepsilon \in L^2(\Omega)$ is a given function, k_0 is a positive constant, and $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative.

In (12.5), u_ε and p_ε are the control functions which act on the system through the set of boundaries of the holes. We interpret the functions u_ε and p_ε as Dirichlet and Neumann boundary controls, respectively. We say that the control functions u_ε and p_ε are admissible if the following conditions hold:

$$p_\varepsilon \in L^2(\Gamma_\varepsilon^N),$$
$$u_\varepsilon \in U_\varepsilon = \left\{ a|_{\Gamma_\varepsilon^D} \ : \ a \in H_0^1(\Omega) \cap H^2(\Omega),\ \|a\|_{H^2(\Omega)} \le C_0 \right\}. \tag{12.6}$$

Now, the OCP can be formulated as follows: Given $z_\varepsilon \in L^2(\Omega)$ and $C_0 > 0$, find a triple $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ such that

$$I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) = \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon), \tag{12.7}$$

where the cost functional I_ε and the set of admissible triplets Ξ_ε are defined as

$$\begin{aligned} I_\varepsilon &= \int_{\Omega_\varepsilon} |\nabla y_\varepsilon|^2 \,\mathrm{d}x + \int_{\Omega_\varepsilon} |y_\varepsilon - z_\varepsilon|^2 \,\mathrm{d}x \\ &\quad + \int_{\Gamma_\varepsilon^N} p_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{n-1} + \beta(\varepsilon) \int_{\Gamma_\varepsilon^D} u_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{n-1}, \\ \Xi_\varepsilon &= \{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in H^1(\Gamma_\varepsilon^D) \times L^2(\Gamma_\varepsilon^N) \times H^1(\Omega_\varepsilon; \Sigma_\varepsilon) \ : \\ &\qquad (u_\varepsilon, p_\varepsilon, y_\varepsilon) \text{ satisfies (12.5)–(12.6)}\}. \end{aligned} \tag{12.8}$$

Here,

$$H^1(\Omega_\varepsilon; \Sigma_\varepsilon) = \left\{ y_\varepsilon \in H^1(\Omega_\varepsilon) \ : \ y_\varepsilon = 0 \text{ on } \Sigma_\varepsilon \right\},$$
$$\beta(\varepsilon) = \varepsilon^{n/(2-n)} \ \text{ if } \ n \ge 3, \qquad \beta(\varepsilon) = \varepsilon^2 \exp(\varepsilon^{-2}) \ \text{ if } \ n = 2.$$

In the sequel, the OCP (12.5)–(12.7) will be called the $\mathbb{P}_\varepsilon$-problem.

Throughout this chapter, we suppose that Ω is a measurable set in the sense of Jordan; the small parameter ε varies in a strictly decreasing sequence of positive numbers which converges to 0; Q and K are compact subsets of Y such that $0 \in \operatorname{int} K \cap \partial Q$; the set Q has Lipschitz boundary ∂Q, $\operatorname{int} Q$ is a strongly connected set, $Q \subset \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1 \ge 0\}$, and its boundary ∂Q contains the origin; $A = B(\mathbf{0}, r_0)$ is an open ball centered at the origin with a radius $r_0 < 1/2$, so that $A \subset\subset Y$ and $K \subset\subset A$ (see Fig. 6.2 for a 2D example); $C_0 > 0$ is a constant independent of ε; the functions $f_\varepsilon \in L^2(\Omega)$, and $z_\varepsilon \in L^2(\Omega)$ are such that $f_\varepsilon \rightharpoonup f$ and $z_\varepsilon \to z^\partial$ in $L^2(\Omega)$ as $\varepsilon \to 0$. For any subset $E \subset \Omega$, we denote by $|E|$ its n-dimensional Lebesgue measure $\mathcal{L}_n(E)$, whereas $|\partial E|_H$ denotes the $(n-1)$-dimensional Hausdorff measure of the manifold ∂E on $\mathbb{R}^n$. We suppose that the sets $K \cap \partial Q^\varsigma$ and $\partial Q \setminus (K \cap \partial Q^\varsigma)$ have nonzero capacity for any $\varsigma > 1$, where

$$Q^\varsigma = \{\varsigma x, \ \forall\, x = (x_1, \ldots, x_n) \in Q\}$$

is the homothetic stretching of Q by a factor of ς. Hence, $|K \cap \partial Q^\varsigma|_H \ne 0$ for all $\varsigma > 1$.

Let $\mathcal{M}_b(\Omega)$ be the space of bounded Borel measures on Ω with values in $[0, +\infty]$. Let $\mathcal{M}_0^+(\Omega)$ be the cone of all non-negative Borel measures μ on Ω such that $\mu(B) = 0$ for every set $B \subseteq \Omega$ with $\operatorname{cap}(B, \Omega) = 0$ and $\mu(B) =$

$\inf\{\mu(U) \ : \ U \text{ quasi open}, B \subseteq U\}$ for every Borel set $B \subseteq \Omega$. Note that if $\mu \in \mathcal{M}_0^+(\Omega)$, then the functions of $H^1(\Omega)$ are defined μ-almost everywhere and are μ-measurable in Ω; hence, the space $H^1(\Omega) \cap L^2(\Omega, d\mu)$ is well defined.

We now give the variational formulation of the boundary value problem (12.5). For this, we set

$$H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon) = \left\{ y_\varepsilon \in H^1(\Omega_\varepsilon) \ : \ y_\varepsilon = 0 \text{ on } \Gamma_\varepsilon^D \cup \Sigma_\varepsilon \right\}$$

and fix a function $a_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $\|a_\varepsilon\|_{H^2(\Omega)} \le C_0$ and $a_\varepsilon|_{\Gamma_\varepsilon^D} = u_\varepsilon$ (the so-called prototype of a control function $u_\varepsilon \in U_\varepsilon$). It is clear that for every admissible control $u_\varepsilon \in U_\varepsilon$, there exists at least one prototype. Then the variational version of the problem (12.5) consists in looking for a function $y_\varepsilon = y_\varepsilon(u_\varepsilon, p_\varepsilon)$ such that

$$\begin{aligned} &y_\varepsilon - a_\varepsilon \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon), \\ &\int_{\Omega_\varepsilon} (\nabla y_\varepsilon \cdot \nabla \varphi) \, \mathrm{d}x + \int_{\Omega_\varepsilon} y_\varepsilon \, \varphi \, \mathrm{d}x + k_0 \int_{\Gamma_\varepsilon^N} y_\varepsilon \, \varphi \, \mathrm{d}\mathcal{H}^{n-1} \\ &\quad = \int_{\Omega_\varepsilon} f_\varepsilon \, \varphi \, \mathrm{d}x + \int_{\Gamma_\varepsilon^N} p_\varepsilon \, \varphi \, \mathrm{d}\mathcal{H}^{n-1}, \quad \forall \, \varphi \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon). \end{aligned} \tag{12.9}$$

Note that the trace of y_ε equals to u_ε almost everywhere on Γ_ε^D. In the sequel, we call the function y_ε the weak solution to the problem (12.5) and identify y_ε with its quasi-continuous representative [82].

For the weak solution of the problem (12.5), we have the following existence and uniqueness result.

Theorem 12.1. *For every fixed ε and for any control functions $u_\varepsilon \in H^1(\Gamma_\varepsilon^D)$ and $p_\varepsilon \in L^2(\Gamma_\varepsilon^N)$ the boundary value problem (12.5) has a unique weak solution $y_\varepsilon \in H^1(\Omega_\varepsilon; \Sigma_\varepsilon)$ such that the a priori norm-estimate*

$$\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C \left[\|f_\varepsilon\|_{L^2(\Omega)} + D_1(\varepsilon) \|u_\varepsilon\|_{H^1(\Gamma_\varepsilon^D)} + D_2(\varepsilon) \|p_\varepsilon\|_{L^2(\Gamma_\varepsilon^N)} \right] \tag{12.10}$$

holds true, where C, $D_1(\varepsilon)$, and $D_2(\varepsilon)$ are some positive constants and C is independent of ε.

Proof. Since $\partial\Omega_\varepsilon$ is Lipschitz continuous and there exists a constant $\gamma > 0$ depending only on Ω_ε such that

$$\|y_\varepsilon\|_{H^{1/2}(\partial\Omega_\varepsilon)} \le \gamma \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)},$$

it follows that the bilinear form

$$< \nabla y_\varepsilon, \nabla v_\varepsilon >_{L^2(\Omega_\varepsilon)} + < y_\varepsilon, v_\varepsilon >_{L^2(\Omega_\varepsilon)} + k_0 < y_\varepsilon, v_\varepsilon >_{H^{1/2}(\partial\Omega_\varepsilon)}$$

is coercive on the space $H^1(\Omega_\varepsilon)$. Hence, the same assertion holds for the space $H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$. Thus, the required result immediately follows from [167, Theorems 1.1 and 2.2, Chap. IV]. □

We are now in a position to prove the existence of a solution to the $\mathbb{P}_\varepsilon$-problem.

Theorem 12.2. *For every value of ε, the OCP $\mathbb{P}_\varepsilon$ has a unique solution $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$.*

Proof. Let $\tau_\varepsilon = \big(w_{H^1(\Gamma_\varepsilon^D)}\big) \times \big(w_{L^2(\Gamma_\varepsilon^N)}\big) \times \big(w_{H^1(\Omega_\varepsilon)}\big)$ be the basic topology on the set of admissible triplets $\Xi_\varepsilon \subset H^1(\Gamma_\varepsilon^D) \times L^2(\Gamma_\varepsilon^N) \times H^1(\Omega_\varepsilon)$, where by $w_{(\cdot)}$ we denote the weak topologies of the corresponding Banach spaces. It is easy to see that this set is nonempty, convex, and τ_ε-closed for every ε. Hence, we may choose a minimizing sequence $\big\{(u_\varepsilon^k, p_\varepsilon^k, y_\varepsilon^k)\big\}_{k=1}^\infty \subset \Xi_\varepsilon$ such that

$$\lim_{k\to\infty} I_\varepsilon(u_\varepsilon^k, p_\varepsilon^k, y_\varepsilon^k) = \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon)\in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon).$$

The boundedness of the sequence $\big\{I_\varepsilon(u_\varepsilon^k, p_\varepsilon^k, y_\varepsilon^k)\big\}_{k=1}^\infty$ implies the boundedness of $\big\{p_\varepsilon^k \in L^2(\Gamma_\varepsilon^N)\big\}_{k=1}^\infty$. Let $\big\{a_\varepsilon^k\big\} \subset H_0^1(\Omega) \cap H^2(\Omega)$ be a sequence of the corresponding prototypes for $\big\{u_\varepsilon^k\big\} \subset U_\varepsilon$. Since this sequence is bounded, we may always suppose that $a_\varepsilon^k \rightharpoonup a_\varepsilon^0$ in $H^2(\Omega)$. Then the compact imbedding $H^2(\Omega) \hookrightarrow H^1(\Gamma_\varepsilon^D)$ implies that $u_\varepsilon^k \to u_\varepsilon^0 = a_\varepsilon^0|_{\Gamma_\varepsilon^D}$ strongly in $H^1(\Gamma_\varepsilon^D)$. As a result, using (12.10), we see that the sequence $\big\{(u_\varepsilon^k, p_\varepsilon^k, y_\varepsilon^k)\big\}_{k=1}^\infty \subset \Xi_\varepsilon$ is relatively sequentially τ_ε-compact. Hence, we may extract subsequences (still indexed by k) such that

$$p_\varepsilon^k \rightharpoonup p_\varepsilon^0 \ \text{ in } \ L^2(\Gamma_\varepsilon^N), \ u_\varepsilon^k \to u_\varepsilon^0 \ \text{ in } \ H^1(\Gamma_\varepsilon^D), \ y_\varepsilon^k \to y_\varepsilon^0 \ \text{ in } \ L^2(\Omega_\varepsilon)$$

for some $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in H^1(\Gamma_\varepsilon^D) \times L^2(\Gamma_\varepsilon^N) \times H^1(\Omega_\varepsilon)$. Taking this fact into account and passing to the limit in (12.9) as $k \to \infty$, we deduce $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$. Finally, using the lower τ_ε-semicontinuity of the cost functional I_ε, we obtain

$$I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \le \liminf_{k\to\infty} I_\varepsilon(u_\varepsilon^k, p_\varepsilon^k, y_\varepsilon^k) = \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon)\in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon).$$

Thus, we have shown that the triplet $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ is a solution for the problem $\mathbb{P}_\varepsilon$. The uniqueness of such a solution is an obvious consequence of the property of strict convexity of I_ε. This concludes the proof. □

12.2 Reformulation of the original problem in terms of singular measures

We begin this section with the description of the geometry of the perforated domain Ω_ε as we did it in Sect. 6.9. We describe the class of admissible solutions to the problem $\mathbb{P}_\varepsilon$ in terms of singular periodic Borel measures on $\mathbb{R}^n$.

Let us denote by K^λ and Q^h the homothetic contractions of the sets K and Q, respectively, by factors of λ^{-1} and h^{-1}. In what follows, it is assumed

that $0 < \lambda << h < 1$. Let the sets $\Gamma^{\lambda,h}$ and $\Lambda^{\lambda,h}$ be defined as follows (see (6.131)):

$$\Gamma^{\lambda,h} = K^{\lambda} \cap \partial Q^h, \quad \Lambda^{\lambda,h} = \partial Q^h \setminus \Gamma^{\lambda,h}. \tag{12.11}$$

Let $\mu^{\lambda,h}$ and $\nu^{\lambda,h}$ be the normalized periodic Borel measures on $\mathbb{R}^n$ with the periodicity cell Y such that $\mu^{\lambda,h}$ is concentrated on $\Gamma^{\lambda,h}$, $\nu^{\lambda,h}$ is concentrated on $\Lambda^{\lambda,h}$, and both of these measures are proportional to the $(n-1)$-dimensional Hausdorff measure. Since these measures are concentrated and uniformly distributed on the corresponding sets, it follows that $\mu^{\lambda,h}(Y \setminus \Gamma^{\lambda,h}) = 0$.

It is easy to see that for any function $\varphi \in C^{\infty}(\mathbb{R}^n)$, we have relations (6.132) and (6.133). We introduce the scaling measures $\mu_{\varepsilon}^{\lambda,h}$ and $\nu_{\varepsilon}^{\lambda,h}$ by setting

$$\mu_{\varepsilon}^{\lambda,h}(B) = \varepsilon^n \mu^{\lambda,h}(\varepsilon^{-1}B), \quad \nu_{\varepsilon}^{\lambda,h}(B) = \varepsilon^n \nu^{\lambda,h}(\varepsilon^{-1}B)$$

for every Borel set $B \subset \mathbb{R}^n$, and we relate the parameters λ, h, and ε by the rule

$$\begin{gathered} h(\varepsilon) = \varepsilon^{n/(n-1)}, \quad \lambda(\varepsilon) = \varepsilon^{n/(n-2)} \text{ if } n \geq 3, \\ \lambda(\varepsilon) = \exp(-\varepsilon^{-2}) \text{ if } n = 2. \end{gathered} \tag{12.12}$$

Then

$$\int_{\varepsilon Y} \mathrm{d}\mu_{\varepsilon}^{\lambda,h} = \varepsilon^n \int_Y \mathrm{d}\mu^{\lambda,h} = \varepsilon^n, \quad \int_{\varepsilon Y} \mathrm{d}\nu_{\varepsilon}^{\lambda,h} = \varepsilon^n \int_Y \mathrm{d}\nu^{\lambda,h} = \varepsilon^n.$$

This means that the measures $\mu_{\varepsilon}^{\lambda,h}$ and $\nu_{\varepsilon}^{\lambda,h}$ weakly converge to the Lebesgue measure: $\mathrm{d}\mu_{\varepsilon}^{\lambda,h} \rightharpoonup \mathrm{d}x$, $\mathrm{d}\nu_{\varepsilon}^{\lambda,h} \rightharpoonup \mathrm{d}x$, that is, for every $\varphi \in C_0^{\infty}(\mathbb{R}^n)$, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu_{\varepsilon}^{\lambda,h} = \int_{\mathbb{R}^n} \varphi \, \mathrm{d}x, \quad \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\nu_{\varepsilon}^{\lambda,h} = \int_{\mathbb{R}^n} \varphi \, \mathrm{d}x. \tag{12.13}$$

Remark 12.3. It is easy to see that the scaling measures $\mu_{\varepsilon}^{\lambda,h}$ and $\nu_{\varepsilon}^{\lambda,h}$ belong to the class $\mathcal{M}_0^+(\Omega)$. Indeed, in view of our initial supposition, $\operatorname{cap}(K \cap \partial Q^{h/\lambda}) \neq 0$, so $\mu^{\lambda,h}(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\operatorname{cap}(B, \Omega) = 0$. Hence, the first property of the cone $\mathcal{M}_0^+(\Omega)$ for the measure $\mu_{\varepsilon}^{\lambda(\varepsilon),h(\varepsilon)}$ is valid. As for the second property, the measure $\mu^{\lambda,h}$ is concentrated on $\Gamma^{\lambda,h}$ and proportional there to the $(n-1)$-dimensional Hausdorff measure. Thus, $\mu^{\lambda,h} \in \mathcal{M}_0^+(\Omega)$. Since the scaling measure $\mu_{\varepsilon}^{\lambda(\varepsilon),h(\varepsilon)}$ inherits these properties from $\mu^{\lambda,h}$, it follows that $\mu_{\varepsilon}^{\lambda(\varepsilon),h(\varepsilon)} \in \mathcal{M}_0^+(\Omega)$ for every $\varepsilon > 0$. The same conclusion is true for the measure $\nu^{\lambda,h}$. Hence, the spaces $H_0^1(\Omega) \cap H^2(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_{\varepsilon}^{\lambda,h})$ and $H^1(\Omega; \Sigma_{\varepsilon}) \cap L^2(\Omega, \mathrm{d}\nu_{\varepsilon}^{\lambda,h})$ are well defined.

Now, we turn back to the definition of the set of admissible solutions of the problem $\mathbb{P}_{\varepsilon}$ (see (12.1)). We see that

$$\Gamma_{\varepsilon}^D = \bigcup_{\mathbf{k} \in \Theta_{\varepsilon}} \left[K^{\lambda(\varepsilon)} \cap \partial Q^{h(\varepsilon)} + \varepsilon \mathbf{k} \right],$$

$$\Gamma_{\varepsilon}^N = \bigcup_{\mathbf{k} \in \Theta_{\varepsilon}} \left[\partial Q^{h(\varepsilon)} \setminus (K^{\lambda(\varepsilon)} \cap \partial Q^{h(\varepsilon)}) + \varepsilon \mathbf{k} \right].$$

Then, using properties (6.132) and (6.133) and setting

$$\sigma(\varepsilon) = \left(h(\varepsilon)^{n-1}|\partial Q|_H - \lambda(\varepsilon)^{n-1}|K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H\right), \tag{12.14}$$

the term $\displaystyle\int_{\Gamma_\varepsilon^N} p_\varepsilon\, \varphi\, \mathrm{d}\mathcal{H}^{n-1}$ of the integral identity (12.9) can be rewritten in the form

$$\begin{aligned}\int_{\Gamma_\varepsilon^N} p_\varepsilon\, \varphi\, \mathrm{d}\mathcal{H}^{n-1} &= \sum_{\mathbf{k}\in\Theta_\varepsilon} \int_{\partial Q^{h(\varepsilon)}\setminus \Gamma^{\lambda(\varepsilon),h(\varepsilon)}+\varepsilon\mathbf{k}} p_\varepsilon\, \varphi\, \mathrm{d}\mathcal{H}^{n-1} \\ &= \sigma(\varepsilon) \sum_{\mathbf{k}\in\Theta_\varepsilon} \int_{\varepsilon(Y+\mathbf{k})} \widehat{p}_\varepsilon\, \varphi\, \mathrm{d}\nu^{\lambda(\varepsilon),h(\varepsilon)}(x/\varepsilon) \\ &= \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega \widehat{p}_\varepsilon\, \varphi\, \mathrm{d}\nu_\varepsilon^{\lambda,h} \end{aligned} \tag{12.15}$$

for every function

$$\varphi \in C_0^\infty(\mathbb{R}^n; \Sigma_\varepsilon \cup \Gamma_\varepsilon^D) = \left\{\psi \in C_0^\infty(\mathbb{R}^n) \ : \ \psi = 0 \text{ on } \ \Sigma_\varepsilon \cup \Gamma_\varepsilon^D\right\}.$$

Here, $\widehat{p}_\varepsilon$ is a function of $L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$ taking the same values as $p_\varepsilon \in L^2(\Gamma_\varepsilon^N)$ on Γ_ε^N. It is clear that for every boundary control $p_\varepsilon \in L^2(\Gamma_\varepsilon^N)$, one can find a function $\widehat{p}_\varepsilon \in L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$ such that $p_\varepsilon = \widehat{p}_\varepsilon$ on Γ_ε^N. Hence,

$$\int_{\Gamma_\varepsilon^N} p_\varepsilon^2\, \mathrm{d}\mathcal{H}^{n-1} = \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega \widehat{p}_\varepsilon{}^2\, \mathrm{d}\nu_\varepsilon^{\lambda,h}, \tag{12.16}$$

where

$$\varepsilon^{-n}\sigma(\varepsilon) = \begin{cases} |\partial Q|_H - \varepsilon^{n/(n-2)}|K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H & \text{if } n \geq 3, \\ |\partial Q|_H - \dfrac{1}{\varepsilon^2}\exp(-\dfrac{1}{\varepsilon^2})|K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H & \text{if } n = 2. \end{cases} \tag{12.17}$$

By analogy, we obtain

$$k_0 \int_{\Gamma_\varepsilon^N} y_\varepsilon\, \varphi\, \mathrm{d}\mathcal{H}^{n-1} = \varepsilon^{-n}\sigma(\varepsilon) k_0 \int_\Omega \breve{y}_\varepsilon\, \varphi\, \mathrm{d}\nu_\varepsilon^{\lambda,h}, \tag{12.18}$$

$$\int_{\Gamma_\varepsilon^D} u_\varepsilon^2\, \mathrm{d}\mathcal{H}^{n-1} = \varepsilon^{-n}\lambda(\varepsilon)^{n-1}|K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H \int_\Omega a_\varepsilon^2\, \mathrm{d}\mu_\varepsilon^{\lambda,h}. \tag{12.19}$$

Here, $\breve{y}_\varepsilon \in H^1(\Omega; \Sigma_\varepsilon)$ is an extension of the weak solution y_ε to the problem (12.5) to the whole of domain Ω, and the function $a_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})$ is a prototype of the Dirichlet control $u_\varepsilon \in U_\varepsilon$ (see (12.6)).

Remark 12.4. In view of our initial assumptions, the measure $\nu_\varepsilon^{\lambda,h}$ is supported on a set with nonzero capacity for every $\varepsilon > 0$. Since every element v of the space $H^1(\Omega; \Sigma_\varepsilon) \cap L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$ can be interpreted as a quasi-continuous

function, it is reasonable to suppose that for every element $v \in H^1(\Omega)$, one can find a sequence $\{v_k \in C(\overline{\Omega})\}_{k\in\mathbb{N}}$ such that

$$\sup_{k\in\mathbb{N}} \limsup_{\varepsilon\to 0} \int_{\mathcal{V}_k} (v - v_k)^2 \, \mathrm{d}\nu_\varepsilon^{\lambda,h} < +\infty \quad \text{and} \quad \lim_{k\to\infty} \mathrm{cap}(\mathcal{V}_k, \Omega) = 0,$$

where $\mathcal{V}_k = \{x \in \Omega \,:\, v \neq v_k\}$. We assume that the same property is valid for the elements of the space $H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})$.

So the original boundary value problem (12.5) can be rewritten in the form

$$\begin{aligned}\int_\Omega \chi_\varepsilon (\nabla \breve{y}_\varepsilon \cdot \nabla\varphi)\,\mathrm{d}x + \int_\Omega \chi_\varepsilon \breve{y}_\varepsilon \varphi \,\mathrm{d}x + k_0 \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega \breve{y}_\varepsilon \,\varphi \,\mathrm{d}\nu_\varepsilon^{\lambda(\varepsilon),h(\varepsilon)} \\ = \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega p_\varepsilon \,\varphi \,\mathrm{d}\nu_\varepsilon^{\lambda(\varepsilon),h(\varepsilon)} \\ + \int_\Omega \chi_\varepsilon f_\varepsilon \,\varphi \,\mathrm{d}x, \ \forall \varphi \in H^1(\Omega;\ \Gamma_\varepsilon^D \cup \Sigma_\varepsilon), \qquad (12.20)\end{aligned}$$

$$\breve{y}_\varepsilon - a_\varepsilon \in H^1(\Omega;\ \Gamma_\varepsilon^D \cup \Sigma_\varepsilon), \qquad (12.21)$$

whereχ_ε denotes the characteristic function of the domain Ω_ε.

We can reformulate the original OCP $\mathbb{P}_\varepsilon$ (12.7)–(12.1) as follows: Find some $(a_\varepsilon{}^0, p_\varepsilon{}^0, y_\varepsilon{}^0) \in \mathbb{X}_\varepsilon$ such that

$$\widehat{I}_\varepsilon(a_\varepsilon{}^0, p_\varepsilon{}^0, y_\varepsilon{}^0) = \inf_{(a_\varepsilon, p_\varepsilon, y_\varepsilon)\in \widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon), \qquad (12.22)$$

where

$$\begin{aligned}\mathbb{X}_\varepsilon = \left[H_0^1(\Omega) \cap H^2(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})\right] \\ \times L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h}) \times \left[H^1(\Omega;\Sigma_\varepsilon) \cap L^2(\Omega, D\nu_\varepsilon^{\lambda,h})\right],\end{aligned}$$

$$\begin{aligned}\widehat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon) = \int_\Omega \chi_\varepsilon |\nabla \breve{y}_\varepsilon|^2 \,\mathrm{d}x \\ + \int_\Omega |\chi_\varepsilon \breve{y}_\varepsilon - z_\varepsilon^\partial|^2 \,\mathrm{d}x + \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega p_\varepsilon^2 \,\mathrm{d}\nu_\varepsilon^{\lambda,h} \\ + |K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H \int_\Omega a_\varepsilon^2 \,\mathrm{d}\mu_\varepsilon^{\lambda,h}, \qquad (12.23)\end{aligned}$$

$$\widehat{\Xi}_\varepsilon = \left\{ (a_\varepsilon, p_\varepsilon, y_\varepsilon) \;\middle|\; \begin{array}{c} \breve{y}_\varepsilon - a_\varepsilon \in H^1(\Omega;\ \Gamma_\varepsilon^D \cup \Sigma_\varepsilon), \\ \|a\|_{H^2(\Omega)} \le C_0, \\ \displaystyle\int_\Omega \chi_\varepsilon (\nabla \breve{y}_\varepsilon \cdot \nabla\varphi)\,\mathrm{d}x + \int_\Omega \chi_\varepsilon \breve{y}_\varepsilon \varphi \,\mathrm{d}x \\ \displaystyle + k_0 \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega \breve{y}_\varepsilon \,\varphi \,\mathrm{d}\nu_\varepsilon^{\lambda,h} - \int_\Omega \chi_\varepsilon f_\varepsilon \,\varphi \,\mathrm{d}x \\ \displaystyle = \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega p_\varepsilon \,\varphi \,\mathrm{d}\nu_\varepsilon^{\lambda,h}, \\ \forall \varphi \in H^1(\Omega;\ \Gamma_\varepsilon^D \cup \Sigma_\varepsilon) \end{array} \right\}. \qquad (12.24)$$

We denote by $\widehat{\mathbb{P}}_\varepsilon$ the OCP (12.22)–(12.24). It is clear that $\widehat{\mathbb{P}}_\varepsilon$ has a unique solution $(a_\varepsilon{}^0, p_\varepsilon{}^0, y_\varepsilon{}^0)$ for every ε ([169, 111]). This solution can be viewed as a prototype of the optimal triplet to the $\mathbb{P}_\varepsilon$-problem. Moreover, in this case, the a priori norm estimate (12.10) takes the form

$$\begin{aligned}\|\breve{y}_\varepsilon\|_{H^1(\Omega,\ \chi_\varepsilon\,\mathrm{d}x)} + \sqrt{\varepsilon^{-n}\sigma(\varepsilon)}\,\|y_\varepsilon\|_{L^2(\Omega,\mathrm{d}\nu_\varepsilon^{\lambda,h})} \\ \leq \widehat{C}\left[\sqrt{\varepsilon^{-n}\sigma(\varepsilon)}\ \|p_\varepsilon\|_{L^2(\Omega,\mathrm{d}\nu_\varepsilon^{\lambda,h})} + \|f_\varepsilon\|_{L^2(\Omega)}\right. \\ \left. + \sqrt{|K\cap\partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H}\ \|a_\varepsilon\|_{H_0^1(\Omega)\cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^{\lambda,h})}\right]. \quad (12.25)\end{aligned}$$

We set

$$\varsigma(\varepsilon) = h(\varepsilon)/\lambda(\varepsilon) = \left\{\begin{array}{ll}\exp\left(\dfrac{-n}{n^2-3n+2}\ln\varepsilon\right) & \text{if}\ \ n\geq 3,\\ \varepsilon^2\exp\left(\dfrac{1}{\varepsilon^2}\right) & \text{if}\ \ n=2.\end{array}\right\} \quad (12.26)$$

Then $\varsigma(\varepsilon)\in(1,+\infty)$, $\forall\,\varepsilon$, and $\lim_{\varepsilon\to 0}\varsigma(\varepsilon)=+\infty$. Let $Q^\varsigma(\varepsilon)=\{\varsigma(\varepsilon)x,\ \forall\, x=(x_1,\dots,x_n)\in Q\}$ be the homothetic stretching of the set Q by a factor of $\varsigma(\varepsilon)$. We recall some results of Sect. 6.9 (see Propositions 6.66 and 6.68) that will be useful later.

Proposition 12.5. *There exists an open cone $\Lambda\subset\{x\in\mathbb{R}^n\ :\ x_1>0\}$ such that*

$$\lim_{\varepsilon\to 0}|K\cap\partial Q^\varsigma(\varepsilon)|_H = |K\cap\partial\Lambda|_H. \quad (12.27)$$

This cone can be recovered in an explicit form for the case when the origin belongs to a smooth part of the boundary ∂Q. Moreover, in view of (12.14), we have $\lim_{\varepsilon\to 0}\sigma(\varepsilon)/\varepsilon^n = |\partial Q|_H$.

Proposition 12.6. *Let $\{\rho'_\varepsilon\in\mathbb{R}\}_{\varepsilon>0}$ be numerical sequence such that*

$$\rho'_\varepsilon = \mathcal{L}_n\left(A\setminus Q^{\varsigma(\varepsilon)}\right)/\mathcal{L}_n(A),\quad \forall\,\varepsilon>0. \quad (12.28)$$

Then the sequence $\{\rho'_\varepsilon\}_{\varepsilon>0}$ is monotone and there exists a value $\rho^*\in[1/2,1)$ such that $\lim_{\varepsilon\to 0}\rho'_\varepsilon=\rho^*$.

12.3 Convergence in the variable space $\mathbb{X}_\varepsilon$

Assume that the parameters $\lambda=\lambda(\varepsilon)$ and $h=h(\varepsilon)$ are defined by (12.12). We begin with the following concept.

Definition 12.7. *Let $\left\{v_\varepsilon^{\lambda,h}\in H_0^1(\Omega)\cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^{\lambda,h})\right\}$ be a bounded sequence. We say that this sequence converges weakly in $H_0^1(\Omega)\cap L^2(\Omega,\mathrm{d}\mu_\varepsilon^{\lambda,h})$ to $v\in H^1(\Omega)$ if*

$$v_\varepsilon^{\lambda,h}\rightharpoonup v\ \ \text{in}\ \ H_0^1(\Omega)\ \ \text{and}\ \ v_\varepsilon^{\lambda,h}\rightharpoonup v\ \ \text{in}\ \ L^2(\Omega,\mathrm{d}\mu_\varepsilon^{\lambda,h}). \quad (12.29)$$

In order to check the correctness of this definition, we make use of the following auxiliary statements.

Lemma 12.8. *If $v \in H^1_0(\Omega)$, then*

$$\lim_{\varepsilon\to 0}\int_\Omega v\,\varphi\,\mathrm{d}\mu_\varepsilon^{\lambda,h} = \int_\Omega v\,\varphi\,\mathrm{d}x, \quad \forall\,\varphi \in C_0^\infty(\Omega).$$

Proof. If $v \in C_0(\Omega)$, then the weak convergence $\mathrm{d}\mu_\varepsilon^{\lambda,h} \rightharpoonup \mathrm{d}x$ of the measures immediately implies this relation. Let v be an arbitrary element of $H^1_0(\Omega)$. Then for every $\delta > 0$, there exist a set $\mathcal{A}^\delta \subset \Omega$ and a function $v^\delta \in C_0(\Omega)$ such that $\mathrm{cap}(\mathcal{A}^\delta, \Omega) < \delta$ and $v^\delta = v$ on $\Omega \setminus \mathcal{A}^\delta$. In what follows, we suppose that the set $\mathcal{A}^\delta$ is closed. We now consider the following estimate:

$$\begin{aligned}\Big|\int_\Omega v\,\varphi\,\mathrm{d}\mu_\varepsilon^{\lambda,h} - \int_\Omega v\varphi\,\mathrm{d}x\Big| &\le \Big|\int_\Omega (v-v^\delta)\,\varphi\,\mathrm{d}\mu_\varepsilon^{\lambda,h}\Big| + \Big|\int_\Omega v^\delta\,\varphi\,\mathrm{d}\mu_\varepsilon^{\lambda,h} - \int_\Omega v^\delta\varphi\,\mathrm{d}x\Big| \\ &+ \Big|\int_\Omega (v-v^\delta)\,\varphi\,\mathrm{d}x\Big| = J_1 + J_2 + J_3.\end{aligned}$$

Owing to the weak convergence $\mu_\varepsilon^{\lambda,h} \rightharpoonup dx$, we have $J_2 \to 0$ as $\varepsilon \to 0$. By Lusin's theorem, we may suppose that there is a constant $d_1 > 0$ such that $\|v - v^\delta\|_{L^2(\Omega)} \le d_1\delta$. Hence, $J_3 \le d_1\|\varphi\|_{L^2(\Omega)}\delta$. As for the value J_1, we note that each measure $\mu_\varepsilon^{\lambda,h}$ is supported on a set with nonzero capacity. So, there is a constant $d_2 > 0$ such that $\|v - v^\delta\|_{L^2(\Omega,\mathrm{d}\mu_\varepsilon^{\lambda,h})} \le d_2$ for any $\delta > 0$ small enough (see Remark 12.4). It implies the following estimate:

$$J_3 = \Big|\int_\Omega (v-v^\delta)\,\varphi\,\mathrm{d}\mu_\varepsilon^{\lambda,h}\Big| \le d_2\,\|\varphi\|_{C(\Omega)}\,\big[\mu_\varepsilon^{\lambda,h}(\mathcal{A}^\delta)\big]^{1/2}.$$

As a result, we have the following:

(i) $\limsup_{\varepsilon\to 0}\mu_\varepsilon^{\lambda,h}(\mathcal{A}^\delta) \le |\mathcal{A}^\delta|$ since the convergence $\mu_\varepsilon^{\lambda,h} \rightharpoonup dx$ implies the following (see [258]):

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi\,\mathrm{d}\mu_\varepsilon^{\lambda,h} = \int_\Omega \varphi\,\mathrm{d}x, \quad \forall\,\varphi \in C(\overline{\Omega}), \tag{12.30}$$

$$\limsup_{\varepsilon\to 0}\mu_\varepsilon^{\lambda,h}(F) \le |F| \quad \text{for any compact set } F \subset \Omega. \tag{12.31}$$

(ii) $|\mathcal{A}^\delta| \le d_3\mathrm{cap}^{n/(n-2)}(\mathcal{A}^\delta)$ by the properties of capacity (see [106]).
(iii) $\mathrm{cap}(\mathcal{A}^\delta) < \delta$ by the initial assumption.

Hence, summing up all estimates that were obtained earlier, we conclude

$$\Big|\int_\Omega v\,\varphi\,\mathrm{d}\mu_\varepsilon^{\lambda,h} - \int_\Omega v\varphi\,\mathrm{d}x\Big| \le d\delta$$

for any $\delta > 0$ small enough (here, a constant d does not depend on δ). This completes the proof. □

We now consider a more delicate situation.

Lemma 12.9. *Let* $\{v_\varepsilon^{\lambda,h} \in H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})\}$ *and* $v \in L^2(\Omega)$ *be such that* $v_\varepsilon^{\lambda,h} \rightharpoonup v$ *in* $H_0^1(\Omega)$*, and hence* $v_\varepsilon^{\lambda,h} \to v$ *in* $L^2(\Omega)$*. Then*

$$\lim_{\varepsilon\to 0}\left[\int_\Omega v_\varepsilon^{\lambda,h}\,\mathrm{d}\mu_\varepsilon^{\lambda,h} - \int_\Omega v_\varepsilon^{\lambda,h}\,\mathrm{d}x\right] = 0. \tag{12.32}$$

Proof. As in Lemma 12.8, we introduce two functions $\widetilde{v}_\varepsilon^{\lambda,h} \in C(\Omega)$ and $\widetilde{v} \in C(\Omega)$ such that $\widetilde{v}_\varepsilon^{\lambda,h} = v_\varepsilon^{\lambda,h}$ and $\widetilde{v} = v$ quasi-everywhere. Let us partition the set Ω into cubes εY with edges ε and denote these cubes by εY^j. Then there are points $x_j^{\lambda,h} \in \varepsilon Y^j$ such that

$$\begin{aligned}\int_\Omega \widetilde{v}_\varepsilon^{\lambda,h}\,\mathrm{d}\mu_\varepsilon^{\lambda,h} &= \sum\int_{\varepsilon Y^j} \widetilde{v}_\varepsilon^{\lambda,h}(x)\,\mathrm{d}\mu_\varepsilon^{\lambda,h} + \sum\int_{\Omega\cap\varepsilon Y^j} \widetilde{v}_\varepsilon^{\lambda,h}(x)\,\mathrm{d}\mu_\varepsilon^{\lambda,h}\\ &= \sum \widetilde{v}_\varepsilon^{\lambda,h}(x_j^{\lambda,h})\int_{\varepsilon Y^j}\mathrm{d}\mu_\varepsilon^{\lambda,h} + \sum\int_{\Omega\cap\varepsilon Y^j} \widetilde{v}_\varepsilon^{\lambda,h}(x)\,\mathrm{d}\mu_\varepsilon^{\lambda,h},\end{aligned}$$

where the second sum is calculated over the set of the "boundary" cubes. By the definition of the measure $\mu_\varepsilon^{\lambda,h}$, we have

$$\int_{\varepsilon Y^j}\mathrm{d}\mu_\varepsilon^{\lambda,h} = \varepsilon^n\int_Y \mathrm{d}\mu_\varepsilon^{\lambda,h} = \varepsilon^n.$$

Hence,

$$\int_\Omega \widetilde{v}_\varepsilon^{\lambda,h}\,\mathrm{d}\mu_\varepsilon^{\lambda,h} = \sum \widetilde{v}_\varepsilon^{\lambda,h}(x_j^{\lambda,h})\,\varepsilon^n + \sum\int_{\Omega\cap\varepsilon Y^j} \widetilde{v}_\varepsilon^{\lambda,h}(x)\,\mathrm{d}\mu_\varepsilon^{\lambda,h}. \tag{12.33}$$

It is clear that an analogous representation takes place for the second term in (12.32), namely

$$\begin{aligned}\int_\Omega \widetilde{v}_\varepsilon^{\lambda,h}\,\mathrm{d}x &= \sum \widetilde{v}_\varepsilon^{\lambda,h}(x_j)\int_{\varepsilon Y^j}\mathrm{d}x + \sum\int_{\Omega\cap\varepsilon Y^j} \widetilde{v}_\varepsilon^{\lambda,h}(x)\,\mathrm{d}x\\ &= \sum \widetilde{v}_\varepsilon^{\lambda,h}(x_j)\,\varepsilon^n + \sum\int_{\Omega\cap\varepsilon Y^j} \widetilde{v}_\varepsilon^{\lambda,h}(x)\,\mathrm{d}x\end{aligned} \tag{12.34}$$

for some $x_j \in \varepsilon Y^j$. Note that

$$\left|\sum\int_{\Omega\cap\varepsilon Y^j} \widetilde{v}_\varepsilon^{\lambda,h}(x)\,\mathrm{d}\mu_\varepsilon^{\lambda,h}\right| \le \sup_{j\in D(\varepsilon)}\left(\sup_{x\in\,\Omega\cap\varepsilon Y^j}|\widetilde{v}_\varepsilon^{\lambda,h}(x)|\right)\varepsilon^n\cdot D(\varepsilon),$$

$$\left|\sum\int_{\Omega\cap\varepsilon Y^j} \widetilde{v}(x)\,\mathrm{d}x\right| \le \sup_{j\in D(\varepsilon)}\left(\sup_{x\in\,\Omega\cap\varepsilon Y^j}|\widetilde{v}(x)|\right)\varepsilon^n\cdot D(\varepsilon),$$

where $D(\varepsilon)$ is the quantity of the "boundary" cubes, and $\varepsilon^n D(\varepsilon) \to 0$ by Jordan's measurability property of the set $\partial\Omega$. Moreover, since $v_\varepsilon^{\lambda,h}, v \in H_0^1(\Omega)$, it follows that

$$\sup_{j\in D(\varepsilon)}\left(\sup_{x\in\,\Omega\cap\varepsilon Y^j}|\widetilde{v}(x)|\right)<+\infty\quad\text{and}\quad\sup_{j\in D(\varepsilon)}\left(\sup_{x\in\,\Omega\cap\varepsilon Y^j}|\widetilde{v}_\varepsilon^{\lambda,h}(x)|\right)<+\infty$$

for ε small enough.

Then, substituting (12.33) and (12.34) into (12.32), we come to the following relation:

$$\begin{aligned}
&\lim_{\varepsilon\to0}\left(\int_\Omega v_\varepsilon^{\lambda,h}\,\mathrm{d}\mu_\varepsilon^{\lambda,h}-\int_\Omega v_\varepsilon^{\lambda,h}\,\mathrm{d}x\right)\\
&\quad\le\lim_{\varepsilon\to0}\sum\left(\widetilde{v}_\varepsilon^{\lambda,h}(x_j^{\lambda,h})-\widetilde{v}_\varepsilon^{\lambda,h}(x_j)\right)\varepsilon^n\\
&\quad+\max\left[\sup_{j\in D(\varepsilon)}\left(\sup_{x\in\,\Omega\cap\varepsilon Y^j}|\widetilde{v}_\varepsilon^{\lambda,h}(x)|\right),\sup_{j\in D(\varepsilon)}\left(\sup_{x\in\,\Omega\cap\varepsilon Y^j}|\widetilde{v}(x)|\right)\right]\\
&\quad\times\limsup_{\varepsilon\to0}\left(\varepsilon^n\,D(\varepsilon)\right)+\lim_{\varepsilon\to0}(J_1+J_2)\\
&\quad=\lim_{\varepsilon\to0}\left(\sum\left(\widetilde{v}_\varepsilon^{\lambda,h}(x_j^{\lambda,h})-\widetilde{v}_\varepsilon^{\lambda,h}(x_j)\right)\varepsilon^n\right),
\end{aligned}\tag{12.35}$$

where

$$J_1=\int_\Omega(v_\varepsilon^{\lambda,h}-\widetilde{v}_\varepsilon^{\lambda,h})\,\mathrm{d}\mu_\varepsilon^{\lambda,h},\quad J_2=\int_\Omega(v_\varepsilon^{\lambda,h}-\widetilde{v}_\varepsilon^{\lambda,h})\,\mathrm{d}x$$

and by the arguments of Lemma 12.8 and Remark 12.4, we may suppose that $\lim_{\varepsilon\to0}(J_1+J_2)=0$.

We now use the fact that $v_\varepsilon^{\lambda,h}\to v$ in $L^2(\Omega)$. One has

$$\begin{aligned}
\lim_{\varepsilon\to0}\int_\Omega(v_\varepsilon^{\lambda,h}-v)^2\,\mathrm{d}x&\le2\lim_{\varepsilon\to0}\int_\Omega(\widetilde{v}_\varepsilon^{\lambda,h}-\widetilde{v})^2\,\mathrm{d}x+2\lim_{\varepsilon\to0}J_0(\varepsilon)\\
&=2\lim_{\varepsilon\to0}\sum\left(\widetilde{v}_\varepsilon^{\lambda,h}(x_j^*)-\widetilde{v}(x_j^*)\right)^2\varepsilon^n=0,
\end{aligned}\tag{12.36}$$

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_\Omega\left(v_\varepsilon^{\lambda,h}\right)^2\,\mathrm{d}x-\int_\Omega v^2\,\mathrm{d}x\\
&\qquad=\lim_{\varepsilon\to0}\int_\Omega\left(\widetilde{v}_\varepsilon^{\lambda,h}\right)^2\,\mathrm{d}x-\int_\Omega\widetilde{v}^2\,\mathrm{d}x+\lim_{\varepsilon\to0}J(\varepsilon)\\
&\qquad=\lim_{\varepsilon\to0}\sum\left(\widetilde{v}_\varepsilon^{\lambda,h}(x_j^*)\right)^2\varepsilon^n-\lim_{\varepsilon\to0}\sum\left(\widetilde{v}(x_j)\right)^2\varepsilon^n\\
&\qquad=\lim_{\varepsilon\to0}\sum\left[\left(\widetilde{v}_\varepsilon^{\lambda,h}(x_j^*)\right)^2-\left(\widetilde{v}(x_j)\right)^2\right]\varepsilon^n,
\end{aligned}\tag{12.37}$$

where, as usual, we suppose that the values

$$J_0(\varepsilon)=\int_\Omega(\widetilde{v}_\varepsilon^{\lambda,h}-v_\varepsilon^{\lambda,h}+v-\widetilde{v})^2\,\mathrm{d}x$$

and

$$J(\varepsilon) = \int_\Omega \left[\left(v_\varepsilon^{\lambda,h}\right)^2 - \left(\widetilde{v}_\varepsilon^{\lambda,h}\right)^2 \right] \mathrm{d}x + \int_\Omega \left[v^2 - \widetilde{v}^2 \right] \mathrm{d}x$$

are arbitrarily small.

Hence, by (12.35), the construction of the Riemann sum, and the fact that $v \in H_0^1(\Omega)$, we conclude that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \left| \int_\Omega \widetilde{v}_\varepsilon^{\lambda,h}\, d\mu_\varepsilon^{\lambda,h} - \int_\Omega \widetilde{v}_\varepsilon^{\lambda,h}\, \mathrm{d}x \right| &\le \lim_{\varepsilon\to 0} \left| \sum \left(\left[\widetilde{v}_\varepsilon^{\lambda,h}(x_j^{\lambda,h}) - \widetilde{v}(x_j^{\lambda,h}) \right] \right.\right. \\
&\quad \left.\left. + \left[\widetilde{v}(x_j^{\lambda,h}) - \widetilde{v}(x_j) \right] + \left[\widetilde{v}(x_j) - \widetilde{v}_\varepsilon^{\lambda,h}(x_j) \right] \right) \varepsilon^n \right| \\
&\le 2\sqrt{|\Omega|} \lim_{\varepsilon\to 0} \left\| v_\varepsilon^{\lambda,h} - v \right\|_{L^2(\Omega)} + \limsup_{\varepsilon\to 0} \left| \sum \left[\widetilde{v}(x_j^{\lambda,h}) - \widetilde{v}(x_j) \right] \varepsilon^n \right| \\
&\le \left| \limsup_{\varepsilon\to 0} \left(\sum \widetilde{v}(x_j^{\lambda,h}) \varepsilon^n - \sum \widetilde{v}(x_j) \varepsilon^n \right) \right| \\
&= \left| \int_\Omega \widetilde{v} \mathrm{d}x - \int_\Omega \widetilde{v} \mathrm{d}x \right| = 0.
\end{aligned}$$

□

Taking into account the proofs of the previous lemmas and relations (12.36) and (12.37), the following statement is readily ascertained.

Lemma 12.10. *Let* $\left\{ v_\varepsilon^{\lambda,h} \in H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h}) \right\}$ *and* $v \in H_0^1(\Omega)$ *be such that* $v_\varepsilon^{\lambda,h} \rightharpoonup v$ *in* $H_0^1(\Omega)$. *Then*

$$\lim_{\varepsilon\to 0} \left[\int_\Omega \left(v_\varepsilon^{\lambda,h}\right)^2 \mathrm{d}\mu_\varepsilon^{\lambda,h} - \int_\Omega \left(v_\varepsilon^{\lambda,h}\right)^2 \mathrm{d}x \right] = 0, \tag{12.38}$$

$$\lim_{\varepsilon\to 0} \int_\Omega v^2 \, \mathrm{d}\mu_\varepsilon^{\lambda,h} = \int_\Omega v^2 \, \mathrm{d}x, \quad \forall\, v \in H_0^1(\Omega). \tag{12.39}$$

Remark 12.11. Since the set Ω is bounded and $|\partial\Omega \setminus \Sigma_\varepsilon|_H \sim \varepsilon^{1-n} h^{n-1}(\varepsilon) = \varepsilon$, it follows that $|\Sigma_\varepsilon|_H \to |\partial\Omega|_H$ as $\varepsilon \to 0$. Hence, by properties (12.30) and (12.31), the statements of Lemmas 12.8–12.10 remain valid if the space $H_0^1(\Omega)$ is changed to $H^1(\Omega, \Sigma_\varepsilon)$.

Theorem 12.12. *Every bounded sequence* $\left\{ v_\varepsilon^{\lambda,h} \in H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h}) \right\}$ *is relatively compact with respect to the weak convergence in the variable space* $H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})$.

Proof. Since the sequence $\left\{ v_\varepsilon^{\lambda,h} \right\}$ is bounded in $H_0^1(\Omega)$, we may suppose that there is an element $v \in H_0^1(\Omega)$ such that $v_\varepsilon^{\lambda,h} \to v$ weakly in $H_0^1(\Omega)$. Then the compact imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ implies the strong convergence $v_\varepsilon^{\lambda,h} \to v$ in $L^2(\Omega)$. Hence, for every $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
\left| \int_\Omega v_\varepsilon^{\lambda,h} \varphi \, \mathrm{d}\mu_\varepsilon^{\lambda,h} - \int_\Omega v\varphi \, \mathrm{d}x \right| &\le \left| \int_\Omega v_\varepsilon^{\lambda,h} \varphi \, \mathrm{d}x - \int_\Omega v_\varepsilon^{\lambda,h} \varphi \, \mathrm{d}\mu_\varepsilon^{\lambda,h} \right| \\
&\quad + \left| \int_\Omega v_\varepsilon^{\lambda,h} \varphi \, \mathrm{d}x - \int_\Omega v\varphi \, \mathrm{d}x \right|.
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ on the right-hand part of this inequality, we obtain

$$\left| \int_\Omega v_\varepsilon^{\lambda,h} \, \varphi \, \mathrm{d}\mu_\varepsilon^{\lambda,h} - \int_\Omega v_\varepsilon^{\lambda,h} \, \varphi \, \mathrm{d}x \right| \to 0 \quad \text{(by Lemma 12.9)}$$

and

$$\left| \int_\Omega (v_\varepsilon^{\lambda,h} - v) \, \varphi \, \mathrm{d}x \right| \to 0 \quad \text{as a weak limit in } L^2(\Omega).$$

The proof is complete. □

Using the above results, we introduce the concept of the weak convergence for the following sequences (for comparison, see Definition 12.7):

$$\left\{ y_\varepsilon \in H^1(\Omega_\varepsilon; \Sigma_\varepsilon) \, : \, \breve{y}_\varepsilon \in H^1(\Omega; \Sigma_\varepsilon) \cap L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h}) \right\}_{\varepsilon > 0}.$$

Here, $\breve{y}_\varepsilon$ is some extension of the function y_ε on the whole of Ω. Let us recall that the perforated domain Ω_ε considered here satisfies the so-called "condition of strong connectedness" (see [182]). It means that there exist a family $\{P_\varepsilon\}_{\varepsilon>0}$ of extension operators $P_\varepsilon : H^1(\Omega_\varepsilon; \Sigma_\varepsilon) \to H^1(\Omega; \Sigma_\varepsilon)$ and a constant C independent of ε such that $\|\nabla(P_\varepsilon y_\varepsilon)\|_{L^2(\Omega)} \le C \, \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ for every $y_\varepsilon \in H^1(\Omega_\varepsilon; \Sigma_\varepsilon)$. So, we can assume that $\breve{y}_\varepsilon := P_\varepsilon y_\varepsilon$ for some extension operator with the above properties.

Definition 12.13. *We say that a sequence* $\left\{ y_\varepsilon \in H^1(\Omega_\varepsilon; \Sigma_\varepsilon) \right\}_{\varepsilon>0}$ *is weakly convergent in* $H^1(\Omega; \Sigma_\varepsilon) \cap L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$ *if there exists an element* $y \in H_0^1(\Omega)$ *such that* $\breve{y}_\varepsilon \rightharpoonup y$ *in* $H^1(\Omega)$ *and* $y_\varepsilon \rightharpoonup y$ *in* $L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$.

We are now in a position to verify the correctness of this definition.

Theorem 12.14. *Every bounded sequence* $\left\{ y_\varepsilon \in H^1(\Omega_\varepsilon; \Sigma_\varepsilon) \right\}_{\varepsilon>0}$ *is relatively compact with respect to the weak convergence in the variable space* $H^1(\Omega; \Sigma_\varepsilon) \cap L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$.

Proof. Taking into account Remark 12.11 and the fact that

$$\int_\Omega \breve{y}_\varepsilon \, \varphi \chi_{\Omega_\varepsilon} \, \mathrm{d}x \overset{\varepsilon \to 0}{\longrightarrow} \int_\Omega y \, \varphi \, \mathrm{d}x,$$

$$\int_\Omega y_\varepsilon \, \varphi \, \mathrm{d}\nu_\varepsilon^{\lambda,h} = \int_\Omega \breve{y}_\varepsilon \, \varphi \, \mathrm{d}\nu_\varepsilon^{\lambda,h}, \quad \forall \, \varphi \in C_0^\infty(\Omega),$$

this theorem can be established in complete analogy with the proof of Theorem 12.12. The main difference is the addition of the property $v \in H_0^1(\Omega)$. However, $|\Sigma_\varepsilon|_H \to |\partial\Omega|_H$ as $\varepsilon \to 0$, and we obtain the required result. □

In fact, we can prove a more precise result.

Theorem 12.15. *Let* $\left\{ y_\varepsilon^{\lambda,h} \in H^1(\Omega_\varepsilon; \Sigma_\varepsilon) \cap L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h}) \right\}_{\varepsilon>0}$ *be a bounded sequence such that* $\breve{y}_\varepsilon^{\lambda,h} \rightharpoonup y$ *in* $H^1(\Omega; \Sigma_\varepsilon) \cap L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$. *Then* $y \in H_0^1(\Omega)$ *and* $y_\varepsilon^{\lambda,h} \to y$ *strongly in* $L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$.

Proof. By the criterium of strong convergence in $L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$, to establish the convergence $y_\varepsilon^{\lambda,h} \to y$ in $L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$ it is enough to show that

$$y_\varepsilon^{\lambda,h} \rightharpoonup y \ \text{ in } \ L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h}), \ \lim_{\varepsilon\to 0} \int_\Omega (y_\varepsilon^{\lambda,h})^2 \, \mathrm{d}\nu_\varepsilon^{\lambda,h} = \int_\Omega y^2 \, \mathrm{d}x. \tag{12.40}$$

The first statement in (12.40) is valid by Definition 12.13. In order to prove the second one, we apply the following estimate:

$$\begin{aligned} \left| \int_\Omega (y_\varepsilon^{\lambda,h})^2 \, \mathrm{d}\nu_\varepsilon^{\lambda,h} - \int_\Omega y^2 \, \mathrm{d}x \right| \le & \left| \int_\Omega (y_\varepsilon^{\lambda,h})^2 \, \mathrm{d}\nu_\varepsilon^{\lambda,h} - \int_\Omega (y_\varepsilon^{\lambda,h})^2 \, \mathrm{d}x \right| \\ & + \left| \int_\Omega (\breve{y}_\varepsilon^{\lambda,h})^2 \, \mathrm{d}x - \int_\Omega y^2 \, \mathrm{d}x \right|. \end{aligned} \tag{12.41}$$

The second term on the right-hand side of (12.41) tends to 0 as $\varepsilon \to 0$ by the strong convergence of $\breve{y}_\varepsilon^{\lambda,h}$ to y in $L^2(\Omega)$. The first one is equal to 0 at the limit as $\varepsilon \to 0$ by applying Lemma 12.10, and this concludes the proof. □

Let $\{(a_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible solutions for the original problem. We assume that this sequence is bounded. Then summing up the above given reasonings, we may introduce the following concept of the weak convergence in the variable space $\mathbb{X}_\varepsilon$.

Definition 12.16. *We say that a bounded sequence* $\{(a_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ *is w-convergent to a triplet* $(a, p, y) \in \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times L^2(\Omega) \times H_0^1(\Omega)$ *in the variable space* $\mathbb{X}_\varepsilon$ *as* ε *tends to* 0 *(in symbols,* $(a_\varepsilon, p_\varepsilon, y_\varepsilon) \overset{w}{\rightharpoonup} (a, p, y)$*) if the following hold:*

(i) $a_\varepsilon \rightharpoonup a$ *in* $H^2(\Omega)$ *and* $a_\varepsilon \rightharpoonup a$ *in* $L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})$*;*
(ii) $p_\varepsilon \rightharpoonup p$ *in* $L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$*;*
(iii) $\breve{y}_\varepsilon \rightharpoonup y$ *in* $H^1(\Omega)$ *and* $y_\varepsilon \rightharpoonup y$ *in* $L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$*.*

In view of Theorems 12.12 and 12.14, we come to the following conclusion:

Theorem 12.17. *Every bounded sequence of admissible solutions*

$$\{(a_\varepsilon, p_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$$

to the problems $\widehat{\mathbb{P}}_\varepsilon$ *is relatively compact with respect to the w-convergence in* $\mathbb{X}_\varepsilon$*.*

To conclude this section, we observe that for the characteristic function χ_ε of the perforated domain Ω_ε, the following result is obvious [73].

Lemma 12.18. χ_ε *converges strongly to* 1 *both in* $L^2(\Omega)$ *and in the variable space* $L^2(\Omega, \chi_\varepsilon \, \mathrm{d}x)$ *as* $\varepsilon \to 0$*.*

12.4 Definition of a limit problem and its property

Since our main goal is to study the asymptotic behavior of the problem $\widehat{\mathbb{P}}_\varepsilon$ as $\varepsilon \to 0$, we represent the $\widehat{\mathbb{P}}_\varepsilon$-problem for various values of ε in the form of the following sequence:

$$\left\{ \left\langle \inf_{(a_\varepsilon, p_\varepsilon, y_\varepsilon) \in \widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon) \right\rangle ; \varepsilon > 0 \right\}. \tag{12.42}$$

Then the definition of an appropriate limiting OCP to the family (12.22) can be reduced to the analysis of the limit properties of the sequence (12.42) as $\varepsilon \to 0$.

Definition 12.19. *We say that a minimization problem*

$$\left\langle \inf_{(a,p,y) \in \Xi_0} I_0(a, p, y) \right\rangle \tag{12.43}$$

is the weak variational limit of the sequence (12.42) with respect to the w-convergence in the variable space $\mathbb{X}_\varepsilon$ (or variational w-limit) if the conditions (d)–(dd) of Definition 8.24 hold true with

$$\begin{aligned} \mathbb{X}_\varepsilon &= \left[H_0^1(\Omega) \cap H^2(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h}) \right] \\ &\quad \times L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h}) \times \left[H^1(\Omega; \Sigma_\varepsilon) \cap L^2(\Omega, D\nu_\varepsilon^{\lambda,h}) \right], \\ \mathbb{X} &= \left[H^2(\Omega) \cap H_0^1(\Omega) \right] \times L^2(\Omega) \times H_0^1(\Omega). \end{aligned}$$

As a result, Theorem 8.25 (see also Corollary 8.37 of Theorem 8.35) implies the following properties of a weak variational limit in the sense of Definition 12.19.

Theorem 12.20. *Assume that the constrained minimization problem (12.43) is the weak variational limit of the sequence (12.42) and has a unique solution $(a^0, p^0, y^0) \in \Xi_0$. Let $\left\{ (a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \widehat{\Xi}_\varepsilon \right\}_{\varepsilon > 0}$ be the sequence of optimal triplets for the $\widehat{\mathbb{P}}_\varepsilon$-problems. Then*

$$(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \xrightarrow{w} (a^0, p^0, y^0), \tag{12.44}$$

$$\begin{aligned} \inf_{(a,p,y) \in \Xi_0} I_0(a, p, y) &= I_0\left(a^0, p^0, y^0\right) \\ &= \lim_{\varepsilon \to 0} \inf_{(a_\varepsilon, p_\varepsilon, y_\varepsilon) \in \widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon). \end{aligned} \tag{12.45}$$

12.5 Convergence theorem and correctors

The main question of this section is the homogenization of the boundary value problem (12.5). Let $H^1_{per}(Y)$ be the Sobolev space of Y-periodic functions. We begin with the following result.

Lemma 12.21. *There exists a sequence of functions* $\{w^{\lambda,h}\}_{h>\lambda>0}$ *satisfying the following:*

(H1) $w^{\lambda,h} \in H^1_{per}(Y)$, $w^{\lambda,h} = 0$ *on* $K^\lambda \cap \partial Q^h$, $0 \le w^{\lambda,h} \le 1$;
(H2) $w^{\lambda,h} = 1$ *in* $Y \setminus A^h$;
(H3) $w^{\lambda,h}(x_1, x_2, \dots, x_n) = w^{\lambda,h}(-x_1, x_2, \dots, x_n)$ $\forall\, x \in A^h$, $\forall\, h > \lambda > 0$;
(H4) $w^{\lambda,h} \rightharpoonup 1$ *weakly in* $H^1_{per}(Y)$ *and strongly in* $L^2_{per}(Y)$.

Proof. Let us define

$$\mathcal{A} = \Big\{ \{v^{\lambda,h}\} \,:\, v^{\lambda,h} = 0 \ \text{ on } K^\lambda \cap \partial Q^h,$$
$$v^{\lambda,h}(x_1, x_2, \dots, x_n) = v^{\lambda,h}(-x_1, x_2, \dots, x_n)$$
$$\forall\, x \in A^h,\ \forall\, h > \lambda > 0,\ v^{\lambda,h} \rightharpoonup 1 \text{ in } H^1_{per}(Y), v^{\lambda,h} = 1 \text{ in } \ Y \setminus A^h \Big\},$$

$$\alpha = \inf \left\{ \liminf_{(h>\lambda)\to 0} \int_Y |\nabla v^{\lambda,h}|^2 \,\mathrm{d}x \ :\ \{v^{\lambda,h}\} \in \mathcal{A} \right\}.$$

Note that the set $\mathcal{A}$ is not empty. Indeed, if we define the functions $v^{\lambda,h}$ as

$$v^{\lambda,h} \in H^1_{per}(Y),\ \Delta v^{\lambda,h} = 0 \text{ in } \ A^h \setminus A^\lambda,$$
$$v^{\lambda,h} = 0 \text{ in } \ A^\lambda,\ v^{\lambda,h} = 1 \text{ in } \ Y \setminus A^h,$$

one has $\{v^{\lambda,h}\} \in \mathcal{A}$ immediately. For any $k \in \mathbb{N}$, we consider a sequence $\{v_k^{\lambda,h}\} \in \mathcal{A}$ such that

$$\liminf_{(h>\lambda)\to 0} \int_Y |\nabla v_k^{\lambda,h}|^2 \,\mathrm{d}x < \alpha + 1/k.$$

Let $\widehat{v}_k^{\lambda,h} = T(v_k^{\lambda,h})$, where $T(s) = |s|$ if $-1 \le s \le 1$ and $T(s) = 1$ otherwise. Then $\{\widehat{v}_k^{\lambda,h}\} \in \mathcal{A}$, $0 \le \widehat{v}_k^{\lambda,h} \le 1$, and

$$\liminf_{(h>\lambda)\to 0} \int_Y |\nabla \widehat{v}_k^{\lambda,h}|^2 \,\mathrm{d}x \le \liminf_{(h>\lambda)\to 0} \int_Y |\nabla v_k^{\lambda,h}|^2 \,\mathrm{d}x < \alpha + \frac{1}{k}.$$

By Rellich–Kondrashov's compactness and Lebesgue's dominated convergence theorems, we conclude that the embedding $H^1_{per}(Y) \cap L^\infty(Y) \hookrightarrow L^q(Y)$ $(1 \le q < +\infty)$ is compact. As a result, the sequence $\{\widehat{v}_k^{\lambda,h}\}$ converges strongly to 1 in $L^2(Y)$ as $(h > \lambda) \to 0$ for every fixed k. Then it is possible to define a subsequence (λ_k, h_k) of (λ, h) which is decreasing and tends to 0 such that

$$\int_Y |\nabla \widehat{v}_k^{\lambda_k,h_k}|^2 \,\mathrm{d}x < \alpha + 2/k, \quad \|\widehat{v}_k^{\lambda_k,h_k} - 1\|_{L^2(Y)} < 1/k.$$

Then the desired sequence $\{w^{\lambda,h}\}_{h>\lambda>0}$ is defined by $w^{\lambda,h} = \widehat{v}_k^{\lambda_k,h_k}$. $\square$

From now on, we suppose that each of the functions $w^{\lambda,h}$ satisfying conditions (H1)–(H4) is extended by Y-periodicity onto $\mathbb{R}^n$. We set

$$w_\varepsilon(x) = w^{\lambda(\varepsilon),h(\varepsilon)}(x/\varepsilon), \quad \forall\, x \in \Omega,\ \forall\, \varepsilon > 0.$$

From Lemma 12.21, we have the following:

(P1) $w_\varepsilon \in H^1(\Omega)$, $0 \le w_\varepsilon \le 1$;
(P2) $w_\varepsilon = 0$ on $\Gamma_\varepsilon^D = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \left[K^{\lambda(\varepsilon)} \cap \partial Q^{h(\varepsilon)} + \varepsilon\mathbf{k}\right]$;
(P3) $w_\varepsilon = 1$ in $\Omega \setminus \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \left(A^{h(\varepsilon)} + \varepsilon\mathbf{k}\right)$;
(P4) $w_\varepsilon(x_1, x_2, \ldots, x_n) = w_\varepsilon(-x_1, x_2, \ldots, x_n)$, $\forall\, x \in A^{\lambda(\varepsilon)}$, $\forall\, \mathbf{k} \in \Theta_\varepsilon$, and $\varepsilon > 0$;
(P5) $w_\varepsilon \rightharpoonup 1$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ as $\varepsilon \to 0$.

Note that the sequence $\left\{|\nabla w_\varepsilon|^2\right\}$ is bounded in $L^1(\Omega)$. So that extracting, if necessary, a subsequence, we can suppose the existence of a bounded nonnegative Radon measure μ^* such that $|\nabla w_\varepsilon|^2$ converges to μ^* in the weak sense of the space $\mathcal{M}_b(\Omega)$. Following in many aspects Casado-Díaz [55, Theorem 2.1], the following quite similar result can be proved.

Theorem 12.22. *Let $\left\{w_\varepsilon \in H^1(\Omega)\right\}$ be a sequence satisfying the properties (P1)–(P5). Then the following hold:*

(L1) $|\nabla w_\varepsilon|^2 \to \mu^*$ weakly in $\mathcal{M}_b(\Omega)$, where $\mu^* \in \mathcal{M}_0^+$, that is,

$$\int_\Omega \varphi\, |\nabla w_\varepsilon|^2\, \mathrm{d}x \to \int_\Omega \varphi\, \mathrm{d}\mu^* \quad \text{for any} \quad \varphi \in C_0^\infty(\Omega);$$

(L2) For any functions $v_\varepsilon \in H^1(\Omega; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$ and for any $v \in H_0^1(\Omega)$ such that $v_\varepsilon \rightharpoonup v$ in $H^1(\Omega)$, we have

$$\begin{gathered} v \in L^2(\Omega, d\mu^*), \\ \int_\Omega \varphi\, (\nabla v_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x \to \int_\Omega \varphi\, v\, \mathrm{d}\mu^*, \quad \forall\, \varphi \in C_0^\infty(\Omega). \end{gathered} \tag{12.46}$$

In fact, the measure $\mu^* \in \mathcal{M}_0^+$ that appeared as the weak limit of $|\nabla w_\varepsilon|^2$ in the space $\mathcal{M}_b(\Omega)$ can be recovered in an explicit form. For this, we recall some properties of capacity (see Theorem 2.38).

Lemma 12.23. *Let D be an open subset of $\mathbb{R}^n$ and let B be a compact subset of D. Then the following hold:*

(i) If $\{D_i\}_{i\in\mathbb{N}}$ is an increasing sequence of open sets such that $\cup_{i\in\mathbb{N}} D_i = D$, then $\lim_{i\to\infty} \mathrm{cap}(B, D_i) = \mathrm{cap}(B, D)$.
(ii) If $\{D_i \subset E\}_{i\in\mathbb{N}}$ is a decreasing sequence of compact sets such that $\cap_{i\in\mathbb{N}} D_i = \mathrm{cl}D$, then $\lim_{i\to\infty} \mathrm{cap}(D_i, E) = \mathrm{cap}(D, E)$.
(iii) If $D_1 \subset D_2$, then $\mathrm{cap}(D_1, E) \le \mathrm{cap}(D_2, E)$.
(iv) If $t > 0$, then $\mathrm{cap}(tB, tD) = t^{n-2}\mathrm{cap}(B, D)$.

We now give the recovery result for the measure μ^*.

Lemma 12.24. *Assume that the origin belongs to a smooth part of the boundary ∂Q ($\partial Q(0) \in C^\infty$). Then for a sequence $\{w_\varepsilon \in H^1(\Omega)\}$ which satisfies properties (P1)–(P5), we have $|\nabla w_\varepsilon|^2 \rightharpoonup \mu^*$ weakly in $\mathcal{M}_b(\Omega)$, where*

$$\mu^* = \operatorname{cap}(K \cap \{x \in \mathbb{R}^n \ : \ x_1 = 0\}) \quad \text{if} \quad n \geq 3, \ \ \mu^* = 2\pi \quad \text{if} \quad n = 2. \tag{12.47}$$

Proof. The proof follows standard techniques in such situations (see [106]), and in some aspects, it is similar to the one given in [73]. First, we note that for any function $\varphi \in C_0^\infty(\Omega)$, $\varepsilon > 0$, and every $\mathbf{k} \in \Theta_\varepsilon$ we have the following inequality:

$$\begin{aligned} \varphi(x_{\mathbf{k}}^\varepsilon) \int_{\varepsilon Y + \varepsilon \mathbf{k}} |\nabla w_\varepsilon|^2 \, \mathrm{d}x &\leq \int_{\varepsilon Y + \varepsilon \mathbf{k}} |\nabla w_\varepsilon|^2 \varphi \, \mathrm{d}x \\ &\leq \varphi(y_{\mathbf{k}}^\varepsilon) \int_{\varepsilon Y + \varepsilon \mathbf{k}} |\nabla w_\varepsilon|^2 \, \mathrm{d}x, \end{aligned} \tag{12.48}$$

where $x_{\mathbf{k}}^\varepsilon, y_{\mathbf{k}}^\varepsilon \in \varepsilon Y + \varepsilon \mathbf{k}$.

Let us begin with the case $n \geq 3$. From the definition of the capacity and Theorem 12.22, it readily follows that

$$\int_{\varepsilon Y + \varepsilon \mathbf{k}} |\nabla w_\varepsilon|^2 \, \mathrm{d}x = \operatorname{cap}\Big(K^{\lambda(\varepsilon)} \cap \partial Q^{h(\varepsilon)}, A^{h(\varepsilon)}\Big).$$

Then, taking into account property (iv) of Lemma 12.23 and relation (12.12), we have

$$\begin{aligned} \int_{\varepsilon Y + \varepsilon \mathbf{k}} |\nabla w_\varepsilon|^2 \, \mathrm{d}x &= \operatorname{cap}\Big(\lambda(\varepsilon) \Big[K \cap \partial Q^{\varsigma(\varepsilon)}\Big], A^{h(\varepsilon)}\Big) \\ &= \lambda^{n-2}(\varepsilon) \operatorname{cap}\Big(\Big[K \cap \partial Q^{\varsigma(\varepsilon)}\Big], A^{\varsigma(\varepsilon)}\Big) \\ &= \varepsilon^n \operatorname{cap}\Big(\Big[K \cap \partial Q^{\varsigma(\varepsilon)}\Big], A^{\varsigma(\varepsilon)}\Big), \end{aligned} \tag{12.49}$$

where $\varsigma(\varepsilon) = h(\varepsilon)/\lambda(\varepsilon) = \exp\Big(-n \ln \varepsilon/(n^2 - 3n + 2)\Big)$ for $n \geq 3$.

Now we interpret the sequence $\big\{\operatorname{cap}\big(\big[K \cap \partial Q^{\varsigma(\varepsilon)}\big], A^{\varsigma(\varepsilon)}\big)\big\}_{\varepsilon > 0}$ as a two-parametric one: $\big\{\Lambda_{\delta,\varepsilon} = \operatorname{cap}\big(\big[K \cap \partial Q^{\varsigma(\delta)}\big], A^{\varsigma(\varepsilon)}\big)\big\}_{\delta,\varepsilon > 0}$. Since this sequence is monotone with respect to the parameter ε, it follows that $\lim_{\delta,\varepsilon \to 0} \Lambda_{\delta,\varepsilon} = \lim_{\varepsilon \to 0} \Lambda_{\delta(\varepsilon),\varepsilon}$ for every sequence $\{\delta(\varepsilon)\}$ converging to 0. Then due to the inequality

$$\begin{aligned} &\Big|\operatorname{cap}\Big(\Big[K \cap \partial Q^{\varsigma(\delta)}\Big], A^{\varsigma(\varepsilon)}\Big) - \operatorname{cap}\Big(K \cap D\Big)\Big| \\ &\quad \leq \Big|\operatorname{cap}\Big(\Big[K \cap \partial Q^{\varsigma(\delta)}\Big]\Big) - \operatorname{cap}\Big(K \cap D\Big)\Big| \\ &\quad + \Big|\operatorname{cap}\Big(\Big[K \cap \partial Q^{\varsigma(\delta)}\Big], A^{\varsigma(\varepsilon)}\Big) - \operatorname{cap}\Big(\Big[K \cap \partial Q^{\varsigma(\delta)}\Big]\Big)\Big| \\ &\quad = J'(\delta) + J''(\delta, \varepsilon) \end{aligned} \tag{12.50}$$

and using property (i) of Lemma 12.23, we have $\lim_{\varepsilon\to 0} J''(\delta,\varepsilon) = 0$ for every $\delta > 0$.

In order to examine the limit properties of the sequence $\{J'(\delta)\}_{\delta>0}$, we have to perform its analysis in a more precise form, namely since a part of boundary ∂Q containing the origin is smooth, it follows that there is a neighborhood $\mathcal{U}(0)$ of the origin such that $\mathcal{U}(0)\cap\partial Q$ is a graph of a smooth function whose epigraph contains $\mathcal{U}(0)\cap Q$. So, we may suppose that there is a function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}_{\geq}$ such that $\Psi \in C_0^\infty(\mathbb{R}^{n-1})$ and $x_1 = \Psi(x_2,\dots,x_n)$ for every $x = (x_1, x_2, \dots, x_n) \in \mathcal{U}(0)\cap\partial Q$.

Then the following conclusion is valid: $x = (x_1, x_2, \dots, x_n) \in K \cap \varepsilon^{-1}\partial Q$ for ε small enough if and only if $x \in \varepsilon^{-1}(\mathcal{U}(0)\cap\partial Q)$ and, hence, $x_1 = \varepsilon\Psi(x_2/\varepsilon,\dots,x_n/\varepsilon)$. As a result, for any sufficiently small ε_0, there exists a constant $C' > 0$ such that

$$K\cap\varepsilon^{-1}\partial Q \subset K\cap\Pi_r \quad \forall\,\varepsilon\leq\varepsilon_0, \text{ with } r = C'\varepsilon_0\,\|\Psi\|_{C(\mathbb{R}^{n-1}\cap\mathcal{U}(0))}$$

where $\Pi_r = \{x\in\mathbb{R}^n : 0\leq x_1\leq r\}$. Then by properties (ii)–(iii) of Lemma 12.23, we have the following implication:

$$\lim_{r\to 0}\operatorname{cap}(K\cap\Pi_r) = \operatorname{cap}(K\cap D),$$

$$K\cap\varepsilon^{-1}\partial Q \subset K\cap\Pi_{\varepsilon C'\,\|\Psi\|_{C(\mathbb{R}^{n-1}\cap\,\mathcal{U}(0))}}, \quad \forall\,\varepsilon>0,$$

implies that $\lim_{\varepsilon\to 0}\operatorname{cap}\big(K\cap\varepsilon^{-1}\partial Q\big) = \operatorname{cap}(K\cap D)$. Hence,

$$\begin{aligned} J'(\delta) &= \Big|\operatorname{cap}\Big(\Big[K\cap\partial Q^{\varsigma(\delta)}\Big]\Big) - \operatorname{cap}\Big(K\cap D\Big)\Big| \\ &\leq C'\|\Psi\|_{C(\mathbb{R}^{n-1})\cap\mathcal{U}(0)}\,\varsigma(\delta) \end{aligned} \tag{12.51}$$

for δ small enough.

Summing up relations (12.48) for every $\mathbf{k}\in\Theta_\varepsilon$ and taking into account (12.49)–(12.51), we come to

$$\begin{aligned} &\Big[\operatorname{cap}\big(K\cap D\big) - C'\varsigma(\delta)\|\Psi\|_{C(\mathbb{R}^{n-1})} - J''(\delta,\varepsilon)\Big]\sum_{\mathbf{k}\in\Theta_\varepsilon}\varepsilon^n\varphi(x_{\mathbf{k}}^\varepsilon) \\ &\qquad\leq \sum_{\mathbf{k}\in\Theta_\varepsilon}\int_{\varepsilon Y+\varepsilon\mathbf{k}}|\nabla w_\varepsilon|^2\varphi\,\mathrm{d}x \\ &\qquad\leq \Big[\operatorname{cap}(K\cap D) + C'\varsigma(\delta)\|\Psi\|_{C(\mathbb{R}^{n-1})} + J''(\delta,\varepsilon)\Big] \\ &\qquad\times\sum_{\mathbf{k}\in\Theta_\varepsilon}\varepsilon^n\varphi(y_{\mathbf{k}}^\varepsilon). \end{aligned} \tag{12.52}$$

Therefore, if we consider the construction of the Riemann sum for $\displaystyle\int_\Omega\varphi\,\mathrm{d}x$, setting $\delta = \varepsilon$, and passing to the limit in (12.52) as $\varepsilon\to 0$, we immediately obtain the required result

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla w_\varepsilon|^2\varphi\,\mathrm{d}x = \mathrm{cap}\big(K\cap D\big)\int_\Omega \varphi\,\mathrm{d}x.$$

If $n=2$, then we have a situation similar to the previous one. The only difference concerns the following obvious equality:

$$\begin{aligned}\int_{\varepsilon Y+\varepsilon\mathbf{k}} |\nabla w_\varepsilon|^2\,\mathrm{d}x &= \mathrm{cap}\Big(\exp\big(-1/\varepsilon^2\big)K\cap\partial Q, A\Big)\\ &= \mathrm{cap}\Big(K\cap\exp\big(1/\varepsilon^2\big)\partial Q, \exp\big(1/\varepsilon^2\big)A\Big).\end{aligned}$$

For the sequence

$$\Big\{\Lambda_{\delta,\varepsilon} = \mathrm{cap}\Big(K\cap\exp\big(1/\delta^2\big)\partial Q, \exp\big(1/\varepsilon^2\big)A\Big)\Big\}_{\delta>0,\varepsilon>0},$$

we can apply the above arguments. Therefore, there is a constant $C''>0$ such that for ε small enough,

$$\Big|\Lambda_{\delta,\varepsilon} - \mathrm{cap}\Big(K\cap D, \exp\big(1/\varepsilon^2\big)\,A\Big)\Big| < C''\,\exp\big(-1/\delta^2\big). \tag{12.53}$$

However, as follows from [73] (see Lemma 12.3.3), we have

$$\mathrm{cap}\Big(K\cap D, \exp\big(1/\varepsilon^2\big)A\Big) = 2\pi\varepsilon^2(1+c_\varepsilon),\quad \text{where}\quad \lim_{\varepsilon\to 0} c_\varepsilon = 0. \tag{12.54}$$

Then, summing up relations (12.49) for all $\mathbf{k}\in\Theta_\varepsilon$ and taking into account (12.53) and (12.54), we obtain

$$\begin{aligned}&\Big[2\pi(1+c_\varepsilon) - C''\,\varepsilon^{-2}\exp\big(-1/\delta^2\big)\Big]\sum_{\mathbf{k}\in\Theta_\varepsilon}\varepsilon^2\varphi(x^\varepsilon_{\mathbf{k}})\\ &\qquad\le \sum_{\mathbf{k}\in\Theta_\varepsilon}\int_{\varepsilon Y+\varepsilon\mathbf{k}}|\nabla w_\varepsilon|^2\varphi\,\mathrm{d}x\\ &\qquad\le \Big[2\pi(1+c_\varepsilon) - C''\,\varepsilon^{-2}\exp\big(-1/\delta^2\big)\Big]\sum_{\mathbf{k}\in\Theta_\varepsilon}\varepsilon^2\varphi(y^\varepsilon_{\mathbf{k}}).\end{aligned} \tag{12.55}$$

Setting $\delta=\varepsilon$ and passing to the limit as $\varepsilon\to 0$, we get

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla w_\varepsilon|^2\varphi\,\mathrm{d}x = 2\pi.$$

□

Corollary 12.25. *Under the assumptions of Lemma 12.24 concerning the local smoothness property of the boundary ∂Q, item (L2) of Theorem 12.22 can be made more precise in the following way: For any $v_\varepsilon\in H^1(\Omega;\Gamma_\varepsilon^D\cup\Sigma_\varepsilon)$ and for any $v\in H^1_0(\Omega)$ such that $v_\varepsilon \rightharpoonup v$ in $H^1(\Omega)$, we have*

$$\int_\Omega \varphi\,(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x \to \mu^*\int_\Omega \varphi\,v\,\mathrm{d}x,\quad \forall\,\varphi\in C_0^\infty(\Omega), \tag{12.56}$$

where the multiplier μ^* is defined by (12.47). Moreover, in this case we have (see Proposition 12.5) $|K\cap\partial\Lambda|_H = |K\cap D|_H$.

The following result is crucial in this section.

Theorem 12.26. *Let* $\left\{v_\varepsilon \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)\right\}$ *be a bounded sequence such that* $v_\varepsilon \rightharpoonup v$ *in the variable space* $H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon) \cap L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$. *Let* $\{\rho'_\varepsilon\}_{\varepsilon>0}$ *be the sequence of numbers that was defined in Proposition 12.6 and* ρ^* *be its limit. Then for the sequence* $\left\{w_\varepsilon \in H^1(\Omega)\right\}$ *with properties (P1)–(P5), we have*

$$\begin{gathered} v \in L^2(\Omega, \mathrm{d}\mu^*), \\ \int_{\Omega_\varepsilon} \varphi\, (\nabla v_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x \to \rho^* \int_\Omega \varphi\, v\, \mathrm{d}\mu^*, \quad \forall\, \varphi \in C_0^\infty(\Omega). \end{gathered} \tag{12.57}$$

Proof. Denote by $\breve{v}_\varepsilon \in H^1(\Omega; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$ some extensions of the functions v_ε and define the following sets:

$$\begin{aligned} J_\varepsilon &= \{\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n \ : \ (\varepsilon Y + \varepsilon \mathbf{k}) \subset \Omega\}, \\ A_\varepsilon &= \bigcup_{\mathbf{k} \in \Theta_\varepsilon} \left[A^{h(\varepsilon)} + \varepsilon \mathbf{k}\right] \\ &= \left(\bigcup_{\mathbf{k} \in J_\varepsilon} \left[A^{h(\varepsilon)} + \varepsilon \mathbf{k}\right]\right) \cup \left(\bigcup_{\mathbf{k} \in \Theta_\varepsilon \setminus J_\varepsilon} \left[A^{h(\varepsilon)} + \varepsilon \mathbf{k}\right]\right) \\ &= A'_\varepsilon \cup A''_\varepsilon. \end{aligned}$$

It is clear that for any bounded sequence $\left\{z_\varepsilon \in H^1(\Omega)\right\}$, we have

$$\begin{aligned} &\int_\Omega \varphi\, (\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x \\ &\quad = \int_{\Omega \cap A_\varepsilon} \varphi\, (\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x \\ &\quad = \int_{(\Omega \setminus \Omega_\varepsilon) \cap A_\varepsilon} \varphi\, (\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x + \int_{\Omega_\varepsilon \cap A_\varepsilon} \varphi\, (\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x \\ &\quad = \int_{(\Omega \setminus \Omega_\varepsilon) \cap A'_\varepsilon} \varphi\, (\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x + \int_{\Omega_\varepsilon \cap A'_\varepsilon} \varphi\, (\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x \\ &\quad + \int_{(\Omega \setminus \Omega_\varepsilon) \cap A''_\varepsilon} \varphi\, (\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x + \int_{\Omega_\varepsilon \cap A''_\varepsilon} \varphi\, (\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x. \end{aligned} \tag{12.58}$$

Since each of the sets $(\Omega \setminus \Omega_\varepsilon) \cap A''_\varepsilon$ and $\Omega_\varepsilon \cap A''_\varepsilon$ is located along the boundary $\partial\Omega$, it follows that $\mathcal{L}_n\left((\Omega \setminus \Omega_\varepsilon) \cap A''_\varepsilon\right) \to 0$ and $\mathcal{L}_n\left(\Omega_\varepsilon \cap A''_\varepsilon\right) \to 0 \quad \text{as } \varepsilon \to 0$. Hence, in view of the boundedness of $\left\{z_\varepsilon \in H^1(\Omega)\right\}$ and $\left\{w_\varepsilon \in H^1(\Omega)\right\}$, we conclude

$$\left.\begin{aligned} \int_{(\Omega \setminus \Omega_\varepsilon) \cap A''_\varepsilon} \varphi\, (\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x \ \overset{\varepsilon \to 0}{\longrightarrow}\ 0, \\ \int_{\Omega_\varepsilon \cap A''_\varepsilon} \varphi\, (\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\, \mathrm{d}x \ \overset{\varepsilon \to 0}{\longrightarrow}\ 0, \end{aligned}\right\} \tag{12.59}$$

$$\lim_{\varepsilon\to 0}\left[\int_{\Omega}\varphi\,(\nabla z_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x-\int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon}\varphi\,(\nabla z_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\right.$$
$$\left.-\int_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi\,(\nabla z_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\right]=0. \quad (12.60)$$

For $\left\{v_\varepsilon\in H^1(\Omega_\varepsilon;\Gamma_\varepsilon^D\cup\Sigma_\varepsilon)\right\}$ and $\varphi\in C_0^\infty(\Omega)$, let us define a function $\psi\in C_0^\infty(\Omega)$ in the following way (which is always possible by property (P4) of w_ε and some freedom of choosing of the extension operators $\breve{v}_\varepsilon=P_\varepsilon(v_\varepsilon)$ [54, 73]):

$$\rho''_\varepsilon\int_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi\,(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x=\rho'_\varepsilon\int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon}\psi\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x,\ \forall\,\varepsilon>0.$$

Here,

$$\rho'_\varepsilon=\frac{\mathcal{L}_n(A^{\lambda(\varepsilon)}\setminus Q^{h(\varepsilon)})}{\mathcal{L}_n(A^{\lambda(\varepsilon)})}=\frac{\mathcal{L}_n(A\setminus Q^{h(\varepsilon)/\lambda(\varepsilon)})}{\mathcal{L}_n(A)},$$
$$\rho''_\varepsilon=\frac{\mathcal{L}_n(A^{\lambda(\varepsilon)}\cap Q^{h(\varepsilon)})}{\mathcal{L}_n(A^{\lambda(\varepsilon)})}=\frac{\mathcal{L}_n(A\cap Q^{h(\varepsilon)/\lambda(\varepsilon)})}{\mathcal{L}_n(A)}.$$

It is clear that $\rho'_\varepsilon+\rho''_\varepsilon=1$ for every $\varepsilon>0$. Then

$$\int_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi\,(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x$$
$$=\rho'_\varepsilon\left[\int_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi\,(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x+\int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon}\psi\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\right]$$
$$=\rho'_\varepsilon\left[\int_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x+\int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon}\psi\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\right]$$
$$=\rho'_\varepsilon\left[\int_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x+\int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon}\varphi\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\right]$$
$$+\rho'_\varepsilon\left(\int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon}(\psi-\varphi)\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\right). \quad (12.61)$$

Since the sequences $\{\breve{v}_\varepsilon\}$ and $\{w_\varepsilon\}$ are equibounded in $H^1(\Omega)$, ρ'_ε tends to ρ^* as $\varepsilon\to 0$, and $d_\vartheta=\max\{\,|\psi(x)-\varphi(y)|\ :\ |x-y|<\vartheta\}$ tends to 0 as $\vartheta\to 0$, we easily obtain

$$\rho'_\varepsilon\left|\int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon}(\psi-\varphi)\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\right|$$
$$\le\|\nabla\breve{v}_\varepsilon\|_{L^2(\Omega)}\|\nabla w_\varepsilon\|_{L^2(\Omega)}\,d_{2\lambda(\varepsilon)r_0}\overset{\varepsilon\to 0}{\longrightarrow}0. \quad (12.62)$$

As a result, taking properties (12.59)–(12.62) into account, we come to the following relation:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\varphi\,(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\\
&\quad=\lim_{\varepsilon\to 0}\left[\int_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi\,(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x+\int_{\Omega_\varepsilon\cap A''_\varepsilon}\varphi\,(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\right]\\
&\quad=\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi\,(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\\
&\quad=\lim_{\varepsilon\to 0}\rho'_\varepsilon\cdot\lim_{\varepsilon\to 0}\left[\int_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x+\int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon}\varphi\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\right]\\
&\quad=\rho^*\lim_{\varepsilon\to 0}\int_\Omega\varphi\,(\nabla\breve{v}_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x. \qquad (12.63)
\end{aligned}$$

In order to complete this proof, it only remains to apply property (12.46) of Theorem 12.22 and the fact that $v\in H^1_0(\Omega)$ due to Theorem 12.14. □

Remark 12.27. As follows from Proposition 12.6, the condition $\rho^*\in[1/2,1)$ is always valid. In particular, using the suppositions of Lemma 12.24 concerning the smoothness of the boundary ∂Q in a neighborhood of the origin, we have

$$\rho'_\varepsilon=\frac{|A\setminus Q^{h(\varepsilon)/\lambda(\varepsilon)}|}{|A|}\overset{\varepsilon\to 0}{\longrightarrow}\rho^*=1/2.$$

So, the main result of Theorem 12.26 can be viewed as

$$\int_{\Omega_\varepsilon}\varphi\,(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,\mathrm{d}x\to(1/2)\mu^*\int_\Omega\varphi\,v\,\mathrm{d}x,\quad\forall\,\varphi\in C_0^\infty(\Omega),$$

where μ^* is defined by (12.47).

We are now in a position to prove the main result of this section concerning the passage to the limit as $\varepsilon\to 0$ in the integral identity

$$\begin{aligned}
\int_\Omega\chi_\varepsilon(\nabla\breve{y}_\varepsilon\cdot\nabla\varphi)\,\mathrm{d}x+\int_\Omega\chi_\varepsilon\breve{y}_\varepsilon\varphi\,\mathrm{d}x+k_0\varepsilon^{-n}\sigma(\varepsilon)\int_\Omega\breve{y}_\varepsilon\,\varphi\,\mathrm{d}\nu_\varepsilon^{\lambda,h}&\\
=\int_\Omega\chi_\varepsilon f_\varepsilon\,\varphi\,\mathrm{d}x+\varepsilon^{-n}\sigma(\varepsilon)\int_\Omega p_\varepsilon\,\varphi\,\mathrm{d}\nu_\varepsilon^{\lambda,h},&\\
\forall\,\varphi\in H^1(\Omega;\ \Gamma_\varepsilon^D\cup\Sigma_\varepsilon). \qquad (12.64)&
\end{aligned}$$

Here, $\{(a_\varepsilon,p_\varepsilon,\breve{y}_\varepsilon)\in\mathbb{X}_\varepsilon\}_{\varepsilon>0}$ is an equibounded sequence of admissible triplets and $\sigma(\varepsilon)$ is defined by (12.17).

By Theorem 12.14, this sequence is relatively compact with respect to the weak convergence in the variable space $\mathbb{X}_\varepsilon$. So, we may suppose that there exists a triplet $(a,p,y)\in\big[H^2(\Omega)\cap H^1_0(\Omega)\big]\times L^2(\Omega)\times H^1_0(\Omega)$ such that $(a_\varepsilon,p_\varepsilon,y_\varepsilon)\overset{w}{\to}(a,p,y)$.

Theorem 12.28. *Let ρ^* be a limit of the sequence (12.28) as $\varepsilon \to 0$ and let*

$$\left\{a_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})\right\} \quad \text{and} \quad \left\{p_\varepsilon \in L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})\right\} \tag{12.65}$$

be any bounded sequences of admissible controls for the $\widehat{\mathbb{P}}_\varepsilon$-problems such that

$$a_\varepsilon \rightharpoonup a \quad \text{in} \quad H^2(\Omega) \cap H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h}), \tag{12.66}$$

$$p_\varepsilon \rightharpoonup p \quad \text{in} \quad L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h}). \tag{12.67}$$

Let $\left\{y_\varepsilon = y_\varepsilon(a_\varepsilon, p_\varepsilon) \in H^1(\Omega, \Sigma_\varepsilon) \cap L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})\right\}_{\varepsilon>0}$ be the corresponding solutions to the problem (12.5). Then $(a_\varepsilon, p_\varepsilon, y_\varepsilon) \overset{w}{\rightharpoonup} (a, p, y)$ as $\varepsilon \to 0$, $y - a \in L^2(\Omega, \mathrm{d}\mu^*)$, and y is the unique function in $H_0^1(\Omega)$ satisfying the following integral identity:

$$\begin{aligned}\int_\Omega (\nabla y \cdot \nabla \varphi)\,\mathrm{d}x + (1 + k_0|\partial Q|_H) \int_\Omega y\varphi\,\mathrm{d}x + \rho^* \int_\Omega (y - a)\,\varphi\,\mathrm{d}\mu^* \\ = \int_\Omega f\,\varphi\,dx + |\partial Q|_H \int_\Omega p\,\varphi\,\mathrm{d}x, \forall\, \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu^*).\end{aligned} \tag{12.68}$$

Proof. Let $\left\{w_\varepsilon \in H^1(\Omega)\right\}_{\varepsilon>0}$ be a sequence defined by Theorem 12.22. Let $\varphi \in C_0^\infty(\Omega)$ be a fixed function. It is clear that $w_\varepsilon\varphi \in H^1(\Omega;\ \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$ for every $\varepsilon > 0$. Take $w_\varepsilon\varphi$ as test functions in (12.64). Then the following integral identity holds true for every $\varepsilon > 0$:

$$\begin{aligned}\int_\Omega \chi_\varepsilon(\nabla(\breve{y}_\varepsilon - a_\varepsilon) \cdot \nabla(w_\varepsilon\varphi))\,\mathrm{d}x + \int_\Omega \chi_\varepsilon(\nabla a_\varepsilon \cdot \nabla(w_\varepsilon\varphi))\,\mathrm{d}x \\ + \int_\Omega \chi_\varepsilon \breve{y}_\varepsilon w_\varepsilon\varphi\,\mathrm{d}x + k_0\varepsilon^{-n}\sigma(\varepsilon) \int_\Omega \breve{y}_\varepsilon\, w_\varepsilon\varphi\,\mathrm{d}\nu_\varepsilon^{\lambda,h} \\ = \int_\Omega \chi_\varepsilon f_\varepsilon w_\varepsilon\,\varphi\,\mathrm{d}x + \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega p_\varepsilon w_\varepsilon\,\varphi\,\mathrm{d}\nu_\varepsilon^{\lambda,h}, \\ \forall\, \varphi \in H^1(\Omega;\ \Gamma_\varepsilon^D \cup \Sigma_\varepsilon).\end{aligned} \tag{12.69}$$

Observe that in view of the boundedness of $\{f_\varepsilon \in L^2(\Omega)\}$, by using estimate (12.25) and Theorem 12.17, we may suppose that there is a function $y \in H_0^1(\Omega)$ such that $(a_\varepsilon, p_\varepsilon, y_\varepsilon) \overset{w}{\rightharpoonup} (a, p, y)$ as $\varepsilon \to 0$. We now pass to the limit in (12.69) as $\varepsilon \to 0$. We do it for each term of (12.69) separately. Observe first that

$$\begin{aligned}\int_\Omega \chi_\varepsilon(\nabla(\breve{y}_\varepsilon - a_\varepsilon) \cdot \nabla(w_\varepsilon\varphi))\,\mathrm{d}x = \int_\Omega \chi_\varepsilon\, w_\varepsilon(\nabla(\breve{y}_\varepsilon - a_\varepsilon) \cdot \nabla\varphi)\,\mathrm{d}x \\ + \int_\Omega \chi_\varepsilon\,\varphi(\nabla(\breve{y}_\varepsilon - a_\varepsilon) \cdot \nabla w_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

Take into account that $\chi_\varepsilon w_\varepsilon \to 1$ strongly in $L^2(\Omega)$ (see Lemma 12.18), $\breve{y}_\varepsilon - a_\varepsilon \rightharpoonup v = y - a$ in $H_0^1(\Omega)$, and $v_\varepsilon = \breve{y}_\varepsilon - a_\varepsilon \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$ for every $\varepsilon > 0$. Then, by Theorem 12.26 we have

$$\int_\Omega \chi_\varepsilon\, w_\varepsilon(\nabla(\breve{y}_\varepsilon - a_\varepsilon)\cdot\nabla\varphi)\,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega (\nabla(y-a)\cdot\nabla\varphi)\,\mathrm{d}x, \tag{12.70}$$

$$\int_\Omega \chi_\varepsilon\,\varphi(\nabla(\breve{y}_\varepsilon - a_\varepsilon)\cdot\nabla w_\varepsilon)\,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \rho^* \int_\Omega \varphi\,(y-a)\,\mathrm{d}\mu^*, \tag{12.71}$$

$$y - a \in L^2(\Omega, \mathrm{d}\mu^*).$$

By (12.24), it follows that $\{a_\varepsilon\}$ is bounded in $H^2(\Omega)\cap H_0^1(\Omega)$, so

$$a_\varepsilon \to a \ \text{ weakly in } \ H^2(\Omega) \text{ and hence}$$

$$\nabla a_\varepsilon \to \nabla a \ \text{ strongly in } \ [L^2(\Omega)]^n.$$

Then, due to (12.5) and (12.66)–(12.67) and since $\nabla w_\varepsilon \rightharpoonup 0$ in $[L^2(\Omega)]^n$, we obtain

$$\int_\Omega \chi_\varepsilon(\nabla a_\varepsilon\cdot\nabla(w_\varepsilon\varphi))\,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} 0, \quad \int_\Omega \chi_\varepsilon \breve{y}_\varepsilon w_\varepsilon\varphi\,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega y\,\varphi\,\mathrm{d}x, \tag{12.72}$$

$$k_0\varepsilon^{-n}\sigma(\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} k_0|\partial Q|_H, \quad \int_\Omega \chi_\varepsilon f_\varepsilon w_\varepsilon\,\varphi\,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega f\,\varphi\,\mathrm{d}x, \tag{12.73}$$

$$\int_\Omega \breve{y}_\varepsilon\, w_\varepsilon\varphi\,\mathrm{d}\nu_\varepsilon^{\lambda(\varepsilon),h(\varepsilon)} \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega y\,\varphi\,\mathrm{d}x \text{ as a weak limit in } \ L^2(\Omega,\mathrm{d}\nu_\varepsilon^{\lambda,h}), \tag{12.74}$$

$$\varepsilon^{-n}\sigma(\varepsilon)\int_\Omega p_\varepsilon w_\varepsilon\,\varphi\,\mathrm{d}\nu_\varepsilon^{\lambda(\varepsilon),h(\varepsilon)} \overset{\varepsilon\to 0}{\longrightarrow} |\partial Q|_H \int_\Omega p\,\varphi\,\mathrm{d}x. \tag{12.75}$$

Thus, the required relation (12.68) is established for any function $\varphi \in C_0^\infty(\Omega)$. Moreover, from the fact that $\breve{y}_\varepsilon - a_\varepsilon \in H^1(\Omega, \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$ and $a_\varepsilon \to a$ in $H_0^1(\Omega)$, we conclude that $(\breve{y}_\varepsilon - a_\varepsilon) \rightharpoonup (y-a)$ in $H^1(\Omega)$, and hence $y \in H_0^1(\Omega)$.

To conclude, we note that the integral identity (12.68) can always be interpreted as the variational formulation of the problem

$$\left.\begin{aligned} -\Delta y + (1 + k_0|\partial Q|_H)y + \rho^*(y-a)\mu^* = f + p\,|\partial Q|_H, \\ y \in H_0^1(\Omega), \quad y - a \in L^2(\Omega, d\mu^*), \end{aligned}\right\} \tag{12.76}$$

with respect to which the following result is well known: For every $a \in H^2(\Omega)\cap H_0^1(\Omega)\cap L^2(\Omega, d\mu^*)$, $p \in L^2(\Omega)$, and $f \in L^2(\Omega)$, there exists a unique solution of (12.76) (see [82]). This completes the proof. □

The following statement is a direct consequence of well known results of the theory of boundary value problems [167].

Corollary 12.29. *Let* $(a_1, p_1, y_1), (a_2, p_2, y_2) \in \big[H^2(\Omega)\cap H_0^1(\Omega)\big] \times L^2(\Omega) \times H_0^1(\Omega)$ *be any triplets satisfying relation (12.76). Then there exists a constant* $\hat{C} > 0$ *(*$\hat{C} = \hat{C}(\Omega, |\partial Q|, \rho^*)$*) such that*

$$\|y_1 - y_2\|_{H_0^1(\Omega)\cap L^2(\Omega,d\mu^*)} \leq \hat{C}\Big[\|a_1 - a_2\|_{H^2(\Omega)\cap H_0^1(\Omega)} + \|p_1 - p_2\|_{L^2(\Omega)} + \|a_1 - a_2\|_{L^2(\Omega,d\mu^*)}\Big]. \tag{12.77}$$

12.6 Identification of the limiting optimal control problem

In this section, we show that for the sequence (12.42), there exists a weak variational limit with respect to the w-convergence and it can be recovered in an explicit form. We begin with the following result.

Lemma 12.30. *Let $\{(a_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence of admissible solutions, assumed to be w-convergent to a triplet*

$$(a, p, y) \in \big[H^2(\Omega) \cap H_0^1(\Omega)\big] \times L^2(\Omega) \times H_0^1(\Omega).$$

Then

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |\nabla y_\varepsilon|^2 \,\mathrm{d}x = \int_\Omega |\nabla y|^2 \,\mathrm{d}x + \rho^* \int_\Omega (y-a)^2 \,\mathrm{d}\mu^*, \tag{12.78}$$

where the measure μ^* and value ρ^* are defined in Theorems 12.22 and 12.26, respectively.

Proof. We first observe that

$$\int_{\Omega_\varepsilon} |\nabla y_\varepsilon|^2 \,\mathrm{d}x = \int_\Omega \chi_\varepsilon |\nabla \breve{y}_\varepsilon - \nabla a_\varepsilon|^2 \,\mathrm{d}x + 2\int_\Omega \chi_\varepsilon (\nabla \breve{y}_\varepsilon \cdot \nabla a_\varepsilon) \,\mathrm{d}x - \int_\Omega \chi_\varepsilon |\nabla a_\varepsilon|^2 \,\mathrm{d}x. \tag{12.79}$$

Then taking into account the facts that $\nabla a_\varepsilon \to \nabla a$ in $[L^2(\Omega)]^n$, $\breve{y}_\varepsilon \rightharpoonup y$ in $H^1(\Omega)$, and $\chi_\varepsilon \to 1$ in $L^2(\Omega)$, we have

$$\int_\Omega \chi_\varepsilon (\nabla \breve{y}_\varepsilon \cdot \nabla a_\varepsilon) \,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \nabla y \cdot \nabla a \,\mathrm{d}x, \tag{12.80}$$

$$\int_\Omega \chi_\varepsilon |\nabla a_\varepsilon|^2 \,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega |\nabla a|^2 \,\mathrm{d}x. \tag{12.81}$$

Since $(\breve{y}_\varepsilon - a_\varepsilon) \in H^1(\Omega, \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$ for every $\varepsilon > 0$, it follows that we can take $\breve{y}_\varepsilon - a_\varepsilon$ as a test function φ in (12.64). Then the following equality is ensured:

$$\int_\Omega \chi_\varepsilon |\nabla \breve{y}_\varepsilon - \nabla a_\varepsilon|^2 \,\mathrm{d}x$$
$$= -\int_\Omega \chi_\varepsilon \nabla a_\varepsilon \cdot (\nabla \breve{y}_\varepsilon - \nabla a_\varepsilon)\,\mathrm{d}x$$
$$- \int_\Omega \chi_\varepsilon \breve{y}_\varepsilon (\breve{y}_\varepsilon - a_\varepsilon)\,\mathrm{d}x - k_0 \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega \breve{y}_\varepsilon\,(\breve{y}_\varepsilon - a_\varepsilon)\,\mathrm{d}\nu_\varepsilon^{\lambda,h}$$
$$+ \int_\Omega \chi_\varepsilon f_\varepsilon\,(\breve{y}_\varepsilon - a_\varepsilon)\,\mathrm{d}x + \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega p_\varepsilon\,(\breve{y}_\varepsilon - a_\varepsilon)\,\mathrm{d}\nu_\varepsilon^{\lambda,h}. \tag{12.82}$$

By properties (12.70)–(12.75), we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega \chi_\varepsilon |\nabla \breve{y}_\varepsilon - \nabla a_\varepsilon|^2 \,\mathrm{d}x$$
$$= -\int_\Omega \nabla a \cdot (\nabla y - \nabla a)\,\mathrm{d}x - \int_\Omega y(y-a)\,\mathrm{d}x$$
$$- k_0 |\partial Q|_H \lim_{\varepsilon\to 0} \int_\Omega \breve{y}_\varepsilon\,(\breve{y}_\varepsilon - a_\varepsilon)\,\mathrm{d}\nu_\varepsilon^{\lambda,h} + \int_\Omega f\,(y-a)\,\mathrm{d}x$$
$$+ |\partial Q|_H \lim_{\varepsilon\to 0} \int_\Omega p_\varepsilon\,(\breve{y}_\varepsilon - a_\varepsilon)\,\mathrm{d}\nu_\varepsilon^{\lambda,h}.$$

Since $(\breve{y}_\varepsilon - a_\varepsilon) \in H^1(\Omega, \Gamma_\varepsilon^D \cup \Sigma_\varepsilon) \cap L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$, we have $\breve{y}_\varepsilon - a_\varepsilon \to y - a$ strongly in $L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$ (by Theorem 12.15) and $\breve{y}_\varepsilon \rightharpoonup y$ in $L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$ (by Theorem 12.14). Hence, in view of the definition of the strong convergence in variable spaces, we conclude that

$$\lim_{\varepsilon\to 0} \int_\Omega \breve{y}_\varepsilon\,(\breve{y}_\varepsilon - a_\varepsilon)\,\mathrm{d}\nu_\varepsilon^{\lambda,h} = \int_\Omega y\,(y-a)\,\mathrm{d}x,$$
$$\lim_{\varepsilon\to 0} \int_\Omega p_\varepsilon\,(\breve{y}_\varepsilon - a_\varepsilon)\,\mathrm{d}\nu_\varepsilon^{\lambda,h} = \int_\Omega p\,(y-a)\,\mathrm{d}x.$$

As a result, we get

$$\lim_{\varepsilon\to 0} \int_\Omega \chi_\varepsilon |\nabla \breve{y}_\varepsilon - \nabla a_\varepsilon|^2 \,\mathrm{d}x$$
$$= -\int_\Omega \nabla a \cdot (\nabla y - \nabla a)\,\mathrm{d}x - \int_\Omega y(y-a)\,\mathrm{d}x$$
$$- k_0 |\partial Q|_H \int_\Omega y\,(y-a)\,\mathrm{d}x + \int_\Omega f\,(y-a)\,\mathrm{d}x$$
$$+ |\partial Q|_H \int_\Omega p\,(y-a)\,\mathrm{d}x. \tag{12.83}$$

Let us consider the integral identity (12.68) with the test function $\varphi = y - a$. By rearrangement, we have

$$\int_\Omega |\nabla y - \nabla a|^2 \,\mathrm{d}x + \rho^* \int_\Omega y\,(y-a)\,\mathrm{d}\mu^* - \rho^* \int_\Omega a\,(y-a)\,\mathrm{d}\mu^*$$
$$= -\int_\Omega \nabla a \cdot (\nabla y - \nabla a)\,\mathrm{d}x - (1 + k_0 |\partial Q|_H) \int_\Omega y(y-a)\,\mathrm{d}x$$
$$+ \int_\Omega f\,(y-a)\,\mathrm{d}x + |\partial Q|_H \int_\Omega p\,(y-a)\,\mathrm{d}x. \tag{12.84}$$

The comparison of (12.83) with (12.84) leads to the following equality:

$$\lim_{\varepsilon\to 0} \int_\Omega \chi_\varepsilon |\nabla \breve{y}_\varepsilon - \nabla a_\varepsilon|^2 \,\mathrm{d}x = \int_\Omega |\nabla y - \nabla a|^2 \,\mathrm{d}x + \rho^* \int_\Omega (y-a)^2 \,\mathrm{d}\mu^*.$$

which, together with (12.79)–(12.81), concludes the proof. □

We are now in position to establish the identification result of the weak variational limit for the sequence (12.42).

Theorem 12.31. *For the sequence of constrained minimization problems (12.42), there exists a unique weak variational limit with respect to w-convergence which can be represented in the form (12.43), where the cost functional I_0 and the set of admissible solutions Ξ_0 are defined as follows:*

$$I_0(a,p,y) = \int_\Omega |\nabla y|^2 \,\mathrm{d}x + \int_\Omega |y - z^\partial|^2 \,\mathrm{d}x + \rho^* \int_\Omega (y-a)^2 \,\mathrm{d}\mu^*$$
$$+ |\partial Q|_H \int_\Omega p^2 \,\mathrm{d}x + |K \cap \partial \Lambda|_H \int_\Omega a^2 \,\mathrm{d}x, \tag{12.85}$$

$$\Xi_0 = \left\{ (a,p,y) \;\middle|\; \begin{array}{c} y \in H_0^1(\Omega), \quad p \in L^2(\Omega), \\ a \in H^2(\Omega) \cap H_0^1(\Omega),\ \ y - a \in L^2(\Omega, \mathrm{d}\mu^*), \\ \|a\|_{H^2(\Omega)} \le C_0, \\ \displaystyle\int_\Omega (\nabla y \cdot \nabla \varphi)\,\mathrm{d}x + \rho^* \int_\Omega (y-a)\,\varphi\,\mathrm{d}\mu^* \\ + (1 + k_0 |\partial Q|_H) \displaystyle\int_\Omega y\,\varphi\,\mathrm{d}x = \int_\Omega f\,\varphi\,\mathrm{d}x \\ + |\partial Q|_H \displaystyle\int_\Omega p\,\varphi\,\mathrm{d}x,\ \forall\, \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu^*) \end{array} \right\}. \tag{12.86}$$

Here, Λ is a cone in $\mathbb{R}^n$ (see Proposition 12.5).

Proof. The proof of this theorem is divided into two steps; each of them concerns the verification of the corresponding item of Definition 8.24.

Step 1: *Statement (d) of Definition 8.24 is valid.*

Let $\{(a_k, p_k, \breve{y}_k) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence which is w-convergent to a triplet $(u,p,y) \in \big[H^2(\Omega) \cap H_0^1(\Omega)\big] \times L^2(\Omega) \times H_0^1(\Omega)$. Let $\{\varepsilon_k\}$ be a subsequence of $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and $(a_k, p_k, y_k) \in \widehat{\Xi}_{\varepsilon_k}$ for all $k \in \mathbb{N}$. Then, due to Theorem 12.28, we have that the w-limit triplet (u,p,y) satisfies the integral identity (12.68), and, moreover,

$$C_0 \geq \liminf_{k\to\infty} \|a_k\|_{H^2(\Omega)} \geq \|a\|_{H^2(\Omega)},$$

by the lower semicontinuity of $\|\cdot\|_{H^2(\Omega)}$ with respect to the weak convergence in $H^2(\Omega)$. So, the first inclusion in (8.59) holds true.

We now turn back to the test of inequality (8.59). By the lower semicontinuity of the weak convergence in variable spaces, Proposition 12.5, and relation (12.5), we have

$$\begin{aligned}\lim_{k\to\infty} \Bigg[\varepsilon_k^{-n}\sigma(\varepsilon_k) \int_\Omega p_{\varepsilon_k}^2 \,\mathrm{d}\nu_{\varepsilon_k}^{\lambda(\varepsilon_k),h(\varepsilon_k)} \\ + |K\cap\partial Q^{\varsigma(\varepsilon_k)}|_H \int_\Omega a_{\varepsilon_k}^2 \,\mathrm{d}\mu_{\varepsilon_k}^{\lambda(\varepsilon_k),h(\varepsilon_k)} \Bigg] \\ \geq |\partial Q|_H \int_\Omega p^2\,\mathrm{d}x + |K\cap\partial\Lambda|_H \int_\Omega a^2\,\mathrm{d}x.\end{aligned}$$

To conclude, it remains only to apply Lemma 12.30.

Step 2: *Statement (dd) of Definition 8.24 holds true.*

Let $(a,p,y)\in\Xi_0$ be an admissible triplet for the minimization problem (12.43). As readily follows from (12.85), for any triplet $(a,\hat p,\hat y)\in\Xi_0$ there exists a constant $\gamma>0$ depending on Ω, z^∂, ρ^*, p, y, and $|\partial Q|_H$ such that

$$\begin{aligned}|I_0(a,p,y) - I_0(a,\hat p,\hat y)| \\ \leq \gamma\left(\|y-\hat y\|^2_{H_0^1(\Omega)\cap L^2(\Omega,\mathrm{d}\mu^*)} + \|p-\hat p\,\|^2_{L^2(\Omega)}\right).\end{aligned} \tag{12.87}$$

Let $1>\delta>0$ be a given value. Using the density of the embedding $H^1(\Omega)\hookrightarrow L^2(\Omega)$, we can choose an element $\hat p\in H^1(\Omega)$ such that

$$\|p-\hat p\|_{L^2(\Omega)} < \varrho, \quad \text{where } \varrho < \max\left\{\delta/\sqrt{\gamma(1+\hat C)}, \delta/\hat C\right\}. \tag{12.88}$$

Let $\hat y=\hat y(a,\hat p\,)$ be the corresponding solution of the boundary value problem (12.76). Then due to estimates (12.77) and (12.87), we have

$$\begin{aligned}|I_0(a,p,y) - I_0(a,\hat p,\hat y)| \leq \gamma(1+\hat C)\|p-\hat p\,\|^2_{L^2(\Omega)} < \delta, \\ \|y-\hat y\|_{H_0^1(\Omega)\cap L^2(\Omega,\mathrm{d}\mu^*)} \leq \hat C\|p-\hat p\,\|_{L^2(\Omega)} < \delta.\end{aligned} \tag{12.89}$$

We now construct the δ-realizing sequence $\{(\widehat a_\varepsilon^{\lambda,h},\widehat p_\varepsilon^{\lambda,h},\widehat y_\varepsilon^{\lambda,h})\in\mathbb{X}_\varepsilon\}_{\varepsilon>0}$ as $\widehat a_\varepsilon^{\lambda,h}=a$, $\widehat p_\varepsilon^{\lambda,h}=\hat p$, and we take $\widehat y_\varepsilon^{\lambda,h}$ as the corresponding solution of the original boundary value problem (12.5). Hence, $(\widehat a_\varepsilon^{\lambda,h},\widehat p_\varepsilon^{\lambda,h},\widehat y_\varepsilon^{\lambda,h})\in\widehat\Xi_\varepsilon$ for every $\varepsilon>0$. It is clear that this sequence in equibounded in $\mathbb{X}_\varepsilon$. Moreover, by Lemma 12.8 and Theorems 12.14 and 12.15, we have

$$\widehat a_\varepsilon^{\lambda,h} \rightharpoonup a \ \text{ in } L^2(\Omega,\mathrm{d}\mu_\varepsilon^{\lambda,h}) \ \text{ and } \ \int_\Omega \left(\widehat a_\varepsilon^{\lambda,h}\right)^2 \mathrm{d}\mu_\varepsilon^{\lambda,h} \overset{\varepsilon\to0}{\longrightarrow} \int_\Omega a^2\,\mathrm{d}x, \tag{12.90}$$

$$\widehat p_\varepsilon^{\lambda,h} \rightharpoonup \hat p \ \text{ in } L^2(\Omega,\mathrm{d}\nu_\varepsilon^{\lambda,h}) \ \text{ and } \ \int_\Omega \left(\widehat p_\varepsilon^{\lambda,h}\right)^2 \mathrm{d}\nu_\varepsilon^{\lambda,h} \overset{\varepsilon\to0}{\longrightarrow} \int_\Omega \hat p^{\,2}\,\mathrm{d}x. \tag{12.91}$$

Then, applying Theorem 12.17, we conclude that the sequence

$$\{(\widehat{a}_\varepsilon^{\lambda,h}, \widehat{p}_\varepsilon^{\lambda,h}, \widehat{y}_\varepsilon^{\lambda,h})\}$$

is compact with respect to the w-convergence. Let $(a, \hat{p}, \hat{y}^*)$ be its w-limit. Due to Theorem 12.28, we have $(a, \hat{p}, \hat{y}^*) \in \Xi_0$. Since the boundary value problem (12.76) has a unique solution for every fixed a and $\hat{p}$, it follows that $\hat{y}^* = \hat{y}$, and hence $(\widehat{a}_\varepsilon^{\lambda,h}, \widehat{p}_\varepsilon^{\lambda,h}, \widehat{y}_\varepsilon^{\lambda,h}) \overset{w}{\rightharpoonup} (a, \hat{p}, \hat{y})$ as $\varepsilon \to 0$.

Consequently, the first part of property (dd) of Definition 8.24 is fulfilled. It remains only to verify inequality (8.61). To do so, we use properties (12.90) and (12.91) and Lemma 12.30. Then

$$\lim_{\varepsilon \to 0} \widehat{I}_\varepsilon(\widehat{a}_\varepsilon^{\lambda,h}, \widehat{p}_\varepsilon^{\lambda,h}, \widehat{y}_\varepsilon^{\lambda,h}) = I_0(a, \hat{p}, \hat{y}).$$

To conclude, we apply inequality (12.89). This yields the required result

$$I_0(a, p, y) \geq \lim_{\varepsilon \to 0} \widehat{I}_\varepsilon(\widehat{a}_\varepsilon^{\lambda,h}, \widehat{p}_\varepsilon^{\lambda,h}, \widehat{y}_\varepsilon^{\lambda,h}) - \delta.$$

□

It is now clear that the constrained minimization problem (12.43) can be interpreted as the following OCP:

$$\begin{aligned} &\int_\Omega (\nabla y \cdot \nabla \varphi)\, \mathrm{d}x + \rho^* \int_\Omega (y - a)\, \varphi \, \mathrm{d}\mu^* + (1 + k_0 |\partial Q|_H) \int_\Omega y\, \varphi \, \mathrm{d}x \\ &\quad = \int_\Omega f\, \varphi \, \mathrm{d}x + |\partial Q|_H \int_\Omega p\, \varphi \, \mathrm{d}x, \ \forall\, \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu^*), \end{aligned} \tag{12.92}$$

$$y \in H_0^1(\Omega), \quad p \in L^2(\Omega), \quad a \in H^2(\Omega) \cap H_0^1(\Omega), \tag{12.93}$$

$$y - a \in L^2(\Omega, d\mu^*), \quad \|a\|_{H^2(\Omega)} \leq C_0, \tag{12.94}$$

$$\begin{aligned} I_0(a, p, y) = &\int_\Omega |\nabla y|^2 \, \mathrm{d}x + \int_\Omega |y - z^\partial|^2 \, \mathrm{d}x + \rho^* \int_\Omega (y - a)^2 \, \mathrm{d}\mu^* \\ &+ |\partial Q|_H \int_\Omega p^2 \, \mathrm{d}x + |K \cap \partial \Lambda|_H \int_\Omega a^2 \, \mathrm{d}x \longrightarrow \inf. \end{aligned} \tag{12.95}$$

Note that the parameters ρ^*, $|\partial Q|_H$, $|K \cap \partial \Lambda|_H \in \mathbb{R}$, and the Borel measure μ^* are coming from the geometry of control zones. Moreover, in contrast to $\mathbb{P}_\varepsilon$, the limit control problem (12.92)–(12.95) contains two independent distributed control functions.

Proposition 12.32. *The limit OCP (12.92)–(12.95) has a unique solution.*

Proof. The proof is quite similar to that given in Theorem 12.2. The main difference is the choice of the topology for the space of admissible solutions $\left[H^2(\Omega)\cap H_0^1(\Omega)\right]\times L^2(\Omega)\times H_0^1(\Omega)$ with respect to which the set Ξ_0 and the cost functional I_0 possess the required topological properties, one of which has to guarantee the inclusion (12.94). It is clear that this topology can be taken as $\tau=(w_{H^2(\Omega)\cap H_0^1(\Omega)})\times(w_{L^2(\Omega)})\times(w_{H_0^1(\Omega)})$, where $w_{(\cdot)}$ denotes the weak topology of the corresponding Banach space. Indeed, due to the fact that $\mu^*\in\mathcal{M}_0^+(\Omega)$, the space $H_0^1(\Omega)\cap L^2(\Omega,\mathrm{d}\mu^*)$ is well defined (see Remark 12.3). Hence, if $y_n\rightharpoonup y$ in $H_0^1(\Omega)$ and $a_n\rightharpoonup a$ in $H^2(\Omega)\cap H_0^1(\Omega)$, then $(y_n-a_n)\rightharpoonup(y-a)$ in $L^2(\Omega,\mathrm{d}\mu^*)$. Moreover, it can be easily checked (by passing to the limit in (12.92) and (12.94)) that the set Ξ_0 is τ-closed and the cost functional I_0 is τ-lower semicontinuous. The proof of Theorem 12.2 is then valid without any more modifications. □

Thus, combining the results of Theorem 12.2, Theorem 12.20, and Proposition 12.32, we come to the following conclusion concerning the variational properties of the limit OCP (12.92)–(12.95).

Theorem 12.33. *Let* $\left\{(a_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0)\in\widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ *be the optimal solutions of the problems* $\widehat{\mathbb{P}}_\varepsilon$. *Then*

$$\lim_{\varepsilon\to 0}\widehat{I}_\varepsilon(a_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0)=\inf_{(a,\,p,y)\in\Xi_0}I_0(a,\,p,y)=I_0(a^0,p^0,y^0)\tag{12.96}$$

and

$$(a_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0)\xrightarrow{w}(a^0,p^0,y^0)\ \ \textit{in the variable space}\ \ \mathbb{X}_\varepsilon.\tag{12.97}$$

12.7 Optimality conditions for the limit problem and suboptimal controls for $\mathbb{P}_\varepsilon$-problem

In this section, we derive the optimality conditions for the problem (12.92)–(12.95) from which an optimal triplet may be determined. For this, we use the Lagrange multiplier principle and make use of Theorem 9.35 (for more details, we refer to Ioffe and Tikhomirov [125]). We obtain the weak form of the optimality system equations that an optimal triplet (a^0,p^0,y^0) and Lagrange multipliers must satisfy. As we see later, this optimality system can serve as a basis for the construction of suboptimal solutions to the original problem in perforated domains.

Theorem 12.34. *A triplet*

$$(a^0,p^0,y^0)\in\left[H^2(\Omega)\cap H_0^1(\Omega)\right]\times L^2(\Omega)\times\left[H^2(\Omega)\cap H_0^1(\Omega)\right],$$
$$y^0-a^0\in L^2(\Omega,\mathrm{d}\mu^*)$$

in an optimal solution to the problem (12.92)–(12.95) if and only if there exists a function $\psi \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu^*)$ such that the quadruple (a^0, p^0, y^0, ψ) satisfies the following optimality system:

$$\int_\Omega (\nabla y^0 \cdot \nabla \varphi)\,\mathrm{d}x + \rho^* \int_\Omega (y^0 - a^0)\,\varphi\,\mathrm{d}\mu^* + (1 + k_0|\partial Q|_H) \int_\Omega y^0\,\varphi\,\mathrm{d}x$$
$$= \int_\Omega f\,\varphi\,\mathrm{d}x + |\partial Q|_H \int_\Omega p^0\,\varphi\,\mathrm{d}x, \quad \forall\,\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu^*). \quad (12.98)$$

$$\int_\Omega (\nabla \psi + 2\nabla y^0) \cdot \nabla \phi\,\mathrm{d}x + \int_\Omega \left[2(y^0 - z^\partial) + (1 + k_0|\partial Q|)\,\psi\right] \phi\,\mathrm{d}x$$
$$+ \rho^* \int_\Omega \left(\psi + 2(y^0 - a^0)\right) \phi\,\mathrm{d}\mu^* = 0, \quad \forall\,\phi \in H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu^*), \quad (12.99)$$

$$p^0 = \psi/2 \quad \textit{a.e. in} \quad \Omega, \quad (12.100)$$

$$2|K \cap \partial\Lambda|_H \int_\Omega a^0(a - a^0)\,\mathrm{d}x - \rho^* \int_\Omega \psi(a - a^0)\,\mathrm{d}\mu^*$$
$$- 2\rho^* \int_\Omega (y^0 - a^0)(a - a^0)\,\mathrm{d}\mu^* \geq 0,$$
$$\forall\, a \in \left\{a \in H^2(\Omega) \cap H_0^1(\Omega) \;:\; \|a\|_{H^2(\Omega)} \leq C_0\right\}. \quad (12.101)$$

Proof. Let (a^0, p^0, y^0) be an optimal solution to the problem (12.92)–(12.95). To apply the Lagrange principle, we set (see Theorem 9.35)

$$\mathbb{Y} = H^2(\Omega) \cap H_0^1(\Omega),\ \mathbb{U} = \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times L^2(\Omega),\ \mathbb{V} = L^2(\Omega),$$

and

$$F(a, p, y) = -\Delta y + (1 + k_0|\partial Q|_H)y + \rho^*(y - a)\mu^* - f - |\partial Q|_H\ p.$$

Since $f \in L^2(\Omega)$, it follows that the boundary value problem (12.76) has the unique solution $y \in H^2(\Omega) \cap H_0^1(\Omega)$ for any $a \in H^2(\Omega) \cap H_0^1(\Omega)$ and $p \in L^2(\Omega)$, and, moreover, in this case $y - a \in L^2(\Omega, \mathrm{d}\mu^*)$ (see [82, 167]). Hence, $\operatorname{Im} F'_y = \mathbb{V}$. Thus, all the assumptions of Theorem 9.35 are fulfilled. We now define the Lagrange function as follows:

$$\mathcal{L}(a, p, y, \psi) = \int_\Omega |\nabla y|^2\,\mathrm{d}x + \int_\Omega |y - z^\partial|^2\,\mathrm{d}x$$
$$+ \rho^* \int_\Omega (y - a)^2\,\mathrm{d}\mu^* + |\partial Q|_H \int_\Omega p^2\,\mathrm{d}x + |K \cap \partial\Lambda|_H \int_\Omega a^2\,\mathrm{d}x$$
$$+ \int_\Omega (\nabla y \cdot \nabla \psi)\,\mathrm{d}x - \int_\Omega f\,\psi\,\mathrm{d}x + \rho^* \int_\Omega (y - a)\,\psi\,\mathrm{d}\mu^*$$
$$+ (1 + k_0|\partial Q|_H) \int_\Omega y\,\psi\,\mathrm{d}x - |\partial Q|_H \int_\Omega p\,\psi\,\mathrm{d}x,$$
$$\forall\,\psi \in H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu^*).$$

In accordance with Theorem 9.35, there exists a function $\psi \in H_0^1(\Omega) \cap L^2(\Omega, d\mu^*)$ such that relations (9.125) and (9.126) are valid. In this case, relation (9.126) takes the form (12.100)–(12.101), whereas (9.125) can be written as (12.99). Since $y^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $z^\partial \in L^2(\Omega)$, it follows that the bilinear form

$$\langle \nabla\psi, \nabla\phi \rangle_{L^2(\Omega)} + (1 + k_0 |\partial Q|_H) \langle \psi, \phi \rangle_{L^2(\Omega)} + \rho^* \langle \psi, \phi \rangle_{L^2(\Omega, \mathrm{d}\mu^*)}$$

is coercive on the space $H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu^*)$. So, by the Riesz representation theorem, we immediately conclude that there exists a unique function $\psi \in H_0^1(\Omega) \cap L^2(\Omega)$ satisfying equality (12.99) and such that $\psi \in H^2(\Omega)$. Thus, the first part of Theorem 9.35 is proved, that is, (12.98)–(12.101) are the necessary optimality conditions. Since the mapping $y \to I_0(a, p, y)$ is convex and the mapping $(a, p) \to F(a, p, y)$ is continuous and affine, relations (12.98)–(12.101) are also sufficient optimality conditions for the problem (12.92)–(12.95). As this problem is uniquely solvable, the proof is complete. □

As an evident consequence of this theorem, we have the following result.

Corollary 12.35. *If* (a^0, p^0, y^0) *is an optimal solution to (12.92)–(12.95), then*

$$p^0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^2(\Omega, d\mu^*). \tag{12.102}$$

Now, using (12.102) and applying Lemmas 12.8 and 12.10, we immediately establish the following approximation property for the optimal controls.

Proposition 12.36. *If*

$$p^0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu^*) \quad \text{and} \quad a^0 \in H^2(\Omega) \cap H_0^1(\Omega)$$

are the optimal controls to the homogenized problem (12.92)–(12.95), then

$$(a^0, p^0) \in L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h}) \times L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h}), \quad \forall\, \varepsilon > 0, \tag{12.103}$$

$$a^0 \rightharpoonup a^0 \quad \text{in} \quad L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h}), \quad \lim_{\varepsilon \to 0} \int_\Omega (a^0)^2 \, \mathrm{d}\mu_\varepsilon^{\lambda,h} = \int_\Omega (a^0)^2 \, \mathrm{d}x, \tag{12.104}$$

$$p^0 \rightharpoonup p^0 \quad \text{in} \quad L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h}), \quad \lim_{\varepsilon \to 0} \int_\Omega (p^0)^2 \, \mathrm{d}\nu_\varepsilon^{\lambda,h} = \int_\Omega (p^0)^2 \, \mathrm{d}x. \tag{12.105}$$

The next question we are going to consider in this section concerns the approximation of the optimal solutions of the original problem $\widehat{\mathbb{P}}_\varepsilon$ for ε small enough. We focus our attention on suboptimal solutions which guarantee the closeness of the corresponding value of the cost functional $I_\varepsilon(a_\varepsilon^{sub}, p_\varepsilon^{sub}, y_\varepsilon^{sub})$ to its minimum if ε is small enough (see Definition 11.30). Then Proposition 12.36 leads to the following final result.

Theorem 12.37. *Let*

$$p^0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu^*) \quad \text{and} \quad a^0 \in H^2(\Omega) \cap H_0^1(\Omega)$$

be the optimal controls for the homogenized problem (12.92)–(12.95). Then the sequence of the pairs $\{(a^0, p^0)\}_{\varepsilon>0}$ *is asymptotically suboptimal for the original OCP* $\widehat{\mathbb{P}}_\varepsilon$.

Proof. Let us consider the sequence of triplets $\{(a^0, p^0, \breve{\widetilde{y}}_\varepsilon) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$, where $\widetilde{y}_\varepsilon = \widetilde{y}_\varepsilon(a^0, p^0)$ are the corresponding solutions of the boundary value problem (12.5). Each of these triplets is admissible for the problem $\widehat{\mathbb{P}}_\varepsilon$. Moreover, due to estimate (12.25), this sequence is equibounded in $\mathbb{X}_\varepsilon$. By Theorem 12.17, it is relatively compact with respect to the w-convergence in $\mathbb{X}_\varepsilon$. Hence, taking into account Proposition 12.36 and Theorem 12.28, we deduce that this sequence is w-compact and $(a^0, p^0, \widetilde{y}_\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} (a^0, p^0, y^0)$, where (a^0, p^0, y^0) is an optimal solution to the homogenized problem (12.92)–(12.95).

Let $\{(a^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) \in \widehat{\Xi}_\varepsilon\}_{\varepsilon>0}$ be the optimal solutions to $\widehat{\mathbb{P}}_\varepsilon$. We observe that

$$
\begin{aligned}
&\Big| \inf_{(a_\varepsilon, p_\varepsilon, y_\varepsilon)\in \widehat{\Xi}_\varepsilon} \widehat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon) - \widehat{I}_\varepsilon(a^0, p^0, \widetilde{y}_\varepsilon)\Big| \\
&= \Big|\widehat{I}_\varepsilon(a^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) - \widehat{I}_\varepsilon(a^0, p^0, \widetilde{y}_\varepsilon)\Big| \le \Big|\widehat{I}_\varepsilon(a^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) - I_0(a^0, p^0, y^0)\Big| \\
&+ \Big|I_0(a^0, p^0, y^0) - \widehat{I}_\varepsilon(a^0, p^0, \widetilde{y}_\varepsilon)\Big| \le \Big|\widehat{I}_\varepsilon(a^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) - I_0(a^0, p^0, y^0)\Big| \\
&+ \left|\int_\Omega |\nabla y^0|^2\,\mathrm{d}x + \rho^* \int_\Omega (y^0 - a^0)^2\,\mathrm{d}\mu^* - \int_\Omega \chi_\varepsilon |\nabla \breve{\widetilde{y}}_\varepsilon|^2\,\mathrm{d}x\right| \\
&+ \left|\int_\Omega |y^0 - z^\partial|^2\,\mathrm{d}x - \int_\Omega \chi_\varepsilon |\breve{\widetilde{y}}_\varepsilon - z^\partial|^2\,\mathrm{d}x\right| \\
&+ \left||\partial Q|_H \int_\Omega (p^0)^2\,\mathrm{d}x - \varepsilon^{-n}\sigma(\varepsilon)\int_\Omega (p^0)^2\,\mathrm{d}\nu^{\lambda,h}_\varepsilon\right| \\
&+ \left||K\cap\partial\Lambda|_H \int_\Omega (a^0)^2\,\mathrm{d}x - |K\cap\partial Q^{\varsigma(\varepsilon)}|_H \int_\Omega (a^0)^2\,\mathrm{d}\mu^{\lambda,h}_\varepsilon\right| \\
&= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}
$$

To conclude the proof, we note that for a given $\delta > 0$ one can always find the following:

(i) $\varepsilon_1 > 0$ such that $J_1 < \delta/5$ for all $\varepsilon < \varepsilon_1$ by Theorem 12.33;
(ii) $\varepsilon_2 > 0$ such that $J_2 < \delta/5$ for all $\varepsilon < \varepsilon_2$ by Lemma 12.30;
(iii) $\varepsilon_3 > 0$ such that $J_3 < \delta/5$ for all $\varepsilon < \varepsilon_3$ by the w-convergence $(a^0, p^0, \widetilde{y}_\varepsilon)$ to (a^0, p^0, y^0);
(iv) $\varepsilon_4 > 0$ such that $J_4 < \delta/5$ for all $\varepsilon < \varepsilon_4$ by (12.17) and (12.105);
(v) $\varepsilon_5 > 0$ such that $J_5 < \delta/5$ for all $\varepsilon < \varepsilon_5$ by (12.104) and (12.27).

Thus, as expected, estimate (11.71) is valid for all $\varepsilon < \min_{1\le i\le 5}\{\varepsilon_i\}$. □

It should be stressed that a sequence of asymptotically suboptimal controls for the problem $\widehat{\mathbb{P}}_\varepsilon$ has a particularly simple and attractive form if the optimal

control pair $(a_\varepsilon^0, p_\varepsilon^0)$ for the problem $\widehat{\mathbb{P}}_\varepsilon$ is such that

$$a_\varepsilon^0(x) = a^*\left(\frac{x}{\varepsilon\,\lambda(\varepsilon)}\right), \quad p_\varepsilon^0(x) = p^*\left(\frac{x}{\varepsilon\,h(\varepsilon)}\right),$$

where $a^* \in H^1(K)$ and $p^* \in L^2(\partial Q, d\mathcal{H}^{n-1})$ are some Y-periodic functions. Indeed, in this case by Theorems 6.70 and 12.33, we have

$$a_\varepsilon^0 \rightharpoonup \left(\frac{1}{|K \cap \partial\Lambda|_H} \int_{K\cap\partial\Lambda} a^*\,\mathrm{d}\mathcal{H}^{n-1}\right) = a^0,$$

$$p_\varepsilon^0 \rightharpoonup \left(\frac{1}{|\partial Q|_H} \int_{\partial Q} p^*\,\mathrm{d}\mathcal{H}^{n-1}\right) = p^0$$

in $L^2(\Omega, \mathrm{d}\mu_\varepsilon^{\lambda,h})$ and $L^2(\Omega, \mathrm{d}\nu_\varepsilon^{\lambda,h})$, respectively, where (a^0, p^0) are optimal controls for the homogenized problem (12.92)–(12.95). Hence, the conclusion of Theorem 12.37 can be reformulated as follows: The sequence of constant pairs $\{(a^0, p^0) \in \mathbb{R}^2\}_{\varepsilon>0}$, where

$$p^0 = \left(\frac{1}{|\partial Q|_H} \int_{\partial Q} p^*\,\mathrm{d}\mathcal{H}^{n-1}\right) \quad \text{and} \quad a^0 = \left(\frac{1}{|K \cap \partial\Lambda|_H} \int_{K\cap\partial\Lambda} a^*\,\mathrm{d}\mathcal{H}^{n-1}\right),$$

is a sequence of asymptotically suboptimal controls for the problem $\widehat{\mathbb{P}}_\varepsilon$.

13

Asymptotic Analysis of Elliptic Optimal Control Problems in Thick Multistructures with Dirichlet and Neumann Boundary Controls

In this chapter, we study a class of optimal control problems (OCPs) for a linear elliptic equation in a domain $\Omega_\varepsilon \subset \mathbb{R}^n$ (thick multistructure), whose boundary $\partial\Omega_\varepsilon$ contains a very highly oscillating part with respect to ε, as $\varepsilon \to 0$. We consider this problem assuming that there are two types of the controls active via Neumann and Dirichlet boundary conditions posed on the different parts of the oscillating boundary (for a comparison, see [73, 88]).

By a thick multistructure Ω_ε we mean a domain in $\mathbb{R}^n$, which consists of some domain Ω^+ and a large number of cylinders with axes parallel to Ox_n and ε-periodically distributed along some manifold Σ on the boundary of Ω^+ (see Fig. 13.1). This manifold is called the joint zone and the domain Ω^+ is called the junction's body. Here, ε is a small positive parameter which characterizes the distance between the neighboring cylinders and their thickness. Thus, each attached cylinder has a small cross-section of size ε and its limiting dimension (as $\varepsilon \to 0$) is equal to 1. In view of this, such cylinders will be called thin domains. For the study of boundary value problems in thick multistructures, we refer to [35, 58, 89, 90, 187], where, however, OCPs have not been considered.

Thick multistructures (or thick junctions) are prototypes of widely used engineering constructions as well as many other physical and biological systems with very distinct characteristic scales (microscopic radiators, ferrite-filled rod radiators, and others). The computational calculation of the solutions of boundary value problems is very complicated due to singularities associated with the thick junctions. Asymptotic analysis is one of the main approaches to study boundary value problems in such domains. However, only few works deal with the problem of optimal control of partial differential equations (PDEs) in thick multistructures (see [85, 88, 100, 153]). This can be explained as follows. Thick junctions have a special character of the connectedness: There are points in a thick junction which are at a short distance of order $\mathcal{O}(\varepsilon)$, but the length of all curves, which connect these points in the junction, is of order $\mathcal{O}(1)$. As a result, there are no extension operators $P : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}_{loc}(\mathbb{R}^n)$ that would be uniformly bounded in ε. Addition-

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_13,
© Springer Science+Business Media, LLC 2011

ally, boundary value problems lose coercivity in the limit passage as $\varepsilon \to 0$ and this creates special difficulties in the asymptotic investigation.

13.1 Statement of the problem and basic notation

Let $B = (0,a)^{n-1}$ and C be bounded open smooth domain in $\mathbb{R}^{n-1}$ $(n \geq 2)$ and $C \subset\subset (0,1)^{n-1}$, let $\gamma \in C^1(\overline{B})$ and $\beta \in C^1(\overline{B})$ be functions such that

$$0 < \gamma_0 = \inf_{x' \in B} \gamma(x'), \quad 0 < \beta_0 = \inf_{x' \in B} \beta(x'),$$

and let $\{\varepsilon\}$ be a sequence of positive numbers converging to 0. In the sequel, we will always assume that $\varepsilon = a/N$, where N is a large positive integer. When we write $\varepsilon > 0$, we consider only the elements of this sequence. Let us denote a point of $\mathbb{R}^n$ by $x = (x_1, x_2, \ldots, x_n) = (x', x_n)$ and introduce the following sets:

$$\begin{aligned}
\Omega &= \{(x', x_n) \in \mathbb{R}^n \ : \ -\beta(x') < x_n < \gamma(x'),\ x' \in B\},\\
\Sigma &= B \times \{0\},\\
\Omega^+ &= \{(x', x_n) \in \mathbb{R}^n \ : \ x' \in B,\ 0 < x_n < \gamma(x')\},\\
\Omega^- &= \{(x', x_n) \in \mathbb{R}^n \ : \ x' \in B,\ -\beta(x') < x_n < 0\},\\
\Gamma_0 &= \{(x', x_n) \in \mathbb{R}^n \ : \ x' \in B,\ x_n = -\beta(x')\},\\
\theta_\varepsilon &= \{\mathbf{k} = (k_1, k_2, \ldots, k_{n-1}) \in \mathbb{N}^{n-1} \ : \ \varepsilon C + \varepsilon \mathbf{k} \subset\subset B\},\\
\Omega_\varepsilon &= \Omega^+ \cup \{(x', x_n) \ : \ x' \in \textstyle\bigcup_{\mathbf{k} \in \theta_\varepsilon} (\varepsilon C + \varepsilon \mathbf{k}),\ -\beta(x') < x_n \leq 0\},\\
\Omega_\varepsilon^- &= \Omega_\varepsilon \cap \Omega^-.
\end{aligned} \tag{13.1}$$

Then, in view of our previous description, the set Ω_ε is a thick multistructure which consists of the junction's body Ω^+ and a large number N^{n-1} of the thin cylinders $G_\varepsilon^{\mathbf{k}}$ with axis Ox_n and ε-periodically distributed on the basis Σ of Ω^+ (see Fig. 13.1 for a 3D example). Here, each cylinder $G_\varepsilon^{\mathbf{k}}$ is obtained with an ε-homothety in the first $(n-1)$ variables, that is,

$$G_\varepsilon^{\mathbf{k}} = \{(x', x_n) \ : \ x' \in \varepsilon C + \varepsilon \mathbf{k},\ -\beta(x') < x_n \leq 0\}.$$

It is easy to see that $\Omega_\varepsilon = \Omega^+ \cup \big(\bigcup_{\mathbf{k} \in \theta_\varepsilon} G_\varepsilon^{\mathbf{k}}\big)$.

Denote by Γ_ε the union of the lower bases

$$\Gamma_\varepsilon^{\mathbf{k}} = \{(x', x_n) \ : \ x' \in \varepsilon \cdot C + \varepsilon \mathbf{k},\ x_n = -\beta(x')\}$$

of the thin cylinders $G_\varepsilon^{\mathbf{k}}$ when $\mathbf{k} \in \theta_\varepsilon$ and by S_ε the union of their boundaries along the axis

$$Ox_n \ : \ S_\varepsilon^{\mathbf{k}} = \{(x', x_n) \ : \ x' \in \varepsilon \partial C + \varepsilon \mathbf{k},\ -\beta(x') < x_n < 0\}.$$

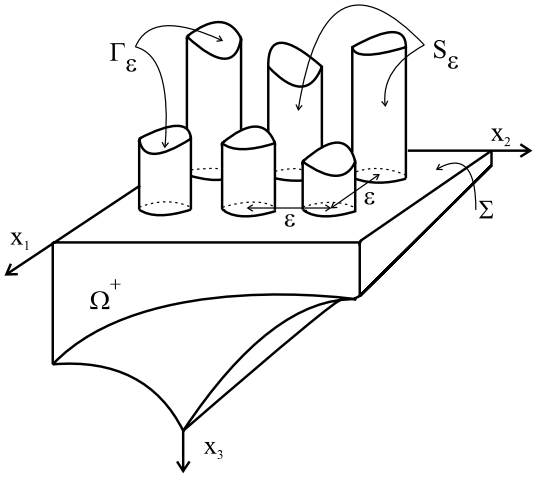


Fig. 13.1. 3D thick multistructure

In Ω_ε, we consider the following optimal boundary control problem:

$$I_\varepsilon = \int_{\Omega^+} (y_\varepsilon - q_0)^2 \mathrm{d}x + \int_{\Gamma_\varepsilon} u_\varepsilon^2 \, \mathrm{d}\mathcal{H}^{n-1} + \varepsilon \int_{S_\varepsilon} p_\varepsilon^2 \, \mathrm{d}\mathcal{H}^{n-1} \longrightarrow \inf, \tag{13.2}$$

$$\left.\begin{aligned} -\Delta_x \, y_\varepsilon(x) + y_\varepsilon &= f_\varepsilon(x), && x \in \Omega_\varepsilon, \\ \partial_\nu y_\varepsilon(x) &= \varepsilon \, p_\varepsilon(x), && x \in S_\varepsilon, \\ y_\varepsilon(x) &= u_\varepsilon(x), && x \in \Gamma_\varepsilon, \\ \partial_\nu y_\varepsilon(x) &= 0, && x \in \partial\Omega_\varepsilon \setminus (\Gamma_\varepsilon \cup S_\varepsilon), \end{aligned}\right\} \tag{13.3}$$

$$u_\varepsilon \in U_\varepsilon = \left\{ u \in H^{1/2}(\Gamma_\varepsilon; \mathcal{H}^{n-1}), \quad \|u\|_{L^2(\Gamma_\varepsilon;\mathcal{H}^{n-1})} \le \mathbf{C_u} \right\}, \tag{13.4}$$

$$p_\varepsilon \in P_\varepsilon = \left\{ p \in L^2(S_\varepsilon; \mathcal{H}^{n-1}), \quad \varepsilon \|p\|^2_{L^2(S_\varepsilon;\mathcal{H}^{n-1})} \le \mathbf{C_p} \right\}, \tag{13.5}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$, which coincides with ordinary $(n-1)$-dimensional surface area, $q_0 \in L^2(\Omega^+)$ and $f_\varepsilon \in L^2(\Omega)$ are given functions such that

$$f_\varepsilon \longrightarrow f_0 \quad \text{strongly in} \quad L_2(\Omega) \quad \text{as} \quad \varepsilon \to 0, \tag{13.6}$$

$\partial_\nu = \partial/\partial\nu$ is the outward normal derivative, $\mathbf{C_u} > 0$ and $\mathbf{C_p} > 0$ are some fixed constants that are independent of $\varepsilon, u_\varepsilon$ and p_ε, U_ε and P_ε are the sets of admissible controls, u_ε and p_ε are control functions, and $y_\varepsilon = y_\varepsilon(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)$ is the corresponding weak solution of the boundary value problem (13.3). In the sequel, for simplicity, we will use the following notation: $H^{1/2}(\Gamma_\varepsilon) = H^{1/2}(\Gamma_\varepsilon; \mathcal{H}^{n-1})$, $L^2(\Gamma_\varepsilon) = L^2(\Gamma_\varepsilon; \mathcal{H}^{n-1})$, and $L^2(S_\varepsilon) = L^2(S_\varepsilon; \mathcal{H}^{n-1})$.

It is well known that for every fixed ε and for any control functions $u_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ and $p_\varepsilon \in L^2(S_\varepsilon)$, the boundary value problem (13.3) admits a unique solution $y_\varepsilon \in H^1(\Omega_\varepsilon)$ such that the following integral identity

$$\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla \varphi \, \mathrm{d}x + \int_{\Omega_\varepsilon} y_\varepsilon \, \varphi \, \mathrm{d}x = \int_{\Omega_\varepsilon} f_\varepsilon(x) \, \varphi(x) \, \mathrm{d}x + \varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi \, \mathrm{d}\mathcal{H}^{n-1}, \quad \forall \, \varphi \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon) \tag{13.7}$$

holds and the trace of y_ε equals u_ε on Γ_ε. The function y_ε is called the weak solution to the problem (13.3). Here,

$$H^1(\Omega_\varepsilon; \Gamma_\varepsilon) = \{\varphi \in H^1(\Omega_\varepsilon) : \ \varphi = 0 \ \text{ on } \ \Gamma_\varepsilon\}.$$

In addition, if $u \in H^{1/2}(\Gamma_\varepsilon)$ and $p_\varepsilon \in L^2(S_\varepsilon)$, then the weak solution satisfies the a priori norm estimate

$$\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le c_1 \Big(\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{H^{1/2}(\Gamma_\varepsilon)} + \sqrt{\varepsilon} \, \|p_\varepsilon\|_{L^2(S_\varepsilon)} \Big). \tag{13.8}$$

Remark 13.1. Since the thick junction Ω_ε is a nonconvex domain with a nonsmooth boundary, it follows that solutions of boundary value problems in such a domain have only minimal H^1-smoothness. So it makes no sense to take admissible boundary controls with more smoothness than $u \in H^{1/2}(\Gamma_\varepsilon; \mathcal{H}^{n-1})$ and $p \in L^2(S_\varepsilon; \mathcal{H}^{n-1})$. For a comparison, we refer to [101], where the class of admissible Dirichlet controls was taken as

$$\Theta_\varepsilon = \Big\{ u = \vartheta|_{\Gamma_\varepsilon} \ : \ \vartheta \in H^{3/2}(\Gamma_0; \mathcal{H}^{n-1}), \quad \|u\|_{H^{3/2}(\Gamma_0; \mathcal{H}^{n-1})} \le \mathbf{C_u} \Big\}.$$

The aim of our research in this chapter is to study the asymptotic behavior of this OCP as $\varepsilon \to 0$, that is, when the number of attached thin cylinders infinitely increases and their thickness vanishes. The approach we propose gives the possibility of replacing the original OCP by some limit problem defined in a more simple domain. We show that an optimal control for the limit problem can be taken as a prototype for the modeling of suboptimal controls to the original one. It should be stressed here that if the small parameter ε is changed, then all components of this control problems (the domain Ω_ε, the constraint sets U_ε and P_ε, the cost functional I_ε, and the set, where we seek its infimum) are changed as well. Let us observe also that the volume of the material included in the set Ω_ε^- does not converge to 0 as $\varepsilon \to 0$. Moreover, $\lim_{\varepsilon \to 0} \Omega_\varepsilon^- = \Omega^-$ in the Hausdorff metric and $|\Omega_\varepsilon^-| \to |\Omega^-|$, that is, the set Ω^- is filled up by the thin cylinders in the limit passage as $\varepsilon \to 0$. It produces the fact that the Neumann boundary controls p_ε will transform (as $\varepsilon \to 0$) to some distributed control function $p \in L^2(\Omega^-)$ on the right-hand side of the homogenized equation.

13.2 Description of the optimal control problem in terms of singular measures

Following the standard approach, we will describe the geometry of the set S_ε, and hence the class of admissible boundary controls, in the terms of singular measures on $\mathbb{R}^n$ which do not satisfy the regularity property with respect to the corresponding Lebesgue measures.

Let μ_0 be a periodic finite positive Borel measure on $\mathbb{R}^{n-1}$. Let $\square = [0,1)^{n-1}$ be the cell or torus of periodicity for μ_0. We assume that the Borel measure μ_0 is the probability measure on $\mathbb{R}^{n-1}$, concentrated and uniformly distributed on the set ∂C, so $\int_\square \mathrm{d}\mu_0 = 1$.

Remark 13.2. By definition, we have $\mu_0(\square \backslash \partial C) = 0$. Therefore, any functions, taking the same values on the set ∂C, coincide as elements of $L^2(\square, \mathrm{d}\mu_0)$. Here, the Lebesgue space $L^2(\square, \mathrm{d}\mu_0)$ with respect to the measure μ_0 is defined in a usual way with the corresponding norm

$$\|f\|^2_{L^2(\square,\,\mathrm{d}\mu_0)} = \int_\square |f(x)|^2 \,\mathrm{d}\mu_0$$

(we adopt the standard notation $L^2(\square)$ when μ_0 is the Lebesgue measure).

We set $\square_n = \square \times [0,1) = [0,1)^n$ and consider the measure $\mathrm{d}\mu = \mathrm{d}\mu_0 \times \mathrm{d}x_n$ on $\square_n$. It is easy to see that this measure is concentrated on the set $\partial C \times [0,1)$, and for any smooth function g, we have

$$\int_{\square_n} g \,\mathrm{d}\mu = \int_0^1 \int_\square g \,\mathrm{d}x_n \,\mathrm{d}\mu_0 \;=\; \left[\mathcal{H}^{n-1}(\partial C \times [0,1))\right]^{-1} \int_{\partial C \times [0,1)} g \,\mathrm{d}\mathcal{H}^{n-1}.$$

However, as follows from the properties of the Hausdorff measure, we have $\mathcal{H}^{n-1}(\partial C \times [0,1)) = \mathcal{H}^{n-2}(\partial C)$ (see [106]). In what follows, we use the notation $|\partial C|_H = \mathcal{H}^{n-2}(\partial C)$ for $(n-2)$-dimensional Hausdorff measure of the set ∂C. Then the previous relation can be rewritten in the form

$$\int_{\square_n} g \,\mathrm{d}\mu = \int_0^1 \int_\square g \,\mathrm{d}x_n \,\mathrm{d}\mu_0 \;=\; |\partial C|_H^{-1} \int_{\partial C \times (0,1)} g \,\mathrm{d}\mathcal{H}^{n-1}. \tag{13.9}$$

In order to clarify relation (13.9), we consider the planar thick multistructure $\Omega_\varepsilon \subset \mathbb{R}^2$. Then $n = 2$ and the set C is some part of the segment $(0,1)$. For instance, let $C = \{x_1 \in (0,1) \;:\; |x_1 - 1/2| < h/2\}$, where $h \in (0,1)$ is a fixed value. In this case, $|\partial C|_H = 2$ and the 1-periodic measure μ_0 on $\mathbb{R}^1$ can be defined as

$$\mu_0(B) = \frac{1}{|\partial C|_H} \left(\delta_{M_1} + \delta_{M_2}\right)(B) = \frac{1}{2} \left(\delta_{M_1} + \delta_{M_2}\right)(B)$$

for any Borel set $B \subseteq [0,1)$, where $M_i = \frac{1}{2} + \left(i - \frac{3}{2}\right)h$, $i = 1,2$. Here, δ_{M_i} is the Dirac measures located at the points M_i. Thus, the multiplier $|\partial C|_H^{-1}$ in (13.9) is equal to $1/2$.

Let Λ be any Borel set of $\mathbb{R}^n$. We introduce the scaling measure μ_ε by the rule $\mu_\varepsilon(\Lambda) = \varepsilon^n \mu(\varepsilon^{-1}\Lambda)$. This measure has a period ε, and, moreover, since $\mu(\varepsilon\square_n) = \varepsilon\,\mu_0(\varepsilon\square)$, it follows that

$$\mu_\varepsilon(\varepsilon\square_n) = \varepsilon^n \int_0^\varepsilon \int_{\varepsilon\square} \mathrm{d}\mu_0(x'/\varepsilon)\,\mathrm{d}(x_n/\varepsilon) = \varepsilon^n \int_0^1 \int_{\square} \mathrm{d}\mu_0\,\mathrm{d}x_n = \varepsilon^n.$$

As a result, the measure μ_ε weakly converges to Lebesgue measure on $\mathbb{R}^n$ as $\varepsilon \to 0$ (in symbols $\mathrm{d}\mu_\varepsilon \rightharpoonup \mathrm{d}x$), that is,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi\,\mathrm{d}\mu_\varepsilon = \int_{\mathbb{R}^n} \varphi\,\mathrm{d}x \quad \text{for all functions} \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

Now, we turn back to the definition of a weak solution $y_\varepsilon \in H^1(\Omega_\varepsilon)$ of the boundary value problem (13.3) (see (13.7)). Since the Sobolev space $H^1(\Omega_\varepsilon)$ can be viewed as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\left(\int_{\Omega_\varepsilon} (y^2 + |\nabla y|^2)\,\mathrm{d}x\right)^{1/2},$$

it follows that a function $y_\varepsilon \in H^1(\Omega_\varepsilon)$ is a weak solution of the above-mentioned problem whenever

$$\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla\varphi\,\mathrm{d}x + \int_{\Omega_\varepsilon} y_\varepsilon\,\varphi\,\mathrm{d}x = \int_{\Omega_\varepsilon} f_\varepsilon(x)\,\varphi(x)\,\mathrm{d}x + \varepsilon \int_{S_\varepsilon} p_\varepsilon\varphi\,\mathrm{d}\mathcal{H}^{n-1} \quad (13.10)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$, where $C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$ is the set of all functions of $C_0^\infty(\mathbb{R}^n)$ such that $\varphi|_{\Gamma_\varepsilon} = 0$.

Let us consider the last term on the left-hand side of identity (13.10). Using the notation of Sect. 13.1, we get

$$\begin{aligned}
\varepsilon \int_{S_\varepsilon} p_\varepsilon\varphi\,\mathrm{d}\mathcal{H}^{n-1} &= \varepsilon \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\partial C+\mathbf{k}_j)} \int_{-\beta(x')}^0 p_\varepsilon\varphi\,\mathrm{d}\mathcal{H}^{n-2}\,\mathrm{d}x_n \\
&= \varepsilon|\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square+\mathbf{k}_j)} \int_{-\beta(x')}^0 \widehat{p}_\varepsilon\varphi\,\varepsilon^{n-2}\,\mathrm{d}\mu_0(x'/\varepsilon)\,\mathrm{d}x_n \\
&= |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square+\mathbf{k}_j)} \int_{-\beta(x')}^0 \widehat{p}_\varepsilon\varphi\;\varepsilon^n\,\mathrm{d}\mu_0(x'/\varepsilon)\,\mathrm{d}(x_n/\varepsilon) \\
&= |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\left\{(x',x_n)\,:\,\left|\begin{smallmatrix} x'\in\varepsilon(\square+\mathbf{k}_j) \\ -\beta(x')<x_n<0 \end{smallmatrix}\right.\right\}} \widehat{p}_\varepsilon\varphi\,\mathrm{d}\mu_\varepsilon \\
&= |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon\varphi\,\mathrm{d}\mu_\varepsilon. \qquad (13.11)
\end{aligned}$$

Here, a function $\widehat{p}_\varepsilon$ is defined as follows: $\widehat{p}_\varepsilon = p_\varepsilon$ on $S_\varepsilon \equiv \mathrm{spt}(\mu_\varepsilon)$ and $\widehat{p}_\varepsilon = 0$ on $\Omega^- \setminus S_\varepsilon$. Hence, $\widehat{p}_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)$. It is clear that any functions, taking the same values on the set S_ε, coincide as elements of $L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)$. In view of this, any function $\widehat{p}_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)$ with the property

$$|\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, \mathrm{d}\mu_\varepsilon = \varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi \, \mathrm{d}\mathcal{H}^{n-1}, \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n),$$

we call a prototype of the boundary control $p_\varepsilon \in L^2(S_\varepsilon)$. We note also that the integral $\int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, \mathrm{d}\mu_\varepsilon$ is well defined for every function $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$. Indeed, since the set Ω^- is bounded and $\widehat{p}_\varepsilon \, \mathrm{d}\mu_\varepsilon$ is a Radon measure, it follows that $\int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, \mathrm{d}\mu_\varepsilon$ is a linear continuous functional on $C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$.

Definition 13.3. *We say that a function $y_\varepsilon \in H^1(\Omega_\varepsilon)$ is a weak solution of the boundary value problem (13.3) if for given functions $u_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ and $\widehat{p}_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)$ the integral identity*

$$\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla \varphi \, \mathrm{d}x + \int_{\Omega_\varepsilon} y_\varepsilon \varphi \, \mathrm{d}x = |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, \mathrm{d}\mu_\varepsilon + \int_{\Omega_\varepsilon} f_\varepsilon(x) \varphi(x) \, \mathrm{d}x \quad (13.12)$$

holds for every $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$ and the trace of y_ε equals u_ε on Γ_ε.

Let us introduce the set

$$\widehat{P}_\varepsilon = \{\widehat{p} \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon) \ : \ \|\widehat{p}\|^2_{L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)} \le |\partial C|_H^{-1} \mathbf{C_p}\}. \quad (13.13)$$

Let $p_\varepsilon \in P_\varepsilon$ be any admissible boundary control. Let $\widehat{p}_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)$ be its prototype. Then, in a manner similar to (13.11), we may derive the relation

$$\varepsilon \int_{S_\varepsilon} p_\varepsilon^2 \, \mathrm{d}\mathcal{H}^{n-1} = |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon{}^2 \, \mathrm{d}\mu_\varepsilon. \quad (13.14)$$

In view of this result, we can reformulate the original optimal boundary control problem (13.2)–(13.5) as follows: Find a triplet $(u_\varepsilon^0, \widehat{p}_\varepsilon{}^0, y_\varepsilon^0)$ such that

$$(u_\varepsilon^0, \widehat{p}_\varepsilon{}^0, y_\varepsilon^0) \in \mathbb{X}_\varepsilon \equiv H^{1/2}(\Gamma_\varepsilon) \times L^2(\Omega^-, \mathrm{d}\mu_\varepsilon) \times H^1(\Omega_\varepsilon)$$

and

$$\begin{aligned} I_\varepsilon(u_\varepsilon^0, \widehat{p}_\varepsilon{}^0, y_\varepsilon^0) &= \inf_{(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \\ &= \inf_{(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} \Big[\int_{\Omega^+} (y_\varepsilon - q_0)^2 \mathrm{d}x + \int_{\Gamma_\varepsilon} u_\varepsilon^2 \, \mathrm{d}\mathcal{H}^{n-1} \\ &\qquad + |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon{}^2 \, \mathrm{d}\mu_\varepsilon \Big], \end{aligned} \quad (13.15)$$

where the set $\Xi_\varepsilon \subset \mathbb{X}_\varepsilon$ has the form

$$\Xi_\varepsilon = \left\{ (u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \;\middle|\; \begin{array}{c} u_\varepsilon \in U_\varepsilon, \quad \widehat{p}_\varepsilon \in \widehat{P}_\varepsilon, \quad y_\varepsilon \in H^1(\Omega_\varepsilon), \\ y_\varepsilon|_{\Gamma_\varepsilon} = u_\varepsilon \quad \mathcal{H}^{n-1} \text{ a. e. on } \Gamma_\varepsilon, \\ \displaystyle\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla \varphi \, \mathrm{d}x + \int_{\Omega_\varepsilon} y_\varepsilon \varphi \, \mathrm{d}x \\ = |\partial C|_H \displaystyle\int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, \mathrm{d}\mu_\varepsilon + \int_{\Omega_\varepsilon} f_\varepsilon(x) \varphi(x) \, \mathrm{d}x, \\ \forall\, \varphi \in C_0^\infty(\mathbb{R}^n, \Gamma_\varepsilon) \end{array} \right\}. \quad (13.16)$$

In the sequel, the OCP (13.15)–(13.16) will be called the $\widehat{\mathbb{P}}_\varepsilon$-problem. Due to the norm estimate (13.8), we have the following obvious result.

Proposition 13.4. *Let $u_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ and $\widehat{p}_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)$ be any admissible control functions for the $\widehat{\mathbb{P}}_\varepsilon$-problem. Then there exists a constant $c > 0$, independent of ε, such that*

$$\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le c \left(\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{H^{1/2}(\Gamma_\varepsilon)} + \|\widehat{p}_\varepsilon\|_{L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)} \right) \quad (13.17)$$

for all $\varepsilon > 0$.

Definition 13.5. *The triplet $(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \mathbb{X}_\varepsilon$ is called admissible for the problem $\widehat{\mathbb{P}}_\varepsilon$ if $u_\varepsilon \in U_\varepsilon$, $\widehat{p}_\varepsilon \in \widehat{P}_\varepsilon$, and $y_\varepsilon = y_\varepsilon(u_\varepsilon, \widehat{p}_\varepsilon)$ is the corresponding weak solution of the problem (13.3) whose trace on Γ_ε is equal to u_ε.*

It is easy to see that the set Ξ_ε of all admissible solutions for $\widehat{\mathbb{P}}_\varepsilon$ is nonempty, convex, and τ_ε-closed for every $\varepsilon > 0$, where by τ_ε we denote the product of the weak topologies of $H^{1/2}(\Gamma_\varepsilon)$, $L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)$, and $H^1(\Omega_\varepsilon)$. Note that the cost functional (13.15) is convex and lower τ_ε-semicontinuous, but not τ_ε-coercive. So, in general, we cannot assert the solvability of this problem [111, 169]. Thus, the OCPs we consider are such that the existence of an optimal solution for the control problem $\widehat{\mathbb{P}}_\varepsilon$ must be assumed. This is untypical for the majority of investigations in this field. However, if an optimal solution to the $\mathbb{P}_\varepsilon$ problem exists, then this solution is unique. Additionally, for every $\delta > 0$, a triplet $(u_\varepsilon^\delta, \widehat{p}_\varepsilon{}^\delta, y_\varepsilon^\delta) \in \Xi_\varepsilon$ (the so-called approximate solution) can be always found such that

$$\left| I_\varepsilon(u_\varepsilon^\delta, \widehat{p}_\varepsilon{}^\delta, y_\varepsilon^\delta) - \inf_{(u,p,y) \in \Xi_\varepsilon} I_\varepsilon(u, p, y) \right| \le \delta. \quad (13.18)$$

13.3 The choice of topology

Following the concept of variational convergence, we represent the $\widehat{\mathbb{P}}_\varepsilon$-problem for various values of ε in the form of the sequence of corresponding constrained minimization problems

$$\left\{ \left\langle \inf_{(u,p,y)\in\Xi_\varepsilon} I_\varepsilon(u,p,y) \right\rangle ;\ \varepsilon = a/N \to 0 \right\}, \tag{13.19}$$

where the cost functional $I_\varepsilon : \Xi_\varepsilon \to \mathbb{R}$ and the sets of admissible solutions are defined in (13.15) and (13.16), respectively. Then the study of asymptotic behavior of the OCP (13.15)–(13.16) as $\varepsilon \to 0$ can be reduced to the analysis of the limit properties of the sequences (13.19). However, since for every fixed ε the admissible solution $(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon)$ belongs to the functional space

$$\mathbb{X}_\varepsilon = H^{1/2}(\Gamma_\varepsilon) \times L^2(\Omega^-, \mathrm{d}\mu_\varepsilon) \times H^1(\Omega_\varepsilon)$$

depending on the small parameter ε, we focus our attention on the convergence formalism in these variable spaces.

Let us denote by $\widetilde{u}_\varepsilon$ the trivial extension by 0 of a function $u_\varepsilon \in L^2(\Gamma_\varepsilon)$ to the set Γ_0 (see (13.1)). Then $\widetilde{u}_\varepsilon \in L^2(\Gamma_0)$.

Definition 13.6. *We say that a sequence* $\left\{u_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)\right\}_{\varepsilon>0}$ *weakly converges to a function* u^* *with respect to the space* $L^2(\Gamma_0)$ *if* $\widetilde{u}_\varepsilon \rightharpoonup u^*$ *in* $L^2(\Gamma_0)$.

Using the ideas of Sect. 7.2 (see Proposition 7.12) and the convergence concept in variable spaces (see Sect. 6.2), we state the following result.

Remark 13.7. For brevity, hereinafter in this chapter we restrict ourself to the case when the function $\beta \in C^1(\overline{B})$ is kept constant $\beta(x) \equiv \beta_0$, $\forall\, x \in B$. Then

$$\Gamma_0 = \{(x', x_n) \in \mathbb{R}^n \ :\ x' \in B,\ x_n = -\beta_0\},$$

$$\Gamma_\varepsilon = \bigcup_{\mathbf{k}\in\theta_\varepsilon} \{(x', x_n) \ :\ x' \in \varepsilon C + \varepsilon\mathbf{k},\ x_n = -\beta_0\},$$

$$G_\varepsilon = \bigcup_{\mathbf{k}\in\theta_\varepsilon} \{(x', x_n) \ :\ x' \in \varepsilon C + \varepsilon\mathbf{k},\ -\beta_0 < x_n < 0\}.$$

Proposition 13.8. *For every bounded sequence* $\{u_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)\}_{\varepsilon>0}$ *such that* $\widetilde{u}_\varepsilon \rightharpoonup u_*$ *in* $L^2(\Gamma_0)$, *the following inequality holds:*

$$\liminf_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} u_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{n-1} \geq |C|^{-1} \int_{\Gamma_0} u_*^2 \,\mathrm{d}\mathcal{H}^{n-1}. \tag{13.20}$$

Proof. Let χ_{Γ_0} and χ_{G_ε} be the characteristic functions of the sets Γ_0 and G_ε, respectively. Then $\Gamma_\varepsilon = \Gamma_0 \cap \overline{G}_\varepsilon$. Let $\chi_{\Gamma_\varepsilon} \in L^\infty(\Gamma_0)$ be the characteristic function of the Dirichlet control zone Γ_ε, that is,

$$\chi_{\Gamma_\varepsilon}(x') = \begin{cases} 1 & \text{if } x' \in \Gamma_\varepsilon, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall\, x' \in \Gamma_0.$$

To begin, let us show that

$$\chi_{\Gamma_\varepsilon} \overset{*}{\rightharpoonup} |C| \ \text{ in } \ L^\infty(\Gamma_0) \ \text{ as } \ \varepsilon \to 0. \tag{13.21}$$

Indeed, in view of $\varepsilon\,[0,1)^{n-1}$-periodicity of this characteristic function and the mean value property, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_0}\varphi(x')\chi_{\Gamma_\varepsilon}(x')\,\mathrm{d}\mathcal{H}^{n-1} &= \lim_{\varepsilon\to 0}\int_B \varphi(x')\chi_{\cup_{\mathbf{k}\in\theta_\varepsilon}(\varepsilon C+\varepsilon\mathbf{k})}(x')\,\mathrm{d}x'\\ &= \int_B \varphi(x')\,\mathrm{d}x'\left(\int_{(0,1)^{n-1}}\chi_C(x')\,\mathrm{d}x'\right)\\ &= |C|\int_B\varphi(x')\,\mathrm{d}x'\\ &= |C|\int_{\Gamma_0}\varphi(x')\,\mathrm{d}\mathcal{H}^{n-1},\quad \forall\,\varphi\in C_0^\infty(\Gamma_0).\end{aligned}$$

Thus, statement (13.21) holds true.

Let ν_ε be a Radon measure on Γ_0 defined as the surface measure $\mathrm{d}\nu_\varepsilon = \chi_{\Gamma_\varepsilon}\mathrm{d}\mathcal{H}^{n-1}$. Then, in view of (13.21), we have

$$\mathrm{d}\nu_\varepsilon \rightharpoonup \mathrm{d}\nu = |C|\,\mathrm{d}\mathcal{H}^{n-1}\ \text{ in the sense of Definition 2.29.} \tag{13.22}$$

Let $\{u_\varepsilon\in H^{1/2}(\Gamma_\varepsilon)\}_{\varepsilon>0}$ be a bounded sequence of admissible controls such that $\widetilde{u}_\varepsilon \rightharpoonup u_*$ in $L^2(\Gamma_0)$. Hence,

$$\int_{\Gamma_0}\widetilde{u}_\varepsilon\varphi\,\mathrm{d}\mathcal{H}^{n-1}\longrightarrow\int_{\Gamma_0}u_*\varphi\,\mathrm{d}\mathcal{H}^{n-1}\ \text{ as }\ \varepsilon\to 0,\quad \forall\,\varphi\in C_0^\infty(\Gamma_0). \tag{13.23}$$

On the other hand, since $u_\varepsilon\in U_\varepsilon$ for every $\varepsilon>0$, it follows that

$$\sup_{\varepsilon>0}\int_{\Gamma_0}\widetilde{u}_\varepsilon^2\,\mathrm{d}\nu_\varepsilon = \sup_{\varepsilon>0}\int_{\Gamma_0}\widetilde{u}_\varepsilon^2\chi_{\Gamma_\varepsilon}\mathrm{d}\mathcal{H}^{n-1} = \sup_{\varepsilon>0}\|u_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)}\le \mathbf{C_u}^2,$$

that is, the sequence $\left\{\widetilde{u}_\varepsilon\in L^2(\Gamma_0,\mathrm{d}\nu_\varepsilon)\right\}_{\varepsilon>0}$ is bounded. So, by Proposition 6.15, we can suppose that this sequence is compact with respect to the weak convergence in the variable space $L^2(\Gamma_0,\mathrm{d}\nu_\varepsilon)$. Hence, there exists an element $u_0\in L^2(\Gamma_0,\mathrm{d}\nu)$ such that passing if necessary to a subsequence, we get

$$\begin{aligned}\int_{\Gamma_0}\widetilde{u}_\varepsilon\varphi\,\mathrm{d}\mathcal{H}^{n-1} &= \int_{\Gamma_0}\widetilde{u}_\varepsilon\varphi\,\mathrm{d}\nu_\varepsilon\longrightarrow\int_{\Gamma_0}u_0\varphi\,\mathrm{d}\nu\\ &= |C|\int_{\Gamma_0}u_0\varphi\,\mathrm{d}\mathcal{H}^{n-1}\ \text{ as }\ \varepsilon\to 0,\quad \forall\,\varphi\in C_0^\infty(\Gamma_0).\end{aligned} \tag{13.24}$$

Having compared the results of limit passages (13.23) and (13.24), we deduce that

$$u_0 = |C|^{-1}u_*. \tag{13.25}$$

As a result, inequality (13.20) is a direct consequence of Proposition 6.16. Indeed, by (6.28) and (13.24), we have

$$\liminf_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} u_\varepsilon^2\,\mathrm{d}\mathcal{H}^{n-1} = \liminf_{\varepsilon\to 0}\int_{\Gamma_0}\widetilde{u}_\varepsilon^2\,\mathrm{d}\nu_\varepsilon \geq \int_{\Gamma_0} u_0^2\,\mathrm{d}\nu$$
$$= \int_{\Gamma_0}|C|^{-2}u_*^2|C|\,\mathrm{d}\mathcal{H}^{n-1} = |C|^{-1}\int_{\Gamma_0} u_*^2\,\mathrm{d}\mathcal{H}^{n-1}.$$

This concludes the proof. □

In fact, in view Proposition 13.8 and representation (13.4), we can deduce the following lemma.

Lemma 13.9. *Every sequence of admissible controls* $\{u_\varepsilon \in U_\varepsilon\}_{\varepsilon>0}$ *is relatively compact with respect to the convergence in the sense of Definition 13.6. Moreover, its weak* L^2*-limit* u^* *satisfies the following condition:*

$$\|u^*\|_{L^2(\Gamma_0)} \leq \sqrt{|C|}\mathbf{C_u} < \mathbf{C_u}.$$

Now, we give the convergence formalism of the sequence of boundary controls $\{\widehat{p}_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$. First, we recall that the sequence of Borel measures $\{\mathrm{d}\mu_\varepsilon\}$ weakly converges to Lebesgue measure $\mathrm{d}x$. Let

$$\{\widehat{p}_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$$

be any bounded sequence, that is,

$$\limsup_{\varepsilon\to 0}\int_{\Omega^-}\widehat{p}_\varepsilon^2\,\mathrm{d}\mu_\varepsilon < +\infty.$$

Definition 13.10. *We say that a bounded sequence* $\{\widehat{p}_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ *is weakly convergent if there exists an element* $p^* \in L^2(\Omega^-)$ *such that*

$$\lim_{\varepsilon\to 0}\int_{\Omega^-}\varphi\widehat{p}_\varepsilon\,\mathrm{d}\mu_\varepsilon = \int_{\Omega^-}\varphi p^*\,\mathrm{d}x \quad \textit{for any function} \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

Note that the class of test functions in Definition 13.10 can be essentially extended (see Sect. 6.2)

Proposition 13.11. *If a bounded sequence* $\{\widehat{p}_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ *converges weakly to* $p^* \in L^2(\Omega^-)$ *and a function* $\psi \in L^2(\Omega^-)$ *is such that* $|\psi(x)| \leq const$ *almost everywhere on* Ω^-*, then*

$$\lim_{\varepsilon\to 0}\int_{\Omega^-}\psi\widehat{p}_\varepsilon\,\mathrm{d}\mu_\varepsilon = \int_{\Omega^-}\psi p^*\,\mathrm{d}x, \quad \lim_{\varepsilon\to 0}\int_{\Omega^-}\psi\,\mathrm{d}\mu_\varepsilon = \int_{\Omega^-}\psi\,\mathrm{d}x.$$

Proposition 13.12. *Let* $\{\widehat{p}_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$ *be any bounded sequence. Then the following hold:*

(i) This sequence is relatively weakly compact.

(ii) If $\widehat{p}_\varepsilon \rightharpoonup p^*$ *in* $L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)$ *as* $\varepsilon \to 0$, *then*

$$\liminf_{\varepsilon\to 0} \int_{\Omega^-} \widehat{p}_\varepsilon^{\,2}\, \mathrm{d}\mu_\varepsilon \geq \int_{\Omega^-} (p^*)^2\, \mathrm{d}x.$$

Then, in view of representations (13.5) and (13.14), the following result is obvious.

Lemma 13.13. *Every sequence of admissible controls* $\{\widehat{p}_\varepsilon \in \widehat{P}_\varepsilon\}_{\varepsilon>0}$ *is relatively weakly compact, and its weak limit* $p^* \in L^2(\Omega^-)$ *satisfies the condition*

$$\|p^*\|^2_{L^2(\Omega^-)} \leq |\partial C|_H^{-1}\, \mathbf{C}_{\mathbf{P}}.$$

Now, we consider the following type of sequences $\{y_\varepsilon \in H^1(\Omega_\varepsilon)\}_{\varepsilon>0}$. Following the idea of Brizzi and Chalot [35] (see also [74, 88]), we extend each of the functions y_ε by 0 onto the whole of domain Ω, namely

$$\widetilde{y}_\varepsilon(x) := \begin{cases} y_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon, \\ 0 & \text{if } x \in \Omega \setminus \Omega_\varepsilon. \end{cases} \tag{13.26}$$

We set $y_\varepsilon^+(x) := y_\varepsilon(x)$ if $x \in \Omega^+$, $y_\varepsilon^-(x) := y_\varepsilon(x)$ if $x \in \Omega_\varepsilon^-$ and $\widetilde{y}_\varepsilon^-(x) := \widetilde{y}_\varepsilon(x)$ if $x \in \Omega^-$. Then, using the fact that the boundaries of the thin rods S_ε are rectilinear with respect to x_n we have

$$\partial_{x_n}(\widetilde{y}_\varepsilon^-) = \widetilde{\partial_{x_n}(y_\varepsilon^-)} \quad \text{in} \quad \Omega^-. \tag{13.27}$$

This means that $\widetilde{y}_\varepsilon^- \in W_2^{(0,1)}(\Omega^-)$, where $W_2^{(0,1)}(\Omega^-)$ is the anisotropic Sobolev space $\{v \in L^2(\Omega^-) :\ \partial_{x_n} v \in L^2(\Omega^-)\}$.

Let χ_C be the characteristic function of the set C. We suppose that this function is $\square$-periodically extended on the entire space $\mathbb{R}^{n-1}$. It is well known that

$$\chi_C(\cdot/\varepsilon) \overset{*}{\rightharpoonup} |C| \ \text{ in } \ L^\infty(B) \ \text{ as } \ \varepsilon \to 0.$$

Here, $B = (0,a)^{n-1}$ and $|C|$ is the $(n-1)$-dimensional Lebesgue measure of C. Then the following facts are valid:

$$\chi_{\Omega_\varepsilon^-} \overset{*}{\rightharpoonup} |C| \quad \text{in} \quad L^\infty(\Omega^-) \quad \text{as} \quad \varepsilon \to 0, \tag{13.28}$$

$$\chi_{\Omega_\varepsilon \cap \Sigma} \overset{*}{\rightharpoonup} |C| \quad \text{in} \quad L^\infty(\Sigma; \mathcal{H}^{n-1}) \quad \text{as} \quad \varepsilon \to 0, \tag{13.29}$$

where $\chi_{\Omega_\varepsilon^-}$ and $\chi_{\Omega_\varepsilon \cap \Sigma}$ are the characteristic functions of the sets Ω_ε^- and $\overline{\Omega}_\varepsilon \cap \Sigma$, respectively.

Thus, if for a sequence $\{y_\varepsilon \in H^1(\Omega_\varepsilon)\}_{\varepsilon>0}$ there exists a constant $\mathbf{C} > 0$ independent of ε such that $\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq \mathbf{C}$, then

$$\|y_\varepsilon^+\|_{H^1(\Omega^+)} + \|\widetilde{y}_\varepsilon^-\|_{W_2^{(0,1)}(\Omega^-)} \leq \mathbf{C}$$

and there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ (still denoted by ε) and elements $y_0^+ \in H^1(\Omega^+)$, $y_0^- \in L^2(\Omega^-)$ such that

$$\left.\begin{aligned} y_\varepsilon^+ &\rightharpoonup y_0^+ && \text{in } H^1(\Omega^+),\\ \widetilde{y}_\varepsilon^- &\rightharpoonup v = |C|(|C|^{-1}v) =: |C|y_0^- && \text{in } L^2(\Omega^-),\\ \partial_{x_n}\widetilde{y}_\varepsilon^- &\rightharpoonup |C|\,\partial_{x_n} y_0^- && \text{in } L^2(\Omega^-). \end{aligned}\right\} \tag{13.30}$$

The last limit is an evident consequence of (13.27). Moreover, similarly as in [35], it is easy to prove the relation

$$y_0^+ = y_0^- \quad \text{a.e. on } \Sigma, \tag{13.31}$$

that is, the traces of the limit functions coincide on Σ.

Definition 13.14. *We say that a sequence $\{y_\varepsilon \in H^1(\Omega_\varepsilon)\}_{\varepsilon>0}$ is weakly convergent to a function $y_* = (y_*^+, y_*^-)$ with respect to the space $H^1(\Omega^+) \times W_2^{(0,1)}(\Omega^-)$ as ε tends to 0 (in symbols, $y_\varepsilon \rightsquigarrow y_* = (y_*^+, y_*^-)$) if the following hold:*

(a) $y_\varepsilon^+ \rightharpoonup y_*^+$ in $H^1(\Omega^+)$;
(b) $\widetilde{y}_\varepsilon^- \rightharpoonup |C|\,y_*^-$ in $W_2^{(0,1)}(\Omega^-)$.

Let us introduce the space

$$\mathcal{V}(\Omega) = \left\{ y \in L^2(\Omega) \;:\; \frac{\partial y}{\partial x_n} \in L^2(\Omega^-),\ y \in H^1(\Omega^+) \right\} \tag{13.32}$$

and endow it with the norm

$$\|y\|^2_{\mathcal{V}(\Omega)} = |C| \left(\|y\|^2_{L^2(\Omega^-)} + \left\| \frac{\partial y}{\partial x_n} \right\|^2_{L^2(\Omega^-)} \right) + \|y\|^2_{H^1(\Omega^+)}.$$

It is easy to see that each limit function y_* in the sense of Definition 13.14 satisfies the condition $y_* \in \mathcal{V}(\Omega)$. Moreover, as follows from [74], any function of this space admits a trace on Σ, and the space $H^1(\Omega)$ is dense in $\mathcal{V}(\Omega)$. As a result, each element of the space $\mathcal{V}(\Omega)$ has the following properties: Let $y_* = (y_*^+, y_*^-) \in \mathcal{V}(\Omega)$ be a weak limit of $\{y_\varepsilon \in H^1(\Omega_\varepsilon)\}_{\varepsilon>0}$ with respect to the space $H^1(\Omega^+) \times W_2^{(0,1)}(\Omega^-)$ and let $\gamma_n \in L^2(\Omega^-)$ be a weak limit in $L^2(\Omega^-)$ of the corresponding sequence of partial derivatives $\{\partial_{x_n}\widetilde{y}_\varepsilon^-\}_{\varepsilon>0}$. Then

$$\gamma_n = |C| \cdot \partial y_*^- / \partial x_n \quad \text{a.e. in } \Omega^-, \tag{13.33}$$

$$y_*^+ = y_*^-, \quad \mathcal{H}^{n-1}\text{-a.e. on } \Sigma. \tag{13.34}$$

Remark 13.15. Note that assertions (13.30), (13.31), (13.33), and (13.34) can be proved without the assumption that y_ε is a solution of the boundary value problem (13.3). Therefore, for any pair (y_*^+, y_*^-), which is a weak limit in the sense of Definition 13.14, we have $y_*^+(\cdot) = y_*^-(\cdot)$ $\mathcal{H}^{n-1}$-almost everywhere on Σ. Moreover, any function $y_* = (y_*^+, y_*^-)$ from $\mathcal{V}(\Omega)$ has a trace on the manifold

$$\Gamma_0 = \{(x', x_n) \in \mathbb{R}^n \ : \ x' \in B, \ x_n = -\beta(x')\}$$

such that $y_*^-|_{\Gamma_0} \in \mathcal{B}_2^{(0,1/2)}(\Gamma_0)$ (see [243, 244]). Here, $\mathcal{B}_2^{(0,1/2)}$ is the Besov space [111].

Definition 13.16. *We say that a sequence of triplets* $\{(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ *is w-convergent to* (u, p, y^+, y^-) *(in symbols,* $(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \xrightarrow{w} (u, p, y^+, y^-)$*) with respect to the space*

$$\mathbb{Y}_0 \equiv L^2(\Gamma_0) \times L^2(\Omega^-) \times H^1(\Omega^+) \times W_2^{(0,1)}(\Omega^-)$$

as ε *tends to* 0 *if the following hold:*

(i) $u_\varepsilon \to u$ *in the sense of Definition 13.6;*
(ii) $\widehat{p}_\varepsilon \to p$ *in the sense of Definition 13.10;*
(iii) $y_\varepsilon \rightsquigarrow (y^+, y^-)$ *in the sense of Definition 13.14.*

As a result, we can give the following conclusion:

Proposition 13.17. *Let* $\{(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *be any sequence of admissible triplets for the* $\widehat{\mathbb{P}}_\varepsilon$*-problem such that* $\sup_{\varepsilon>0} \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} < +\infty$*. Then there exist a subsequence* $\{(u_{\varepsilon'}, \widehat{p}_{\varepsilon'}, y_{\varepsilon'})\}_{\varepsilon'>0}$ *and a quadruple* $(u, p, y^+, y^-) \in \mathbb{Y}_0$ *such that* $u \in \widehat{U}_0$*,* $p \in \widehat{P}_0$*, and* $(u_{\varepsilon'}, \widehat{p}_{\varepsilon'}, y_{\varepsilon'}) \xrightarrow{w} (u, p, y^+, y^-)$*, where the sets* $\widehat{U}_0$ *and* $\widehat{P}_0$ *are defined as*

$$\widehat{U}_0 = \left\{u \in L^2(\Gamma_0) \ : \ \|u\|_{L^2(\Gamma_0)} \le \sqrt{|C|} \cdot \mathbf{C_u}\right\},$$

$$\widehat{P}_0 = \left\{p \in L^2(\Omega^-) \ : \ \|p\|^2_{L^2(\Omega^-)} \le |\partial C|^{-1}_H \cdot \mathbf{C_p}\right\}.$$

Remark 13.18. As follows from Proposition 13.17, any uniformly bounded sequence of suboptimal triplets $\left\{(u_\varepsilon^\delta, \widehat{p}_\varepsilon^{\,\delta}, y_\varepsilon^\delta) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ in

$$\mathbb{Y}_\varepsilon \equiv L^2(\Gamma_\varepsilon) \times L^2(\Omega^-, \mathrm{d}\mu_\varepsilon) \times H^1(\Omega_\varepsilon)$$

for $\widehat{\mathbb{P}}_\varepsilon$ is relatively compact with respect to the topology associated with w-convergence on $\mathbb{Y}_0$. So, this topology can be chosen as the most natural one for the limit analysis of the original optimal boundary control problem.

13.4 Definition of the limit problem and its properties

We begin this section with some notions that will be useful for the definition of a limit boundary control problem. Let u be any element of $L^2(\Gamma_0)$. Since the embedding $H^{1/2}(\Gamma_0) \hookrightarrow L^2(\Gamma_0)$ is dense with respect to the strong topology for $L^2(\Gamma_0)$, it follows that for a given value $\gamma > 0$, there is an element $q \in H^{1/2}(\Gamma_0)$ such that $\|u - q\|_{L^2(\Gamma_0)} \le \gamma$. Let us consider the sequence $\{\widetilde{q}_\varepsilon = |C|^{-1}\chi_\varepsilon q\}_{\varepsilon>0}$. It is clear that $q_\varepsilon = |C|^{-1}q\big|_{\Gamma_\varepsilon} \in H^{1/2}(\Gamma_\varepsilon)$ for every $\varepsilon > 0$, and, moreover, $\widetilde{q}_\varepsilon \rightharpoonup |C|^{-1}|C|q = q$ in $L^2(\Gamma_0)$. Thus, the space $L^2(\Gamma_0)$ possesses *the weak approximation property* with respect to the family of sets $\{\Gamma_\varepsilon\}_{\varepsilon>0}$, that is, for every $\gamma > 0$ and any $u \in L^2(\Gamma_0)$, there exist an element $q \in H^{1/2}(\Gamma_0)$ and a sequence $\{q_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)\}_{\varepsilon>0}$ such that

$$\sup_{\varepsilon>0} \|q_\varepsilon\|_{H^{1/2}(\Gamma_\varepsilon)} < +\infty, \tag{13.35}$$

$$\|u - q\|_{L^2(\Gamma_0)} \le \gamma \quad \text{and} \quad q_\varepsilon \rightharpoonup q \quad \text{with respect to} \quad L^2(\Gamma_0). \tag{13.36}$$

In this case, the sequence $\{q_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)\}_{\varepsilon>0}$ is called γ-realizing.

Taking this observation into account, we introduce the following concept of a limit minimization problem for the family (13.19) (for a comparison, see Definition 8.24).

Definition 13.19. *We say that a minimization problem on* $\mathbb{Y}_0$

$$\left\langle \inf_{(u,p,y^+,y^-)\in\,\Xi_0} I_0(u,p,y^+,y^-) \right\rangle \tag{13.37}$$

is the variational limit of the sequence (13.19) with respect to the w-convergence (or variational w-limit) if the following conditions are satisfied:

(i) If a sequence $\{(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ *w-converges to a* (u,p,y^+,y^-)*, and there exists a subsequence* $\{\varepsilon_k\}$ *of* $\{\varepsilon > 0\}$ *such that* $\varepsilon_k \to 0$ *as* $k \to \infty$ *and* $(u_{\varepsilon_k}, \widehat{p}_{\varepsilon_k}, y_{\varepsilon_k}) \in \Xi_{\varepsilon_k}$ *for all* k*, then*

$$(u,p,y^+,y^-) \in \Xi_0, \tag{13.38}$$

$$I_0(u,p,y^+,y^-) \le \liminf_{k\to\infty} I_{\varepsilon_k}(u_{\varepsilon_k}, \widehat{p}_{\varepsilon_k}, y_{\varepsilon_k}). \tag{13.39}$$

(ii) For every quadruple $(u,p,y^+,y^-) \in \Xi_0$ *and any value* $\gamma > 0$*, there exists a* γ*-realizing sequence* $\{(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ *such that*

$$(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon, \ \forall\, \varepsilon > 0, \quad (u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \overset{w}{\to} (v,q,z^+,z^-), \tag{13.40}$$

$$\|(u,p,y^+,y^-) - (v,q,z^+,z^-)\|_{\mathbb{Y}_0} \le \gamma, \tag{13.41}$$

$$I_0(u,p,y^+,y^-) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) - \gamma. \tag{13.42}$$

In what follows, conditions (i)–(ii) together with relations (13.38), (13.40) and (13.41) will be used as a basis for the recovery of the set of admissible solutions Ξ_0, whereas inequalities (13.39) and (13.42) will be exploited for the identification of the limit cost functional $I_0 : \Xi_0 \to \mathbb{R}$. Note that in this case, the set Ξ_0 is the sequential w-limit of the sequence $\{\Xi_\varepsilon \subset \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ in the Kuratowski sense (or $K(w)$-limit, for simplicity).

Following the arguments of the proof of Theorem 8.25, it is easy to show that the variational w-convergence of the sequence (13.19) to the problem (13.37) implies the convergence of the minimum values of I_ε on Ξ_ε to the minimum of I_0 on Ξ_0, and moreover in this case, every w-cluster point of any sequence of subminimizers for I_ε is a minimizer of I_0.

Theorem 13.20. *Assume that the constrained minimization problem (13.37) is the variational w-limit of the sequence (13.19) and this problem has a unique solution $(u^0, p^0, (y^0)^+, (y^0)^-)$ in Ξ_0. Let $\{(u_\varepsilon^{\delta_\varepsilon}, \widehat{p}_\varepsilon^{\delta_\varepsilon}, y_\varepsilon^{\delta_\varepsilon}) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of approximate solutions to the $\mathbb{P}_\varepsilon$ problems (see (13.18)) such that $sup_{\varepsilon>0}\|y_\varepsilon^{\delta_\varepsilon}\|_{H^1(\Omega_\varepsilon)} < +\infty$ and $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. Then*

$$(u_\varepsilon^{\delta_\varepsilon}, \widehat{p}_\varepsilon^{\delta_\varepsilon}, y_\varepsilon^{\delta_\varepsilon}) \xrightarrow{w} (u^0, p^0, (y^0)^+, (y^0)^-), \tag{13.43}$$

$$\begin{aligned} \inf_{(u,\,p,\,y^+,\,y^-)\in\,\Xi_0} I_0(u,p,y^+,y^-) &= I_0\left(u^0, p^0, (y^0)^+, (y^0)^-\right) \\ &= \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^{\delta_\varepsilon}, \widehat{p}_\varepsilon^{\delta_\varepsilon}, y_\varepsilon^{\delta_\varepsilon}) \\ &= \lim_{\varepsilon\to 0} \inf_{(u,p,y)\in\,\Xi_\varepsilon} I_\varepsilon(u,p,y). \end{aligned} \tag{13.44}$$

Definition 13.21. *We say that the family of optimal control problems (13.2)–(13.5) has a limit, as ε tends to 0, with respect to the w-convergence if for the corresponding sequence of constrained minimization problems (13.19), there exists a variational limit which can be recovered in the form of some OCP.*

13.5 Analytical representation of the limit set of admissible solutions

The main object of our consideration in this section is the sequence of the sets of admissible solutions

$$\left\{\Xi_\varepsilon \in \mathbb{X}_\varepsilon \equiv H^{1/2}(\Gamma_\varepsilon) \times L^2(\Omega^-, \mathrm{d}\mu_\varepsilon) \times H^1(\Omega_\varepsilon)\right\}_{\varepsilon>0} \tag{13.45}$$

to the original problem $\widehat{\mathbb{P}}_\varepsilon$. Our aim is to study its w-limiting properties in the sense of Definition 13.19. In spite of the fact that we have no result concerning the $K(w)$-compactness of the sequences of sets in the scale $\{\mathbb{X}_\varepsilon\}$, we will show that the $K(w)$-limit set exists for the sequence (13.45) and it can be recovered in an implicit analytical form. As it follows from Definition 13.19, the main difficulty is connected with the passage to the limit in the integral identity (13.12) for the problem (13.3) as $\varepsilon \to 0$.

Proposition 13.22. *Let $w_* \in L^\infty(\Omega^-)$ and $p_* \in L^2(\Omega^-)$ be elements such that*

$$L^\infty(\Omega^-) \ni w_\varepsilon \longrightarrow w_* \quad \text{strongly in} \quad L^\infty(\Omega^-), \tag{13.46}$$

$$L^2(\Omega^-, \mathrm{d}\mu_\varepsilon) \ni p_\varepsilon \longrightarrow p_* \quad \text{weakly in the sense of Definition 13.10.} \tag{13.47}$$

Then

$$\lim_{\varepsilon\to 0} \int_{\Omega^-} w_\varepsilon\, p_\varepsilon\, \varphi \,\mathrm{d}\mu_\varepsilon = \int_{\Omega^-} w_*\, p_*\, \varphi \,\mathrm{d}x, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

Proof. First, we note that for every $\varepsilon > 0$, each of the integrals $\int_{\Omega^-} w_\varepsilon\, p_\varepsilon\, \varphi \,\mathrm{d}\mu_\varepsilon$ is well defined. Indeed, using the Cauchy–Schwartz inequality, we have

$$\left| \int_{\Omega^-} w_\varepsilon\, p_\varepsilon\, \varphi \,\mathrm{d}\mu_\varepsilon \right| \le \|w_\varepsilon\|_{L^\infty(\Omega^-)}\, |\mu_\varepsilon(\overline{\Omega^-})|^{1/2}\, \|p_\varepsilon\|_{L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)} \cdot \sup_{x\in\Omega^-} |\varphi(x)|.$$

Since the sequence of Borel measures $\{\mathrm{d}\mu_\varepsilon\}$ is weakly convergent to the Lebesgue measure $\mathrm{d}x$, the sequences

$$\{w_\varepsilon \in L^\infty(\Omega^-)\}_{\varepsilon>0}, \quad \{p_\varepsilon \in L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0}$$

are bounded, and for every compact set $K \subset \mathbb{R}^n$, the inequality

$$\limsup_{\varepsilon\to 0} \mu_\varepsilon(K) \le |K|$$

holds (see Lemma 2.32); it follows that

$$\left| \int_{\Omega^-} w_\varepsilon\, p_\varepsilon\, \varphi \,\mathrm{d}\mu_\varepsilon \right| \le C \sup_{x\in\Omega^-} |\varphi(x)|.$$

Thus, $(w_\varepsilon\, p_\varepsilon)\,\mathrm{d}\mu_\varepsilon$ is a Radon measure. Moreover, from this, we immediately have that $p_\varepsilon w \longrightarrow p_* w$ weakly in $\{L^2(\Omega^-, \mathrm{d}\mu_\varepsilon)\}$ for any function $w \in L^\infty(\Omega^-)$. Then taking (13.46) and (13.47) into account and passing to the limit in the inequality

$$\begin{aligned}
\left| \int_{\Omega^-} w_\varepsilon\, p_\varepsilon \,\mathrm{d}\mu_\varepsilon - \int_{\Omega^-} w_*\, p_* \,\mathrm{d}x \right| &\le \left| \int_{\Omega^-} p_\varepsilon\, (w_\varepsilon - w_*) \,\mathrm{d}\mu_\varepsilon \right| \\
&\quad + \left| \int_{\Omega^-} w_*\, p_\varepsilon \,\mathrm{d}\mu_\varepsilon - \int_{\Omega^-} w_*\, p_* \,\mathrm{d}x \right| \\
&\le \|w_\varepsilon - w_*\|_{L^\infty(\Omega^-)} \int_{\Omega^-} |p_\varepsilon\, \varphi| \,\mathrm{d}\mu_\varepsilon \\
&\quad + \left| \int_{\Omega^-} w_*\, p_\varepsilon \,\mathrm{d}\mu_\varepsilon - \int_{\Omega^-} w_*\, p_* \,\mathrm{d}x \right|
\end{aligned}$$

as ε tends to 0, we immediately obtain the required result. □

Let us introduce the anisotropic Sobolev space

$$\mathcal{V}(\Omega;\Gamma_0)=\big\{v\in L^2(\Omega):\\ \partial_{x_n}v\in L^2(\Omega),\quad v\in H^1(\Omega^+),\quad v=0\ \mathcal{H}^{n-1}\text{ a.e. on }\ \Gamma_0\big\}$$

with the scalar product

$$(y,v)_{\mathcal{V}(\Omega;\Gamma_0)}=\int_{\Omega^+}(y\,v+\nabla y\cdot\nabla v)\,\mathrm{d}x\\ +|C|\int_{\Omega^-}(y\,v+\partial_{x_n}y\,\partial_{x_n}v)\,\mathrm{d}x. \tag{13.48}$$

Note that each function of this space has a trace on the manifold Σ (see [243, 244]), and the space $H^1(\Omega;\Gamma_0)$ is dense in $\mathcal{V}(\Omega;\Gamma_0)$. Moreover, for any function $v\in\mathcal{V}(\Omega)$, relation (13.31) holds true.

Theorem 13.23. *Let $\{u_\varepsilon\in U_\varepsilon\}_{\varepsilon>0}$ and $\{\widehat{p}_\varepsilon\in\widehat{P}_\varepsilon\}_{\varepsilon>0}$ be any sequences of admissible boundary controls for the $\widehat{\mathbb{P}}_\varepsilon$ problems. Let $u_0\in L^2(\Gamma_0)$ and $p_0\in L^2(\Omega^-)$ be their weak limits in the sense of Definitions 13.6 and 13.10, respectively. Let $\big\{y_\varepsilon\in H^1(\Omega_\varepsilon)\big\}$ be the corresponding weak solutions to the problem (13.3) such that $\sup_{\varepsilon>0}\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}<+\infty$. Then $(u_\varepsilon,\widehat{p}_\varepsilon,y_\varepsilon)\xrightarrow{w}(u_0,p_0,v_0^+,v_0^-)$ as $\varepsilon\to0$, where*

$$\widehat{v}_0(x)=\begin{cases}v_0^+(x) & \text{if } x\in\Omega^+,\\ v_0^-(x) & \text{if } x\in\Omega^-\end{cases} \tag{13.49}$$

is the unique weak solution in $\mathcal{V}(\Omega)$ of the following limit problem:

$$\left.\begin{aligned}-\Delta_x v_0^+(x)+v_0^+(x)&=f_0(x), && x\in\Omega^+,\\ -\partial^2_{x_nx_n}v_0^-(x)+v_0^-&=f_0(x)+|\partial C|_H\,|C|^{-1}\cdot p_0(x), && x\in\Omega^-,\\ \partial_\nu v_0^+(x)&=0, && x\in\partial\Omega^+\setminus\Sigma,\\ v_0^-&=|C|^{-1}u_0, && \text{on }\ \Gamma_0,\\ v_0^+&=v_0^-, && \text{on }\ \Sigma,\\ \partial_{x_n}v_0^+&=|C|\,\partial_{x_n}v_0^-, && \text{on }\ \Sigma.\end{aligned}\right\} \tag{13.50}$$

Proof. As follows from Proposition 13.17, there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and a quadruple

$$(u_0,p_0,v_0^+,v_0^-)\in L^2(\Gamma_0)\times L^2(\Omega^-)\times H^1(\Omega^+)\times W_2^{(0,1)}(\Omega^-)$$

such that

$$(u_\varepsilon,\widehat{p}_\varepsilon,y_\varepsilon)\xrightarrow{w}(u_0,p_0,v_0^+,v_0^-)\quad\text{as}\quad\varepsilon\to0.$$

Similarly as in Sect. 13.3 (see relations (13.30) and (13.31)), we prove that

$$\begin{aligned} y_\varepsilon^+ &\rightharpoonup v_0^+ && \text{in } H^1(\Omega^+),\\ \widetilde{y}_\varepsilon^- &\rightharpoonup |C|\, v_0^- && \text{in } L^2(\Omega^-),\\ v_0^+ &= v_0^- && \mathcal{H}^{n-1}\text{-almost everywhere on } \Sigma,\\ \partial_{x_n} v_0^+ &= |C| \partial_{x_n} v_0^- && \mathcal{H}^{n-1}\text{-almost everywhere on } \Sigma. \end{aligned} \tag{13.51}$$

In addition, there exist functions $\gamma_i \in L^2(\Omega^-)$ $(i = 1, \dots, n-1)$ such that

$$\widetilde{\nabla_x y_\varepsilon^-} \rightharpoonup \left(\gamma_1, \dots, \gamma_{n-1}, |C| \frac{\partial v_0^-}{\partial x_n}\right) \quad \text{in } \left[L^2(\Omega^-)\right]^n. \tag{13.52}$$

Let us prove that the function v_0^- satisfies the boundary condition

$$v_0^- = |C|^{-1} u_0 \quad \mathcal{H}^{n-1}\text{-almost everywhere on } \Gamma_0, \tag{13.53}$$

and give a sense to this equality.

In view of our initial suppositions, we have

$$\widetilde{y}_\varepsilon^- = \widetilde{u}_\varepsilon \quad \mathcal{H}^{n-1}\text{-a.e. on } \Gamma_0, \forall\, \varepsilon > 0, \quad \widetilde{u}_\varepsilon \rightharpoonup u_0 \quad \text{in } L^2(\Gamma_0), \tag{13.54}$$

$$\widetilde{y}_\varepsilon^- \rightharpoonup |C|\, v_0^- \quad \text{in } W_2^{(0,1)}(\Omega^-). \tag{13.55}$$

Then, as follows from the integral identity

$$\begin{aligned} \int_{\Gamma_0} \widetilde{y}_\varepsilon^- \psi \, \mathrm{d}\mathcal{H}^{n-1} &= -\int_{\Omega^-} \frac{\partial \widetilde{y}_\varepsilon^-}{\partial x_n} \psi \, \mathrm{d}x \\ &\quad - \int_{\Omega^-} \widetilde{y}_\varepsilon^- \frac{\partial \psi}{\partial x_n} \, \mathrm{d}x, \quad \forall \psi \in C_0^\infty(\mathbb{R}^n; \partial\Omega^- \setminus \Gamma_0), \end{aligned}$$

where

$$C_0^\infty(\mathbb{R}^n; \partial\Omega^- \setminus \Gamma_0) = \{\varphi \in C_0^\infty(\mathbb{R}^n) \ : \ \varphi = 0 \ \text{ on } \partial\Omega^- \setminus \Gamma_0\},$$

we get $(\widetilde{y}_\varepsilon^-)|_{\Gamma_0} \longrightarrow |C| \ \left(v_0^-\right)\big|_{\Gamma_0}$ weakly in $L^2(\Gamma_0)$. Hence,

$$\begin{aligned} \int_{\Gamma_0} u_0 \psi \, \mathrm{d}\mathcal{H}^{n-1} &= \lim_{\varepsilon \to 0} \int_{\Gamma_0} \widetilde{u}_\varepsilon \psi \, \mathrm{d}\mathcal{H}^{n-1} = \lim_{\varepsilon \to 0} \int_{\Gamma_0} \widetilde{y}_\varepsilon^- \psi \, \mathrm{d}\mathcal{H}^{n-1} \\ &= -|C| \int_{\Omega^-} \frac{\partial v_0^-}{\partial x_n} \psi \, \mathrm{d}x - |C| \int_{\Omega^-} v_0^- \frac{\partial \psi}{\partial x_n} \, \mathrm{d}x \\ &= |C| \int_{\Gamma_0} v_0^- \psi \, \mathrm{d}\mathcal{H}^{n-1}, \quad \forall \psi \in C_0^\infty(\mathbb{R}^n; \partial\Omega^- \setminus \Gamma_0), \end{aligned} \tag{13.56}$$

and we obtain the required relation (13.53).

Now, let us show that the function v_0 is the unique weak solution of the boundary value problem (13.50). With this aim, we rewrite the integral identity (13.12) in the following way:

$$\int_{\Omega^+} \nabla y_\varepsilon^+ \cdot \nabla\varphi \,\mathrm{d}x + \int_{\Omega^-} \widetilde{\nabla y_\varepsilon^-} \cdot \nabla\varphi \,\mathrm{d}x + \int_{\Omega^+} y_\varepsilon^+ \varphi \,\mathrm{d}x + \int_{\Omega^-} \widetilde{y}_\varepsilon^- \varphi \,\mathrm{d}x$$
$$= \int_{\Omega^+} f_\varepsilon \,\varphi \,\mathrm{d}x + |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \,\varphi \,\mathrm{d}\mu_\varepsilon$$
$$+ \int_{\Omega^-} \chi_{\Omega_\varepsilon^-} f_\varepsilon \,\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_0). \tag{13.57}$$

Passing to the limit in (13.57) as $\varepsilon \to 0$ and taking (13.51) into account (see also (13.28)), we obtain

$$\int_{\Omega^+} \nabla v_0^+ \cdot \nabla\varphi \,\mathrm{d}x + \int_{\Omega^-} \sum_{i=1}^{n-1} \gamma_i \,(\partial\varphi/\partial x_i) \,\mathrm{d}x$$
$$+ |C| \int_{\Omega^-} (\partial v_0^-/\partial x_n)\,(\partial\varphi/\partial x_n) \,\mathrm{d}x + \int_{\Omega^+} v_0^+ \varphi \,\mathrm{d}x$$
$$+ |C| \int_{\Omega^-} v_0^- \varphi \,\mathrm{d}x$$
$$= |\partial C|_H \int_{\Omega^-} p_0 \,\varphi \,\mathrm{d}x + \int_{\Omega^+} f_0 \,\varphi \,\mathrm{d}x$$
$$+ |C| \int_{\Omega^-} f_0 \,\varphi \,\mathrm{d}x, \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_0). \tag{13.58}$$

In order to prove that $\gamma_i = 0$ a.e. on Ω^- for all $i \in \{1, \ldots, n-1\}$, let us fix $i \in \{1, \ldots, n-1\}$ and let w_ε^i be a sequence in $W^{1,\infty}(\Omega^-)$ satisfying the following conditions:

$$w_\varepsilon^i \longrightarrow x_i \quad \text{strongly in } L^\infty(\Omega^-), \tag{13.59}$$
$$D\, w_\varepsilon^i = 0 \quad \text{a.e. on } \Omega_\varepsilon^- \tag{13.60}$$

for every ε. The existence of such a sequence is proved in [35, 74].

Further, we take the test functions $\varphi = w_\varepsilon^i \,\phi$ and $\varphi = x_i \,\phi$ with $\phi \in C_0^\infty(\Omega^-)$ in (13.57). By virtue of (13.60), we get

$$\int_{\Omega^-} \widetilde{\nabla y_\varepsilon^-} \cdot \nabla\phi \, w_\varepsilon^i \,\mathrm{d}x + \int_{\Omega^-} \widetilde{y}_\varepsilon^- \phi \, w_\varepsilon^i \,\mathrm{d}x$$
$$= |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \,\phi \, w_\varepsilon^i \,\mathrm{d}\mu_\varepsilon$$
$$+ \int_{\Omega^-} \chi_{\Omega_\varepsilon^-} f_\varepsilon \,\phi \, w_\varepsilon^i \,\mathrm{d}x, \quad \forall\, \phi \in C_0^\infty(\Omega^-), \tag{13.61}$$

$$\int_{\Omega^-} \widetilde{\nabla y_\varepsilon^-} \cdot \nabla(\phi \, x_i) \,\mathrm{d}x + \int_{\Omega^-} \widetilde{y}_\varepsilon^- \phi \, x_i \,\mathrm{d}x$$
$$= |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \,\phi \, x_i \,\mathrm{d}\mu_\varepsilon$$
$$+ \int_{\Omega^-} \chi_{\Omega_\varepsilon^-} f_\varepsilon \,\phi \, x_i \,\mathrm{d}x, \quad \forall\, \phi \in C_0^\infty(\Omega^-), \tag{13.62}$$

for every ε. Then, passing to the limit in (13.61) and (13.62) as $\varepsilon \to 0$ and using the properties (13.28), (13.51), and (13.59) and Proposition 13.22, we have

$$\int_{\Omega^-} \sum_{k=1}^{n-1} \gamma_k \frac{\partial \phi}{\partial x_k} x_i \, \mathrm{d}x + |C| \int_{\Omega^-} \frac{\partial v_0^-}{\partial x_n} \frac{\partial \phi}{\partial x_n} x_i \, \mathrm{d}x + |C| \int_{\Omega^-} v_0^- \phi \, x_i \, \mathrm{d}x$$
$$= |\partial C|_H \int_{\Omega^-} p_0 \, \phi \, x_i \, \mathrm{d}x$$
$$+ |C| \int_{\Omega^-} f_0 \, \phi \, x_i \, \mathrm{d}x, \quad \forall \, \phi \in C_0^\infty(\Omega^-), \tag{13.63}$$

$$\int_{\Omega^-} \sum_{k=1}^{n-1} \gamma_k \frac{\partial (\phi \, x_i)}{\partial x_k} \, \mathrm{d}x + |C| \int_{\Omega^-} \frac{\partial v_0^-}{\partial x_n} \frac{\partial \phi}{\partial x_n} x_i \, \mathrm{d}x + |C| \int_{\Omega^-} v_0^- \phi \, x_i \, \mathrm{d}x$$
$$= |\partial C|_H \int_{\Omega^-} p_0 \, \phi \, x_i \, \mathrm{d}x$$
$$+ |C| \int_{\Omega^-} f_0 \, \phi \, x_i \, \mathrm{d}x, \quad \forall \, \phi \in C_0^\infty(\Omega^-). \tag{13.64}$$

Making comparison (13.63) with (13.64), we conclude that

$$\int_{\Omega^-} \gamma_k \, \phi \, \mathrm{d}x = 0, \quad \forall k \in \{1, \ldots, n-1\}, \ \forall \phi \in C_0^\infty(\Omega^-),$$

that is, $\gamma_i = 0$ a.e. on Ω^-.

Thus, the function v_0 satisfies the identity

$$\int_{\Omega^+} \nabla v_0^+ \cdot \nabla \varphi \, \mathrm{d}x + |C| \int_{\Omega^-} \frac{\partial v_0^-}{\partial x_n} \frac{\partial \varphi}{\partial x_n} \, \mathrm{d}x$$
$$+ \int_{\Omega^+} v_0^+ \varphi \, \mathrm{d}x + |C| \int_{\Omega^-} v_0^- \varphi \, \mathrm{d}x$$
$$= |\partial C|_H \int_{\Omega^-} p_0 \, \varphi \, \mathrm{d}x + \int_{\Omega^+} f_0 \, \varphi \, \mathrm{d}x$$
$$+ |C| \int_{\Omega^-} f_0 \, \varphi \, \mathrm{d}x, \quad \forall \, \varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_0). \tag{13.65}$$

If we take into account that $\widehat{v}_0 = \big(v_0^+, v_0^-\big)$, where $v_0^+ \in H^1(\Omega^+)$ and $v_0^- \in W_2^{(0,1)}(\Omega^-)$, and make use of the definition of the space $\mathcal{V}(\Omega; \Gamma_0)$ (see (13.48)), then identity (13.65) can be rewritten in the form

$$\big(\widehat{v}_0, \varphi\big)_{\mathcal{V}} = \int_{\Omega^+} f_0 \, \varphi \, \mathrm{d}x + |C| \int_{\Omega^-} f_0 \, \varphi \, \mathrm{d}x$$
$$+ |\partial C|_H \int_{\Omega^-} p_0 \, \varphi \, \mathrm{d}x \quad \forall \, \varphi \in \mathcal{V}(\Omega; \Gamma_0). \tag{13.66}$$

As a result, we say that a function $\widehat{v}_0 \in \mathcal{V}(\Omega)$ is a weak solution to the problem (13.50) if it satisfies identity (13.66) and the trace of v_0^- on Γ_0 equals $|C|^{-1}u_0$ in the sense of (13.56). Then, following the standard Hilbert space methods, we obtain the required result: The function $\widehat{v}_0 \in \mathcal{V}(\Omega)$ is the unique weak solution to the problem (13.50). Due to the uniqueness of the solution to the problem (13.50), the above reasoning holds for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. This concludes the proof. □

We are now in a position to state the main result of this section, which deals with the $K(w)$-limit analysis of the set sequence (13.45).

Theorem 13.24. *For the sequence of the sets of admissible solutions for the $\widehat{\mathbb{P}}_\varepsilon$ problems $\{\Xi_\varepsilon\}_{\varepsilon>0}$, there exists a no-empty $K(w)$-limit set $\Xi_0 \subset L^2(\Gamma_0) \times L^2(\Omega^-) \times H^1(\Omega^+) \times W_2^{(0,1)}(\Omega^-)$ with the following representation:*

$$
\begin{aligned}
\Xi_0 = \big\{ & (u, p, v^+, v^-) \\
& \left| \begin{aligned}
u \in \widehat{U}_0 &= \left\{ u \in L^2(\Gamma_0) \ : \ \|u\|_{L^2(\Gamma_0)} \le \sqrt{|C|} \cdot \mathbf{C_u} \right\}, \\
p \in \widehat{P}_0 &= \left\{ p \in L^2(\Omega^-) \ : \ \|p\|^2_{L^2(\Omega^-)} \le |\partial C|^{-1}_H \cdot \mathbf{C_p} \right\}, \\
& -\Delta_x v^+(x) + v^+ = f_0(x), \ x \in \Omega^+, \\
& -\frac{\partial^2 v^-(x)}{\partial x_n^2} + v^-(x) \\
& = f_0(x) + |\partial C|_H \, |C|^{-1} \cdot p(x), \ x \in \Omega^-, \\
& \partial_\nu v^+(x) = 0, \ x \in \partial\Omega^+ \setminus \Sigma, \\
& v^- = |C|^{-1} u \quad on \quad \Gamma_0, \\
& v^+ = v^- \quad on \quad \Sigma, \\
& \partial_{x_n} v^+ = |C| \, \partial_{x_n} v^- \quad on \quad \Sigma
\end{aligned} \right\}. \qquad (13.67)
\end{aligned}
$$

Proof. To obtain representation (13.67), we have to verify conditions (13.38) and (13.40)–(13.41) of Definition 13.19. We also show that a $K(w)$-limit set exists for the whole sequence (13.45). For this, we divide the proof into several steps.

Step 1. Since the set Ξ_0 is nonempty (see the previous lemma), let (u, p, y^+, y^-) be any of its representatives. We begin with the construction of a sequence $\{\widehat{u}_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)\}_{\varepsilon>0}$ such that $\widehat{u}_\varepsilon \to u$ in the sense of Definition 13.6 and $\widehat{u}_\varepsilon \in U_\varepsilon$ for every $\varepsilon < \varepsilon_0$, where $\varepsilon_0 > 0$ is some positive value. For this, we consider two cases: The given quadruple is such that $\|u\|_{L^2(\Gamma_0)} < \sqrt{|C|} \cdot \mathbf{C_u}$, and, in the second case, $\|u\|_{L^2(\Gamma_0)} = \sqrt{|C|} \cdot \mathbf{C_u}$. In the first case, we construct a weakly convergent sequence $\{\widehat{u}_\varepsilon\}_{\varepsilon>0}$ to u as follows: Let $u_\varepsilon \in L^2(\Gamma_\varepsilon)$ be the restriction of $|C|^{-1}u$ on Γ_ε. Then (see the proof of Proposition 13.8)

$$
\widetilde{u}_\varepsilon = |C|^{-1} \chi_{\Gamma_\varepsilon} u \rightharpoonup u \quad \text{in} \quad L^2(\Gamma_0). \qquad (13.68)
$$

Additionally,

$$\|u_\varepsilon\|_{L^2(\Gamma_\varepsilon)} = |C|^{-1}\left(\int_{\Gamma_0}\chi_{\Gamma_\varepsilon}\,u^2\,\mathrm{d}\mathcal{H}^{n-1}\right)^{1/2}$$
$$\longrightarrow \sqrt{|C|^{-1}}\left(\int_{\Gamma_0}u^2\,\mathrm{d}\mathcal{H}^{n-1}\right)^{1/2} < \mathbf{C_u}$$

as $\varepsilon \to 0$ (recall that $\chi^2_{\Gamma_\varepsilon} = \chi_{\Gamma_\varepsilon}$, and $\chi_{\Gamma_\varepsilon} \to |C|$ weakly-$*$ in $L^\infty(\Gamma_0)$ and weakly in $L^2(\Gamma_0)$).

Further, using the fact that the Sobolev space $H^{1/2}(\Gamma_\varepsilon)$ is dense in $L^2(\Gamma_\varepsilon)$, we get the following: For every $\varepsilon > 0$, there is an element $\widehat{u}_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ such that $\|u_\varepsilon - \widehat{u}_\varepsilon\|_{L^2(\Gamma_\varepsilon)} < \varepsilon^2$ and $\widehat{u}_\varepsilon \in U_\varepsilon$ for ε sufficiently small. Then taking into account (13.68), we obtain

$$\left|\int_{\Gamma_0} g\left(\widetilde{(\widehat{u}_\varepsilon)} - u\right)\,\mathrm{d}\mathcal{H}^{n-1}\right| \equiv \left|< g, \widetilde{(\widehat{u}_\varepsilon)} - u >\right|$$
$$\leq \|g\|_{L^2(\Gamma_0)}\|\widehat{u}_\varepsilon - u_\varepsilon\|_{L^2(\Gamma_\varepsilon)}$$
$$+ |< g, \widetilde{u}_\varepsilon - u >| \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

for every $g \in L^2(\Gamma_0)$, that is, $\widehat{u}_\varepsilon \to u$ weakly in the sense of Definition 13.6 and $\widehat{u}_\varepsilon \in U_\varepsilon$ for ε sufficiently small.

Now, we consider the second case when $\|u\|_{L^2(\Gamma_0)} = \sqrt{|C|}\cdot\mathbf{C_u}$. Then there exists a control sequence $\{\overline{u}_\varepsilon \in L^2(\Gamma_0)\}$ satisfying the condition:

$$\overline{u}_\varepsilon \rightharpoonup u \quad \text{in } L^2(\Gamma_0) \text{ and } \|\overline{u}_\varepsilon\|_{L^2(\Gamma_0)} < \sqrt{|C|}\cdot\mathbf{C_u},\ \forall\,\varepsilon > 0.$$

Since the weak topology of $L^2(\Gamma_0)$ is metrizable on the set

$$\widehat{U}_0 = \left\{u \in L^2(\Gamma_0) \,:\, \|u\|_{L^2(\Gamma_0)} \leq \sqrt{|C|}\cdot\mathbf{C_u}\right\},$$

one can construct a sequence $\{w_\varepsilon \in L^2(\Gamma_0)\}_{\varepsilon>0}$ such that each of w_ε is in a convex envelope of a finite number of elements $\{\overline{u}_\varepsilon\}_{\varepsilon>0}$, and $w_\varepsilon \longrightarrow u$ strongly in $L^2(\Gamma_0)$. Note that in this case we also have $\|w_\varepsilon\|_{L^2(\Gamma_0)} < \sqrt{|C|}\cdot\mathbf{C_u}$ for every $\varepsilon > 0$. So, we construct a weakly to u convergent sequence $\{\widehat{u}_\varepsilon\}_{\varepsilon>0}$ as follows: $\widehat{u}_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ are elements such that $\|\widehat{u}_\varepsilon - |C|^{-1}w_\varepsilon\|_{L^2(\Gamma_\varepsilon)} < \varepsilon^2$. Then, since

$$|C|^{-1}\chi_{\Gamma_\varepsilon}w_\varepsilon \rightharpoonup u \quad \text{in } L^2(\Gamma_0)$$

and

$$\||C|^{-1}w_\varepsilon\|_{L^2(\Gamma_\varepsilon)} = \sqrt{|C|^{-2}\int_{\Gamma_0}\chi_{\Gamma_\varepsilon}w_\varepsilon^2\,\mathrm{d}\mathcal{H}^{n-1}}$$
$$\longrightarrow \sqrt{|C|^{-1}\int_{\Gamma_0}u^2\,\mathrm{d}\mathcal{H}^{n-1}} < \mathbf{C_u}$$

(as the limit of the product of weakly and strongly convergent sequences), it follows that $\widetilde{(\widehat{u}_\varepsilon)} \to u$ weakly in $L^2(\Gamma_0)$ and $\widehat{u}_\varepsilon \in U_\varepsilon$ for ε sufficiently small.

Step 2. For the above given element $p \in \widehat{P}_0$, we have

$$p \in L^2(\Omega^-) \quad \text{and} \quad \| p \|^2_{L^2(\Omega^-)} \le |\partial C|_H^{-1} \cdot \mathbf{C_p}.$$

Let $\{\zeta_\varepsilon\}$ be any sequence of smooth functions such that $\zeta_\varepsilon \to p$ strongly in $L^2(\Omega^-)$ as $\varepsilon \to 0$. The following result will be useful later.

Lemma 13.25.

$$\lim_{\varepsilon\to 0} \int_{\Omega^-} \zeta_\varepsilon \, \varphi \, \mathrm{d}\mu_\varepsilon \;=\; \int_{\Omega^-} p \, \varphi \, \mathrm{d}x, \quad \forall \, \varphi \in C_0^\infty(\mathbb{R}^n). \tag{13.69}$$

Proof. Indeed, let us partition Ω^- into cubes with edges ε and denote these cubes by the symbols $\varepsilon\square_n^j$. Then

$$\begin{aligned} \int_{\Omega^-} \zeta_\varepsilon \, \varphi \, \mathrm{d}\mu_\varepsilon \;&= \sum_j \int_{\varepsilon\square_n^j} \zeta_\varepsilon(x) \, \varphi(x) \, \mathrm{d}\mu_\varepsilon \;+ \sum \int_{\Omega^- \cap \, \varepsilon\square_n^j} \zeta_\varepsilon(x) \, \varphi(x) \, \mathrm{d}\mu_\varepsilon \\ &= \sum_j \zeta_\varepsilon(x_j) \, \varphi(x_j) \int_{\varepsilon\square_n^j} \mathrm{d}\mu_\varepsilon \;+ \sum \int_{\Omega^- \cap \, \varepsilon\square_n^j} \zeta_\varepsilon(x) \, \varphi(x) \, \mathrm{d}\mu_\varepsilon, \end{aligned} \tag{13.70}$$

where x_j is a Lebesgue point of p in the cube $\varepsilon\square_n^j$ and the second sum is calculated over the set of boundary cubes. By the definition of the measure μ_ε, we have $\displaystyle\int_{\varepsilon\square_n^j} \mathrm{d}\mu_\varepsilon = \varepsilon^n \int_{\square_n} \mathrm{d}\mu = \varepsilon^n$. Moreover,

$$\left| \sum \int_{\Omega^- \cap \, \varepsilon\square_n^j} \zeta_\varepsilon(x) \, \varphi(x) \, \mathrm{d}\mu_\varepsilon \right| \le \sup_{x \in \, \Omega^-} |\zeta_\varepsilon(x)\varphi(x)| \varepsilon^n \cdot D(\varepsilon),$$

where $D(\varepsilon)$ is the number of boundary cubes and $\varepsilon^n D(\varepsilon) \to 0$ by Jordan's measurability of the set $\partial\Omega^-$. Thus, summarizing the above-cited facts, we have

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\Omega^-} \zeta_\varepsilon \, \varphi \, \mathrm{d}\mu_\varepsilon \;&= \lim_{\varepsilon\to 0} \left[\sum_j \zeta_\varepsilon(x_j) \, \varphi(x_j) \varepsilon^n - \int_{\Omega^-} \zeta_\varepsilon \, \varphi \, \mathrm{d}x \right] \\ + \lim_{\varepsilon\to 0} \int_{\Omega^-} \zeta_\varepsilon \, \varphi \, \mathrm{d}x \;&= \lim_{\varepsilon\to 0} \sum_j \left[\zeta_\varepsilon(x_j) \, \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon\square_n^j} \zeta_\varepsilon \, \varphi \, \mathrm{d}x \right] \varepsilon^n \\ &\quad + \lim_{\varepsilon\to 0} \sum \int_{\Omega^- \cap \, \varepsilon\square_n^j} \zeta_\varepsilon(x) \, \varphi(x) \, \mathrm{d}x + \int_{\Omega^-} p \, \varphi \, \mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \sum_j \left[\zeta_\varepsilon(x_j) \, \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon\square_n^j} \zeta_\varepsilon \, \varphi \, \mathrm{d}x \right] \varepsilon^n \\ &\quad + \int_{\Omega^-} p \, \varphi \, \mathrm{d}x. \end{aligned}$$

Let us suppose that

$$\lim_{\varepsilon\to 0}\sum_{j}\left|\zeta_\varepsilon(x_j)\,\varphi(x_j)-\varepsilon^{-n}\int_{\varepsilon\square_n^j}\zeta_\varepsilon\,\varphi\,\mathrm{d}x\right|\varepsilon^n>0.$$

Then there exist a constant $C^*>0$ and a value $\varepsilon^*>0$ such that

$$\left|\zeta_\varepsilon(x_j)\,\varphi(x_j)-\varepsilon^{-n}\int_{\varepsilon\square_n^j}\zeta_\varepsilon\,\varphi\,\mathrm{d}x\right|\geq C^*,\ \forall\,\varepsilon\in(0,\varepsilon^*)$$

(possibly except for a finite number of indices j for every fixed ε). Hence, wild oscillations are present in the sequence $\{\zeta_\varepsilon\,\varphi\}$. However (see [106, 245]), if we have very rapid fluctuations in the functions $\{\zeta_\varepsilon\,\varphi\}$, then the convergence $\zeta_\varepsilon\,\varphi\ \to\ p\,\varphi$ almost everywhere on Ω^- is excluded. This fact immediately reflects the failure of the strong convergence $\zeta_\varepsilon\,\varphi\to p\,\varphi$ in $L^2(\Omega^-)$ as $\varepsilon\to 0$ (see Valadier's theorem [245]). So, our supposition was wrong, and we get

$$\lim_{\varepsilon\to 0}\sum_{j}\left|\zeta_\varepsilon(x_j)\,\varphi(x_j)-\varepsilon^{-n}\int_{\varepsilon\square_n^j}\zeta_\varepsilon\,\varphi\,\mathrm{d}x\right|\varepsilon^n=0.$$

As a result, we have

$$\lim_{\varepsilon\to 0}\int_{\Omega^-}\zeta_\varepsilon\,\varphi\,\mathrm{d}\mu_\varepsilon\ =\int_{\Omega^-}p\,\varphi\,\mathrm{d}x,$$

which completes the proof of this lemma. □

Taking property (13.69) into account, we note that for the selected sequence $\{\zeta_\varepsilon\}$, there exist a numerical sequence $\{d_\varepsilon\in\mathbb{R}\}_{\varepsilon>0}$ and a value $\varepsilon_0>0$ such that

$$\lim_{\varepsilon\to 0}\sqrt{d}_\varepsilon=1\quad\text{and}\quad d_\varepsilon\|\zeta_\varepsilon\|^2_{L^2(\Omega^-)}\leq|\partial C|_H^{-1}\cdot\mathbf{C_p}\quad\text{for all}\quad\varepsilon<\varepsilon_0.$$

Thus, the desired sequence of admissible boundary controls

$$\{\widehat{p}_\varepsilon\in L^2(\Omega^-,\mathrm{d}\mu_\varepsilon)\}_{\varepsilon>0},$$

which is weakly convergent to p, can be constructed as follows: $\widehat{p}_\varepsilon=\sqrt{d}_\varepsilon\cdot\zeta_\varepsilon$ for every $\varepsilon>0$.

Step 3. Let $\widehat{y}_\varepsilon$ be a weak solution of the boundary value problem (13.3) in the sense of Definition 13.3 corresponding to $\widehat{u}_\varepsilon$ and $\widehat{p}_\varepsilon$. If this sequence satisfies the condition $\sup_{\varepsilon>0}\|\widehat{y}_\varepsilon\|_{H^1(\Omega_\varepsilon)}<+\infty$, then, due to Theorem 13.23, we have $(\widehat{u}_\varepsilon,\widehat{p}_\varepsilon,\widehat{y}_\varepsilon)\xrightarrow{w}(u,p,v_0^-,v_0^+)$, where $\widehat{v}=(v_0^+,v_0^-)$ is a weak solution in $\mathcal{V}(\Omega)$ of the limit problem (13.50). Since this problem has a unique solution we immediately deduce that $(u,p,v_0^-,v_0^+)=(u,p,y^-,y^+)$. Thus, putting $\gamma=0$, we see that for a given quadruple $(u,p,y^+,y^-)\in\Xi_0$, properties (13.40) and (13.41) of Definition 13.19 hold true.

However, if the sequence $\{\widehat{y}_\varepsilon \in H^1(\Omega_\varepsilon)\}_{\varepsilon>0}$ is unbounded, then the sequence $\{(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon, \widehat{y}_\varepsilon)\}_{\varepsilon>0}$ does not need to be relatively w-compact. Let us fix a value $\gamma > 0$. Then by the weak approximation property of the space $L^2(\Gamma_0)$, there exists an element $q \in H^{1/2}(\Gamma_0)$ and a sequence $\{q_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)\}_{\varepsilon>0}$ such that relations (13.35) and (13.36) are valid. Let $\{\widehat{p}_\varepsilon \in \widehat{P}_\varepsilon\}_{\varepsilon>0}$ be the sequence that has been constructed in Step 2. Then $\widehat{p}_\varepsilon \rightharpoonup p$ in $L^2(\Omega^-, d\mu_\varepsilon)$. Let y_ε be the corresponding weak solution of the boundary value problem (13.3) with prescribed q_ε and $\widehat{p}_\varepsilon$. Then, in view of estimate (13.17), we immediately conclude that $\sup_{\varepsilon>0} \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} < +\infty$. Hence, due to Theorem 13.23, we have $(q_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \xrightarrow{w} (q, p, v^-, v^+)$, where $v = (v^+, v^-)$ is a weak solution in $\mathcal{V}(\Omega)$ of the limit problem (13.50) with $u_0 = q$ and $p_0 = p$. Let $y = (y^+, y^-)$ be a weak solution of the problem (13.50) with $u_0 = u$ and $p_0 = p$. Then using standard techniques, we obtain

$$\|y - v\|_{\mathcal{V}(\Omega)} \le \mathbf{C}^* \|u - q\|_{L^2(\Gamma_0)} < \mathbf{C}^* \gamma. \tag{13.71}$$

As a result, we have (see (13.32))

$$(u_\varepsilon, \widehat{p}_\varepsilon, y_\varepsilon) \xrightarrow{w} (q, p, v^-, v^+),$$

$$\|(u, p, y^+, y^-) - (q, p, v^-, v^+)\|_{\mathbb{Y}_0} \le \gamma \sqrt{1 + \mathbf{C}^*}.$$

Therefore, properties (13.40) and (13.41) of Definition 13.19 hold true.

Step 4. We now verify property (13.38) of Definition 13.19. Let

$$\{(u_k, p_k, y_k)\}_{k \in \mathbb{N}}$$

be a w-convergent sequence for which there exists a sequence $\{\varepsilon_k \to 0\}$ such that $(u_k, p_k, y_k) \in \Xi_{\varepsilon_k}$ for all $k \in \mathbb{N}$. Let (u, p, y^+, y^-) be its w-limit. Then, by Proposition 13.8, we immediately have

$$\mathbf{C_0} \ge \liminf_{k \to \infty} \|u_k\|_{L^2(\Gamma_{\varepsilon_k})} \ge \sqrt{|C|^{-1}} \cdot \|u\|_{L^2(\Gamma_0)},$$

that is, $u \in \widehat{U}_0$.

Due to Proposition 13.12 and the definition of constrained set $\widehat{P}_\varepsilon$ (see (13.13)), we also have

$$|\partial C|_H^{-1} \, \mathbf{C_p} \ge \liminf_{k \to 0} \int_{\Omega^-} p_k^2 \, d\mu_{\varepsilon_k} \ge \int_{\Omega^-} p^2 \, dx;$$

that is, $p \in \widehat{P}_0$. In conclusion, it remains only to apply Theorem 13.23. Thus, $(u, p, y^+, y^-) \in \Xi_0$, and we have obtained the required result. □

13.6 Identification of the limiting cost functional

In this section, we show that the cost functional in the limit constrained minimization problem (13.37) can be recovered in the explicit form, and moreover, its analytical representation is different from the original one (13.2).

Theorem 13.26. *For the sequence of constrained minimization problems (13.19), there exists a variational w-limit (13.37) (in the sense of Definition 13.19) as ε tends to 0, where the set Ξ_0 is defined in (13.67), and*

$$I_0(u,p,y^+,y^-) = \int_{\Omega^+} (y^+ - q_0)^2 \,\mathrm{d}x + |C|^{-1} \int_{\Gamma_0} u^2 \,\mathrm{d}\mathcal{H}^{n-1} + |\partial C|_H \int_{\Omega^-} p^2 \,\mathrm{d}x. \tag{13.72}$$

Proof. In order to obtain the result of identification for the limit cost functional $I_0 : L^2(\Gamma_0) \times L^2(\Omega^-) \times H^1(\Omega^+) \times W_2^{(0,1)}(\Omega^-) \to \mathbb{R}$ in the form (13.72), we have to verify conditions (i) and (ii) of Definition 13.19.

Let (u,p,y^+,y^-) be any representative of Ξ_0, and let $\{(u_k,p_k,y_k)\}_{k\in\mathbb{N}}$ be a w-convergent sequence such that $(u_k,p_k,y_k) \xrightarrow{w} (u,p,y^+,y^-)$ and $(u_k,p_k,y_k) \in \Xi_{\varepsilon_k}$ for every $k \in \mathbb{N}$, where $\{\varepsilon_k\}$ is a subsequence of $\{\varepsilon\}$ converging to zero. Then, using Propositions 13.12 and 13.8 and the properties of w-convergence, we get

$$\liminf_{k\to\infty} \int_{\Gamma_{\varepsilon_k}} u_k^2 \,\mathrm{d}\mathcal{H}^{n-1} = \liminf_{k\to\infty} \int_{\Gamma_0} \widetilde{u}_k^2 \,\mathrm{d}\mathcal{H}^{n-1} \ge |C|^{-1} \int_{\Gamma_0} u^2 \,\mathrm{d}\mathcal{H}^{n-1},$$

$$\liminf_{k\to\infty} \int_{\Omega^-} p_k^2 \,\mathrm{d}\mu_{\varepsilon_k} \ge \int_{\Omega^-} p^2 \,\mathrm{d}x.$$

Therefore,

$$\begin{aligned}
\liminf_{k\to\infty} I_{\varepsilon_k}(u_k,p_k,y_k) &= \lim_{k\to\infty} \int_{\Omega^+} (y_k^+ - q_0)^2 \,\mathrm{d}x + \liminf_{k\to\infty} \int_{\Gamma_{\varepsilon_k}} u_k^2 \,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad + |\partial C|_H \liminf_{k\to\infty} \int_{\Omega^-} p_k^2 \,\mathrm{d}\mu_{\varepsilon_k} \\
&\ge \int_{\Omega^+} (y^+ - q_0)^2 \,\mathrm{d}x + |C|^{-1} \int_{\Gamma_0} u^2 \,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad + |\partial C|_H \int_{\Omega^-} p^2 \,\mathrm{d}x,
\end{aligned}$$

that is, property (i) of Definition 13.19 is valid.

To verify inequality (13.42) for an arbitrary quadruple $(u,p,y^+,y^-) \in \Xi_0$ and a positive value γ, we have to construct a special γ-realizing sequence $\{(u_\varepsilon,p_\varepsilon,y_\varepsilon)\}$ that satisfies condition (ii) of Definition 13.19. Let $q \in H^{1/2}(\Gamma_0)$ be an element of the open ball $\{w \in L^2(\Gamma_0) \,:\, \|w-u\|_{L^2(\Gamma_0)} < \gamma\}$ for which there exists a sequence of admissible Dirichlet controls $\{q_\varepsilon \in U_\varepsilon\}_{\varepsilon>0}$ such that

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} q_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{n-1} = |C|^{-1} \int_{\Gamma_0} q^2 \,\mathrm{d}\mathcal{H}^{n-1}.$$

In view of Lusin's theorem, such a choice is always possible.

As for the realizing sequence for the function $p \in L^2(\Omega^-)$, we take any collection of smooth functions $\{p_\varepsilon\}_{\varepsilon>0}$ such that $p_\varepsilon \to p$ strongly in $L^2(\Omega^-)$. Then

$$\lim_{\varepsilon\to 0}\int_{\Omega^-} p_\varepsilon^2\,\mathrm{d}x = \int_{\Omega^-} p^2\,\mathrm{d}x,$$

and we may always suppose that there exists a value $\varepsilon_0 > 0$ such that

$$\int_{\Omega^-} p_\varepsilon^2\,\mathrm{d}x \le \int_{\Omega^-} p^2\,\mathrm{d}x \quad \text{for every} \quad \varepsilon < \varepsilon_0.$$

So, using the same arguments as in the proof of Lemma 13.25, we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega^-} p_\varepsilon^2\,\mathrm{d}\mu_\varepsilon = \int_{\Omega^-} p^2\,\mathrm{d}x \quad \text{and} \quad p_\varepsilon \in \widehat{P}_\varepsilon, \quad \forall\,\varepsilon < \varepsilon_0.$$

Let y_ε be the corresponding weak solutions of the boundary value problem (13.3) in the sense of Definition 13.3. Then, in view of the a priori estimate (13.17) and Theorem 13.23, we have $(q_\varepsilon, p_\varepsilon, y_\varepsilon) \xrightarrow{w} (q, p, v^+, v^-)$. By analogy with Theorem 13.24, we can easily obtain that

$$\|(u, p, y^+, y^-) - (q, p, v^-, v^+)\|_{\mathbb{Y}_0} \le \gamma\sqrt{1 + \mathbf{C}^*}, \tag{13.73}$$

where the constant $\mathbf{C}^*$ is defined in (13.71).

Therefore,

$$\begin{aligned}
\limsup_{\varepsilon\to 0} I_\varepsilon(q_\varepsilon, p_\varepsilon, y_\varepsilon)
&= \lim_{\varepsilon\to 0}\int_{\Omega^+}(y_\varepsilon^+ - q_0)^2\,\mathrm{d}x + \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} q_\varepsilon^2\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad + |\partial C|_H \liminf_{\varepsilon\to 0}\int_{\Omega^-} p_\varepsilon^2\,\mathrm{d}\mu_\varepsilon \\
&= \int_{\Omega^+}(y^+ - q_0)^2\,\mathrm{d}x + |C|^{-1}\int_{\Gamma_0} q^2\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad + |\partial C|_H \int_{\Omega^-} p^2\,\mathrm{d}x = I_0(q, p, v^-, v^+).
\end{aligned} \tag{13.74}$$

However, as immediately follows from (13.72) and (13.17), there is a constant C_* such that

$$\big|I_0(u, p, y^-, y^+) - I_0(q, p, v^-, v^+)\big| \le C_*\|(u, p, y^+, y^-) - (q, p, v^-, v^+)\|_{\mathbb{Y}_0}.$$

Hence, taking (13.73) into account, we conclude that

$$\big|I_0(u, p, y^-, y^+) - I_0(q, p, v^-, v^+)\big| \le \gamma C_*\sqrt{1 + \mathbf{C}^*}.$$

Combining the last estimate and relation (13.74), we obtain the required inequality (13.42). This completes the proof. □

It is easy to see that the constrained minimization problems (13.37) can be recovered in the form of some OCP. So, in view of Theorems 13.24 and 13.26 and Definition 13.21, we can give the following conclusion.

Theorem 13.27. *For the OCP (13.2)–(13.5) under condition (13.6), there exists the unique limit with respect to w-convergence and it can be represented in the form*

$$I_0(u,p,y^+,y^-)* = \int_{\Omega^+}(y^+ - q_0)^2\,\mathrm{d}x \,+\, |C|^{-1}\int_{\Gamma_0} u^2\,\mathrm{d}\mathcal{H}^{n-1} \\ * + \,|\partial C|_H \int_{\Omega^-} p^{\,2}\,\mathrm{d}x \longrightarrow \inf, \tag{13.75}$$

$$u \in \widehat{U}_0 = \left\{ u \in L^2(\Gamma_0) \;:\; \|u\|_{L^2(\Gamma_0)} \le \sqrt{|C|}\,\mathbf{C_u} \right\}, \tag{13.76}$$

$$p \in \widehat{P}_0 = \left\{ p \in L^2(\Omega^-) \;:\; \|p\,\|^2_{L^2(\Omega^-)} \le |\partial C|_H^{-1}\,\mathbf{C_p} \right\}, \tag{13.77}$$

$$\left.\begin{aligned} -\Delta_x y^+(x) + y^+(x) &= f_0(x), && x \in \Omega^+,\\ -\partial^2 y^-(x)/\partial x_n^2 + y^-(x) &= f_0(x) + |\partial C|_H\,|C|^{-1} p\,(x), && x \in \Omega^-,\\ \partial_\nu y^+(x) &= 0, && x \in \partial\Omega^+ \setminus \Sigma,\\ y^- &= |C|^{-1}u, && on\ \ \Gamma_0,\\ y^+ &= y^-, && on\ \ \Sigma,\\ \partial_{x_n} y^+ &= |C|\,\partial_{x_n} y^-, && on\ \ \Sigma. \end{aligned}\right\} \tag{13.78}$$

To end of this section, we give the following observation concerning the variational properties of the homogenized problem (13.75)–(13.78). Since the cost functional $I_0 : \Xi_0 \to \mathbb{R}$ is strictly convex and lower semicontinuous with respect to the product of the weak topologies for $L^2(\Gamma_0)$, $L^2(\Omega^-)$, $H^1(\Omega^+)$, and $W_2^{(0,1)}(\Omega^-)$ and its domain Ξ_0 is a bounded, convex, and closed subset of

$$\mathbb{Y}_0 \equiv L^2(\Gamma_0) \times L^2(\Omega^-) \times H^1(\Omega^+) \times W_2^{(0,1)}(\Omega^-),$$

we conclude that the limiting OCP (13.75)–(13.78) has a unique solution. Let us denote by $(u^0, p^{\,0}, (y^0)^+, (y^0)^-) \in \mathbb{Y}_0$ the optimal quadruple for the limit problem (13.75)–(13.78). Then applying Theorem 13.20, we immediately obtain the following result.

Theorem 13.28. *Let* $\left\{(u_\varepsilon^{\delta_\varepsilon}, \widehat{p}_\varepsilon^{\,\delta_\varepsilon}, y_\varepsilon^{\delta_\varepsilon}) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ *be a sequence of admissible solutions for the* $\mathbb{P}_\varepsilon$ *problems such that*

$$I_\varepsilon(u_\varepsilon^{\delta_\varepsilon}, \widehat{p}_\varepsilon^{\,\delta_\varepsilon}, y_\varepsilon^{\delta_\varepsilon}) - \inf_{(u,p,y)\,\in\,\Xi_\varepsilon} I_\varepsilon(u,p,y) \le \delta_\varepsilon, \tag{13.79}$$

$$\sup_{\varepsilon>0} \|y_\varepsilon^{\delta_\varepsilon}\|_{H^1(\Omega_\varepsilon)} < +\infty, \tag{13.80}$$

and $\delta_\varepsilon \to 0$ *as* $\varepsilon \to 0$. *Assume that* $f_\varepsilon \to f_0$ *strongly in* $L^2(\Omega)$. *Then*

$$(u_\varepsilon^{\delta_\varepsilon}, \widehat{p}_\varepsilon^{\,\delta_\varepsilon}, y_\varepsilon^{\delta_\varepsilon}) \xrightarrow{w} (u^0, p^0, (y^0)^+, (y^0)^-),$$

$$\begin{aligned}\inf_{(u,\,p,\,y^+,\,y^-)\in\,\Xi_0} I_0(u,p,y^+,y^-) &= I_0\left(u^0, p^0, (y^0)^+, (y^0)^-\right)\\ &= \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^{\delta_\varepsilon}, \widehat{p}_\varepsilon^{\,\delta_\varepsilon}, y_\varepsilon^{\delta_\varepsilon})\\ &= \lim_{\varepsilon\to 0}\ \inf_{(u,p,y)\in\,\Xi_\varepsilon} I_\varepsilon(u,p,y).\end{aligned}$$

Note that this result holds true for any bounded sequence of approximate solutions. Additionally, if the original OCP (13.2)–(13.5) has an optimal solution $(u_\varepsilon^0, \widehat{p}_\varepsilon^{\,0}, y_\varepsilon^0)$ for every $\varepsilon > 0$ such that $\sup_{\varepsilon>0} \|y_\varepsilon^0\|_{H^1(\Omega_\varepsilon)} < +\infty$, then, having put $\delta_\varepsilon = 0$ in Theorem 13.28, we obtain the classical variational property for the limit problems.

13.7 Modeling of suboptimal controls

In this section, we deal with the construction of suboptimal controls to the original optimal boundary control problem (13.2)–(13.5). As was mentioned in Sect. 13.2, we cannot assert the solvability of this problem for each ε-level. However, for every $\delta > 0$, there is always a triplet $(u_\varepsilon^\delta, \widehat{p}_\varepsilon^{\,\delta}, y_\varepsilon^\delta) \in \Xi_\varepsilon$ (the so-called approximate solution) such that

$$\left| I_\varepsilon(u_\varepsilon^\delta, \widehat{p}_\varepsilon^{\,\delta}, y_\varepsilon^\delta) - \inf_{(u,p,y)\,\in\Xi_\varepsilon} I_\varepsilon(u,p,y) \right| \le \delta.$$

As follows from Theorem 13.27, the limiting OCP (13.75)–(13.78) has a unique solution. Let $(u^0, p^{\,0}, (y^0)^+, (y^0)^-) \in \mathbb{Y}_0$ be an optimal one to this limit problem. Since $p^0 \in L^2(\Omega^-)$, we cannot just use the restriction of this function as an admissible Neumann boundary control for the original problem (13.2)–(13.5) because a trace of an L^2-function on the $(n-1)$-dimensional surface S_ε is not defined. As for the Dirichlet control u^0, we have a similar situation. Indeed, the restriction $u^0\big|_{\Gamma_\varepsilon}$ is an element of $L^2(\Gamma_\varepsilon)$, whereas admissible Dirichlet controls must be elements of $H^{1/2}(\Gamma_\varepsilon)$.

In view of this, we make use of the idea of classical smoothing. Namely (see Sect. 6.5), let $K_u \in C_0^\infty(\mathbb{R}^{n-1})$ and $K_p \in C_0^\infty(\mathbb{R}^n)$ be nonnegative functions such that

$$\int_{\mathbb{R}^{n-1}} K_u(x')\,\mathrm{d}x' = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} K_p(x)\,\mathrm{d}x = 1,$$

$$K_u(x') = K_u(-x')\ \forall\, x' \in \mathbb{R}^{n-1} \quad \text{and} \quad K_p(x) = K_p(-x)\ \forall\, x \in \mathbb{R}^n.$$

Let $\widetilde{u}^{\,0}$ and $\widetilde{p}^{\,0}$ be the trivial extensions by zero of functions $u^0 \in L^2(\Gamma_0)$ and $p^0 \in L^2(\Omega^-)$ to the sets $\mathcal{O} = \left\{(x', x_n) \in \mathbb{R}^n \ :\ x' \in \mathbb{R}^{n-1},\ x_n = -\beta_0\right\}$ and

$\mathbb{R}^n$, respectively. For every fixed $\varepsilon > 0$, we define the following smoothing operators

$$\left(u^0\right)_\varepsilon = \left(\mathcal{K}_u^\varepsilon u^0\right)(x') = \varepsilon^{-n+1} \int_{\mathcal{O}} K_u\left(\frac{x'-y'}{\varepsilon}\right) \widetilde{u}^0(y', y_n)\, \mathrm{d}\mathcal{H}^{n-1}$$
$$= \varepsilon^{-n+1} \int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\varepsilon}\right) \widetilde{u}^0(y')\, \mathrm{d}y', \quad (13.81)$$

$$\left(p^0\right)_\varepsilon = \left(\mathcal{K}_p^\varepsilon p^0\right)(x) = \varepsilon^{-n} \int_{\mathbb{R}^n} K_p\left(\frac{x-y}{\varepsilon}\right) \widetilde{p}^0(y)\, \mathrm{d}y. \quad (13.82)$$

We begin with the following result:

Proposition 13.29. *For every $\delta > 0$, there exists $\varepsilon(\delta) = a/N$, where $N \in \mathbb{N}$, such that*

$$u_\varepsilon^{sub} = |C|^{-1} \left(u^0\right)_\delta\big|_{\Gamma_\varepsilon} \in U_\varepsilon, \quad (13.83)$$

that is, u_ε^{sub} is an admissible Dirichlet control to the problem (13.2)–(13.5). Moreover,

$$u_{\varepsilon(\delta)}^{sub} \rightharpoonup u^0 \quad \text{as} \quad \delta \to 0 \quad \text{in the sense of Definition 13.6.} \quad (13.84)$$

Proof. First, we note that $u_\varepsilon^{sub} \in H^{1/2}(\Gamma_\varepsilon)$ by the properties of the smoothing operator (13.81). To show that $u_\varepsilon^{sub} \in U_\varepsilon$ for some $\varepsilon > 0$, it remains to prove the estimate

$$\left\|u_\varepsilon^{sub}\right\|_{L^2(\Gamma_\varepsilon)} \le \mathbf{C_u}. \quad (13.85)$$

Following the definition of the smoothing operator (13.81) and using the Cauchy–Bunyakovskiĭ inequality, we have

$$\left\|\left(u^0\right)_\delta\right\|^2_{L^2(\mathbb{R}^{n-1})}$$
$$= \int_{\mathbb{R}^{n-1}} \left(\delta^{-n+1} \int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\delta}\right) \widetilde{u}^0(y')\, \mathrm{d}y'\right)^2 \mathrm{d}x'$$
$$\le \int_{\mathbb{R}^{n-1}} \left(\delta^{-n+1} \int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\delta}\right) \mathrm{d}y'\right)$$
$$\times \left(\delta^{-n+1} \int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\delta}\right) \left(\widetilde{u}^0(y')\right)^2 \mathrm{d}y'\right) \mathrm{d}x'$$
$$= \int_{\mathbb{R}^{n-1}} \left(\delta^{-n+1} \int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\delta}\right) \left(\widetilde{u}^0(y')\right)^2 \mathrm{d}y'\right) \mathrm{d}x'$$
$$\times \int_{\mathbb{R}^{n-1}} \left(\delta^{-n+1} \int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\delta}\right) \mathrm{d}x'\right) \left(\widetilde{u}^0(y')\right)^2 \mathrm{d}y'$$
$$= \int_{\mathbb{R}^{n-1}} \left(\widetilde{u}^0(y')\right)^2 \mathrm{d}y' = \int_{\Gamma_0} \left(u^0(y')\right)^2 \mathrm{d}y' = \|u^0\|^2_{L^2(\Gamma_0)}. \quad (13.86)$$

Due to this estimate, we come to the inequality

$$\begin{aligned}\left\|\left(u^0\right)_\delta\right\|_{L^2(\Gamma_\varepsilon)} &= \left\|\chi_{\Gamma_\varepsilon}\left(u^0\right)_\delta\right\|_{L^2(\Gamma_\varepsilon)} \\ &\le \left\|\chi_{\Gamma_0}\left(u^0\right)_\delta\right\|_{L^2(\mathbb{R}^{n-1})} \le \left\|u^0\right\|_{L^2(\Gamma_0)} \le \sqrt{|C|}\,\mathbf{C_u}, \end{aligned} \tag{13.87}$$

which holds true for every $\varepsilon > 0$. Further, for a given $\delta > 0$, we define a parameter $\varepsilon = a/N$ so that the following inequality would be valid:

$$\frac{1}{|\Gamma_\varepsilon|}\int_{\Gamma_\varepsilon}\left(u^0\right)_\delta^2\,\mathrm{d}x' \le \frac{1}{|\Gamma_0|}\int_{\Gamma_0}\left(u^0\right)_\delta^2\,\mathrm{d}x'.$$

Then, from (13.87) we deduce

$$\left\||C|^{-1}\left(u^0\right)_\delta\right\|^2_{L^2(\Gamma_\varepsilon)} \le \frac{|\Gamma_\varepsilon|}{|\Gamma_0|}\left\||C|^{-1}\left(u^0\right)_\delta\right\|^2_{L^2(\Gamma_0)} \le \frac{|\Gamma_\varepsilon|}{|\Gamma_0|}|C|^{-1}{\mathbf{C_u}}^2. \tag{13.88}$$

Since $|\Gamma_0| = |B| = a^{n-1}$ and

$$|\Gamma_\varepsilon| = \sum_{\mathbf{k}\in\theta_\varepsilon}\int_{\Gamma_\varepsilon^{\mathbf{k}}}\mathrm{d}x' = \sum_{\mathbf{k}\in\theta_\varepsilon}\varepsilon^{n-1}|C| = \left(\frac{a}{\varepsilon}\right)^{n-1}\varepsilon^{n-1}|C| = a^{n-1}|C|,$$

it follows from (13.88) that

$$\left\|u_\varepsilon^{sub}\right\|_{L^2(\Gamma_\varepsilon)} \equiv \left\||C|^{-1}\left(u^0\right)_\delta\right\|_{L^2(\Gamma_\varepsilon)} \le \mathbf{C_u}. \tag{13.89}$$

Thus, for the chosen value of ε, inequality (13.85) is valid, and hence u_ε^{sub} is an admissible Dirichlet control for the $\mathbb{P}_\varepsilon$ problem.

To conclude the proof, we note that

$$\left(u^0\right)_\delta \to u^0 \quad \text{strongly in} \quad L^2(\Gamma_0) \quad \text{as} \quad \delta \to 0, \tag{13.90}$$

by the properties of the classical smoothing. Hence in view of (13.21), (13.90), and the fact that the value of ε depends on a given parameter δ, we have

$$\widetilde{u}_{\varepsilon(\delta)}^{sub} = |C|^{-1}\chi_{\Gamma_{\varepsilon(\delta)}}\left(u^0\right)_\delta \rightharpoonup |C|^{-1}|C|u^0 = u^0 \quad \text{as} \quad \delta \to 0,$$

as a limit of the product of the weakly and strongly convergent sequences in $L^2(\Gamma_0)$. This implies the required conclusion (13.84). □

For the Neumann controls, we have a similar result.

Proposition 13.30. *Let $p^0 \in L^2(\Omega^-)$ be an optimal distributed control for the limit problem (13.75)–(13.78). Then there exists $\varepsilon_0 > 0$ such that*

$$p_\varepsilon^{sub} = \left(p^0\right)_\varepsilon\big|_{S_\varepsilon} \in P_\varepsilon, \quad \forall\,\varepsilon \in (0, \varepsilon_0), \tag{13.91}$$

that is, p_ε^{sub} is an admissible Neumann boundary control to the problem (13.2)–(13.5) for $\varepsilon > 0$ small enough. Moreover,

$$p_\varepsilon^{sub} \rightharpoonup p^0 \quad \textit{in} \quad L^2(\Omega^-, \mathrm{d}\mu_\varepsilon) \quad \textit{as} \quad \varepsilon \to 0. \tag{13.92}$$

Proof. Since $\left(p^0\right)_\varepsilon \in C(\overline{\Omega^-})$, it is clear that $p_\varepsilon^{sub} = \left(p^0\right)_\varepsilon\big|_{S_\varepsilon} \in L^2(S_\varepsilon, \mathcal{H}^{n-1})$. By analogy with (13.86), it is easy to show the validity of the inequality

$$\left\|\left(p^0\right)_\varepsilon\right\|^2_{L^2(\Omega^-)} \le \int_{\Omega^-} \left(p^0\right)^2 \,\mathrm{d}x = \|p^0\|^2_{L^2(\Omega^-)} \le |\partial C|_H^{-1}\mathbf{C_p}. \tag{13.93}$$

Thus, as follows from (13.93), in order to assert that p_ε^{sub} is an admissible control for the problem (13.2)–(13.5), we have to be sure that the estimate

$$\varepsilon \left\|p_\varepsilon^{sub}\right\|^2_{L^2(S_\varepsilon;\mathcal{H}^{n-1})} \le \mathbf{C_p} \tag{13.94}$$

is valid at least for ε small enough. In order to verify this fact, we will show that the value

$$\sigma = \left|\varepsilon|\partial C|_H^{-1} \left\|\left(p^0\right)_\varepsilon\right\|^2_{L^2(S_\varepsilon;\mathcal{H}^{n-1})} - \left\|\left(p^0\right)_\varepsilon\right\|^2_{L^2(\Omega^-)}\right|$$

can be made as small as we wish for ε small enough.

To begin, we note that

$$\left\|\left(p^0\right)_\varepsilon\right\|^2_{L^2(\Omega^-)} = \sum_{j=1}^{N^{n-1}} \int_{\left\{x=(x',x_n)\ :\ \begin{smallmatrix} x'\in\varepsilon(\square+\mathbf{k}_j) \\ -\beta_0<x_n<0 \end{smallmatrix}\right\}} \left(p^0\right)^2_\varepsilon \,\mathrm{d}x. \tag{13.95}$$

Here, $\square = [0,1)^{n-1}$. Further, we make use of the following notation:

$$\begin{aligned} L_\varepsilon^{\mathbf{k}} &= \{(x',x_n) \ :\ x' \in \varepsilon\square + \varepsilon\mathbf{k},\ -\beta_0 < x_n \le 0\}, \\ G_\varepsilon^{\mathbf{k}} &= \{(x',x_n) \ :\ x' \in \varepsilon C + \varepsilon\mathbf{k},\ -\beta_0 < x_n \le 0\}, \\ \Delta_i S_\varepsilon^{\mathbf{k}} &= \{(x',x_n) \ :\ x' \in \varepsilon\partial C + \varepsilon\mathbf{k},\ -\beta_0 + (i-1)\Delta < x_n \le -\beta_0 + i\Delta\}, \end{aligned}$$

where $\Delta = \beta_0/J$.

Since

$$\left(p^0\right)_\varepsilon \in C_0^\infty(\mathbb{R}^n),$$

it means that a wild oscillation of this function is excluded on the cross-sections of thin cylinders $L_\varepsilon^{\mathbf{k}}$ for ε small enough. Hence, for a given $\eta > 0$, there exist an $\varepsilon_\eta > 0$, an integer $J \in \mathbb{N}$, and a collection of points

$$\left\{x_{\varepsilon,i}^{\mathbf{k}} \in \Delta_i S_\varepsilon^{\mathbf{k}}\right\}$$

such that

$$\left|\int_{L_\varepsilon^{\mathbf{k}}} \left(p^0\right)^2_\varepsilon \,\mathrm{d}x - \sum_{i=1}^{J} \left(p^0\right)^2_\varepsilon (x_{\varepsilon,i}^{\mathbf{k}})\mathcal{L}_n(\varepsilon\square)\Delta\right| < \frac{\eta}{2N^{n-1}}, \quad \forall\, \varepsilon \in (0,\varepsilon_\eta).$$

Taking into account the chain of transformations

$$\sum_{i=1}^{J} \left(p^0\right)_\varepsilon^2 (x_{\varepsilon,i}^{\mathbf{k}}) \mathcal{L}_n(\varepsilon\square)\Delta$$

$$= \sum_{i=1}^{J} \left(p^0\right)_\varepsilon^2 (x_{\varepsilon,i}^{\mathbf{k}}) \varepsilon^n \Delta = \sum_{i=1}^{J} \left(p^0\right)_\varepsilon^2 (x_{\varepsilon,i}^{\mathbf{k}}) \varepsilon^n \frac{\mathcal{H}^{n-1}(\Delta_i S_\varepsilon^{\mathbf{k}})}{H^{n-2}(\varepsilon\partial C)}$$

$$= \frac{\varepsilon}{H^{n-2}(\partial C)} \sum_{i=1}^{J} \left(p^0\right)_\varepsilon^2 (x_{\varepsilon,i}^{\mathbf{k}}) \mathcal{H}^{n-1}(\Delta_i S_\varepsilon^{\mathbf{k}})$$

$$= \varepsilon |\partial C|_H^{-1} \sum_{i=1}^{J} \left(p^0\right)_\varepsilon^2 (x_{\varepsilon,i}^{\mathbf{k}}) \mathcal{H}^{n-1}(\Delta_i S_\varepsilon^{\mathbf{k}})$$

and the fact that

$$\left| \sum_{i=1}^{J} \left(p^0\right)_\varepsilon^2 (x_{\varepsilon,i}^{\mathbf{k}}) \mathcal{H}^{n-1}(\Delta_i S_\varepsilon^{\mathbf{k}}) - \int_{S_\varepsilon^{\mathbf{k}}} \left(p^0\right)_\varepsilon^2 \, \mathrm{d}\mathcal{H}^{n-1} \right| < \frac{\eta |\partial C|_H}{2\varepsilon N^{n-1}}$$

for J large enough, we deduce the following: For a given $\eta > 0$, there exists an $\varepsilon_\eta > 0$ such that

$$\left| \int_{L_\varepsilon^{\mathbf{k}}} \left(p^0\right)_\varepsilon^2 \, \mathrm{d}x - \varepsilon |\partial C|_H^{-1} \int_{S_\varepsilon^{\mathbf{k}}} \left(p^0\right)_\varepsilon^2 \, \mathrm{d}\mathcal{H}^{n-1} \right|$$

$$\leq \left| \int_{L_\varepsilon^{\mathbf{k}}} \left(p^0\right)_\varepsilon^2 \, \mathrm{d}x - \varepsilon |\partial C|_H^{-1} \sum_{i=1}^{J} \left(p^0\right)_\varepsilon^2 (x_{\varepsilon,i}^{\mathbf{k}}) \mathcal{H}^{n-1}(\Delta_i S_\varepsilon^{\mathbf{k}}) \right|$$

$$+ \varepsilon |\partial C|_H^{-1} \left| \sum_{i=1}^{J} \left(p^0\right)_\varepsilon^2 (x_{\varepsilon,i}^{\mathbf{k}}) \mathcal{H}^{n-1}(\Delta_i S_\varepsilon^{\mathbf{k}}) \right.$$

$$\left. - \int_{S_\varepsilon^{\mathbf{k}}} \left(p^0\right)_\varepsilon^2 \, \mathrm{d}\mathcal{H}^{n-1} \right| < \frac{\eta}{N^{n-1}} \tag{13.96}$$

for all $\varepsilon \in (0, \varepsilon_\eta)$.

As a result, combining this estimate with representation (13.95), we come to the conclusion that

$$\left| \left\| \left(p^0\right)_\varepsilon \right\|_{L^2(\Omega^-)}^2 - \varepsilon |\partial C|_H^{-1} \left\| \left(p^0\right)_\varepsilon \right\|_{L^2(S_\varepsilon; \mathcal{H}^{n-1})}^2 \right|$$

$$= \left| \sum_{j=1}^{N^{n-1}} \left(\int_{L_\varepsilon^{\mathbf{k}_j}} \left(p^0\right)_\varepsilon^2 \, \mathrm{d}x - \varepsilon |\partial C|_H^{-1} \int_{S_\varepsilon^{\mathbf{k}_j}} \left(p^0\right)_\varepsilon^2 \, \mathrm{d}\mathcal{H}^{n-1} \right) \right| < \eta. \tag{13.97}$$

Since relation (13.93) turns into an equality only in the case when p^0 is a constant, it follows that choosing an appropriate value $\eta > 0$, we can find an $\varepsilon_\eta > 0$ such that for every $\varepsilon \in (0, \varepsilon_\eta)$, the inequality

$$\varepsilon |\partial C|_H^{-1} \left\| \left(p^0\right)_\varepsilon \right\|_{L^2(S_\varepsilon; \mathcal{H}^{n-1})}^2 \leq \left\| p^0 \right\|_{L^2(\Omega^-)}^2 \leq |\partial C|_H^{-1} \mathbf{C}_{\mathbf{p}}$$

holds true. Hence, estimate (13.94) is valid for $\varepsilon > 0$ small enough. To conclude the proof, it remains to use the main property of the classical smoothing,

$$\left(p^{\,0}\right)_{\varepsilon} \to p^{\,0} \quad \text{strongly in} \quad L^2(\Omega^-) \quad \text{as} \quad \varepsilon \to 0,$$

and apply Lemma 13.25. □

As a direct consequence of the proof of Proposition 13.30, which we feel to be interesting per se, we present one observation concerning L^2-functions defined on thick junctions.

Corollary 13.31. *Let Ω_ε be a thick multistructure in $\mathbb{R}^n$, which consists of some domain Ω^+ and a large number of thin cylinders*

$$G_\varepsilon^{\mathbf{k}} = \{(x', x_n) \ : \ x' \in \varepsilon C + \varepsilon \mathbf{k}, \ -\beta_0 < x_n \le 0\}$$

with a small cross-section of the size εC and ε-periodically distributed along some manifold Σ on the boundary of Ω^+ (see Fig. 14.1 for a 3D example). Let $\Omega^- = \Sigma \times (a, b)$ be a domain which is filled up by thin cylinders in the limit passage as $\varepsilon \to 0$. Let $f \in L^2_{loc}(\mathbb{R}^n)$ be a given function. Then

$$\|f\|^2_{L^2(\Omega^-)} \ge \frac{\varepsilon}{\mathcal{H}^{n-2}(\partial C)} \, \|(f)_\varepsilon\|^2_{L^2(S_\varepsilon;\mathcal{H}^{n-1})}$$

for ε small enough, where $(f)_\varepsilon$ denotes the direct smoothing of the function $\chi_{\Omega^-} f$, that is,

$$(f)_\varepsilon = \varepsilon^{-n} \int_{\mathbb{R}^n} K\left(\frac{x-y}{\varepsilon}\right) \chi_{\Omega^-}(y) f(y) \, \mathrm{d}y.$$

Now we are in a position to give the main result of this section.

Theorem 13.32. *Let $(u^0, p^{\,0}, (y^0)^+, (y^0)^-) \in \mathbb{Y}_0$ be an optimal solution to the limit OCP (13.75)–(13.78). Then for every $\eta > 0$, there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$\left| \inf_{(u,p,y) \in \Xi_\varepsilon} I_\varepsilon(u, p, y) - I_\varepsilon(u_\varepsilon^{sub}, p_\varepsilon^{sub}, y^{\,sub}) \right| < \eta, \tag{13.98}$$

where

$$u_\varepsilon^{sub} = |C|^{-1} \left(u^0\right)_\delta \big|_{\Gamma_\varepsilon} \in U_\varepsilon, \tag{13.99}$$

$$p_\varepsilon^{sub} = \left(p^{\,0}\right)_\varepsilon \big|_{S_\varepsilon} \in P_\varepsilon, \tag{13.100}$$

and $y_\varepsilon^{sub} = y_\varepsilon(u_\varepsilon^{sub}, p_\varepsilon^{sub})$ is the corresponding solution of the boundary value problem (13.3), that is, the pair $(u_\varepsilon^{sub}, p_\varepsilon^{sub})$ is a suboptimal control to the original problem (13.2)–(13.5).

Proof. For a given sequence $\{\delta(\varepsilon)\}_{\varepsilon>0}$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$, we construct the following sequence of triplets $\left\{(u_\varepsilon^{sub},p_\varepsilon^{sub},y_\varepsilon^{sub})\in\mathbb{X}_\varepsilon\right\}_{\varepsilon>0}$. By Propositions 13.29 and 13.30, we can suppose that each of these triplets is admissible for the problem $\widehat{\mathbb{P}}_\varepsilon$ (to be more precise, it is so for ε small enough). Moreover, due to estimate (13.17), this sequence is equibounded in $\mathbb{X}_\varepsilon$. Then by Proposition 13.17, this one is relatively compact with respect to the w-convergence in $\mathbb{X}_\varepsilon$. Hence, taking into account Propositions 13.29 and 13.30 and Theorem 13.24, we deduce the following: This sequence is w-compact and

$$(u_\varepsilon^{sub},p_\varepsilon^{sub},y_\varepsilon^{sub})\ \overset{w}{\longrightarrow}\ (u^0,p^{\,0},(y^0)^+,(y^0)^-),$$

where $(u^0,p^{\,0},(y^0)^+,(y^0)^-)\in\mathbb{Y}_0$ is a unique optimal solution to the limit problem (13.75)–(13.78).

Let $\{(u_\varepsilon^{\eta_\varepsilon},\widehat{p}_\varepsilon^{\eta_\varepsilon},y_\varepsilon^{\eta_\varepsilon})\in\Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of approximate solutions to the $\mathbb{P}_\varepsilon$ problems such that

$$\left|I_\varepsilon(u_\varepsilon^{\eta_\varepsilon},\widehat{p}_\varepsilon^{\eta_\varepsilon},y_\varepsilon^{\eta_\varepsilon})-\inf_{(u,p,y)\in\Xi_\varepsilon}I_\varepsilon(u,p,y)\right|\le\eta_\varepsilon\le\frac{\eta}{3}\tag{13.101}$$

and $\sup_{\varepsilon>0}\|y_\varepsilon^{\eta_\varepsilon}\|_{H^1(\Omega_\varepsilon)}<+\infty$, where $\eta_\varepsilon\to 0$ as $\varepsilon\to 0$ and $\eta>0$ is a given value. Then, by Theorem 13.20, we have

$$(u_\varepsilon^{\eta_\varepsilon},\widehat{p}_\varepsilon^{\eta_\varepsilon},y_\varepsilon^{\eta_\varepsilon})\ \overset{w}{\longrightarrow}\ (u^0,p^0,(y^0)^+,(y^0)^-)\ \ \text{as}\ \ \varepsilon\to 0.$$

Further, we observe that

$$\begin{aligned}
&\left|\inf_{(u,p,y)\in\Xi_\varepsilon}I_\varepsilon(u,p,y)-I_\varepsilon(u_\varepsilon^{sub},p_\varepsilon^{sub},y_\varepsilon^{sub})\right|\\
&\qquad\le\left|\inf_{(u,p,y)\in\Xi_\varepsilon}I_\varepsilon(u,p,y)-I_\varepsilon(u_\varepsilon^{\eta_\varepsilon},\widehat{p}_\varepsilon^{\eta_\varepsilon},y_\varepsilon^{\eta_\varepsilon})\right|\\
&\qquad+\left|I_\varepsilon(u_\varepsilon^{\eta_\varepsilon},\widehat{p}_\varepsilon^{\eta_\varepsilon},y_\varepsilon^{\eta_\varepsilon})-I_0\left(u^0,p^0,(y^0)^+,(y^0)^-\right)\right|\\
&\qquad+\left|I_0\left(u^0,p^0,(y^0)^+,(y^0)^-\right)-I_\varepsilon(u_\varepsilon^{sub},p_\varepsilon^{sub},y_\varepsilon^{sub})\right|\\
&\qquad=J_1+J_2+J_3.
\end{aligned}\tag{13.102}$$

We note that for a given $\eta>0$ one can always find the following:

(i) $\varepsilon_1>0$ such that $J_1<\eta/3$ for all $\varepsilon<\varepsilon_1$ by property(13.101);
(ii) $\varepsilon_2>0$ such that $J_2<\eta/3$ for all $\varepsilon<\varepsilon_2$ by Theorem 13.20.

As for the estimate for the term J_3, we make use of the following observations:

(i) By the definition of the w-convergence (see Definition 13.16), we have

$$\left(y_\varepsilon^{sub}\right)^+\rightharpoonup(y^0)^+\ \ \text{in}\ \ H^1(\Omega^+).$$

Hence, $\left(y_\varepsilon^{sub}\right)^+\to(y^0)^+$ in $L^2(\Omega^+)$. Therefore, there exists $\varepsilon_3>0$ such that

$$L_1 = \left| \int_{\Omega^+} (y_\varepsilon^{sub} - q_0)^2 \mathrm{d}x - \int_{\Omega^+} ((y^0)^+ - q_0)^2 \, \mathrm{d}x \right| < \frac{\eta}{12} \tag{13.103}$$

for all $\varepsilon < \varepsilon_3$.

(ii) As follows from Propositions 13.8 and 13.29, the following properties take place:

$$\begin{gathered} \chi_{\Gamma_\varepsilon} \overset{*}{\rightharpoonup} |C| \ \text{ in } \ L^\infty(\Gamma_0) \ \text{ as } \ \varepsilon \to 0, \\ \left(u^0\right)_{\delta(\varepsilon)} \to u^0 \ \text{ strongly in } \ L^2(\Gamma_0) \ \text{ as } \ \varepsilon \to 0, \\ \left(\left(u^0\right)_{\delta(\varepsilon)}\right)^2 \to \left(u^0\right)^2 \ \text{ strongly in } \ L^1(\Gamma_0) \ \text{ as } \ \varepsilon \to 0. \end{gathered}$$

Hence,

$$\begin{aligned} \int_{\Gamma_\varepsilon} \left(u_\varepsilon^{sub}\right)^2 \mathrm{d}\mathcal{H}^{n-1} &= |C|^{-2} \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} \left(\left(u^0\right)_{\delta(\varepsilon)}\right)^2 \mathrm{d}\mathcal{H}^{n-1} \\ &\longrightarrow |C|^{-2} \int_{\Gamma_0} |C| \left(u^0\right)^2 \mathrm{d}\mathcal{H}^{n-1} \end{aligned}$$

as a product of the weakly and strongly convergent sequences. Therefore, there exists $\varepsilon_4 > 0$ such that

$$\begin{aligned} L_2 = \Bigg| &\int_{\Gamma_0} \chi_{\Gamma_\varepsilon} \left(|C|^{-1} \left(u^0\right)_\delta(\varepsilon)\right)^2 \mathrm{d}\mathcal{H}^{n-1} \\ &- |C|^{-1} \int_{\Gamma_0} (u^0)^2 \, \mathrm{d}\mathcal{H}^{n-1} \Bigg| < \frac{\eta}{12}, \quad \forall\, \varepsilon < \varepsilon_4. \end{aligned} \tag{13.104}$$

(iii) Estimate (13.97) implies the existence of $\varepsilon_5 > 0$ such that

$$L_3 = \left| \varepsilon \int_{S_\varepsilon} \left(p_\varepsilon^{sub}\right)^2 \mathrm{d}\mathcal{H}^{n-1} - |\partial C|_H \int_{\Omega^-} \left(p^0\right)_\varepsilon^2 \, \mathrm{d}x \right| < \frac{\eta}{12} \tag{13.105}$$

for every $\varepsilon < \varepsilon_5$.

(iv) Since $\left(p^0\right)_\varepsilon \to p^0$ strongly in $L^2(\Omega^-)$ as $\varepsilon \to 0$ by the properties of the smoothing operator, it follows that there exists $\varepsilon_6 > 0$ such that

$$L_4 = \left| \int_{\Omega^-} \left(p^0\right)_\varepsilon^2 \, \mathrm{d}x - \int_{\Omega^-} \left(p^0\right)^2 \mathrm{d}x \right| < \frac{\eta}{12}, \quad \forall\, \varepsilon < \varepsilon_6. \tag{13.106}$$

To conclude the proof, we note that the term J_3 in (13.102) can be estimated as follows:

$$
\begin{aligned}
J_3 &= \left|I_0\left(u^0, p^0, (y^0)^+, (y^0)^-\right) - I_\varepsilon(u_\varepsilon^{sub}, p_\varepsilon^{sub}, y_\varepsilon^{sub})\right| \\
&\le \left|\int_{\Omega^+} (y_\varepsilon^{sub} - q_0)^2 \mathrm{d}x - \int_{\Omega^+} ((y^0)^+ - q_0)^2 \,\mathrm{d}x\right| \\
&+ \left|\int_{\Gamma_\varepsilon} \left(u_\varepsilon^{sub}\right)^2 \mathrm{d}\mathcal{H}^{n-1} - |C|^{-1} \int_{\Gamma_0} (u^0)^2 \,\mathrm{d}\mathcal{H}^{n-1}\right| \\
&+ \left|\varepsilon \int_{S_\varepsilon} \left(p_\varepsilon^{sub}\right)^2 \mathrm{d}\mathcal{H}^{n-1} - |\partial C|_H \int_{\Omega^-} \left(p^0\right)^2 \mathrm{d}x\right| \\
&\le \left|\int_{\Omega^+} (y_\varepsilon^{sub} - q_0)^2 \,\mathrm{d}x - \int_{\Omega^+} ((y^0)^+ - q_0)^2 \,\mathrm{d}x\right| \\
&+ \left|\int_{\Gamma_0} \chi_{\Gamma_\varepsilon} \left(|C|^{-1} \left(u^0\right)_{\delta(\varepsilon)}\right)^2 \mathrm{d}\mathcal{H}^{n-1} - |C|^{-1} \int_{\Gamma_0} (u^0)^2 \,\mathrm{d}\mathcal{H}^{n-1}\right| \\
&+ \left|\varepsilon \int_{S_\varepsilon} \left(p_\varepsilon^{sub}\right)^2 \mathrm{d}\mathcal{H}^{n-1} - |\partial C|_H \int_{\Omega^-} \left(p^0\right)_\varepsilon^2 \mathrm{d}x\right| \\
&+ |\partial C|_H \left|\int_{\Omega^-} \left(p^0\right)_\varepsilon^2 \mathrm{d}x - \int_{\Omega^-} \left(p^0\right)^2 \mathrm{d}x\right| \\
&= L_1 + L_2 + L_3 + L_4. \qquad (13.107)
\end{aligned}
$$

As a result, combining estimates (13.103)–(13.107), we obtain

$$J_3 < \eta/3 \quad \text{for} \quad \varepsilon < \min\{\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}.$$

Hence, in view of (13.102), we just come to the required conclusion: Inequality (13.98) holds true for all $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}$. This concludes the proof. □

14

Gap Phenomenon in Modeling of Suboptimal Controls to Parabolic Optimal Control Problems in Thick Multistructures

In this chapter, we study the asymptotic behavior of the following class of the parabolic optimal control problems (OCPs)

$$I_\varepsilon(u_\varepsilon, y_\varepsilon) = \int_0^T \int_{\Omega^+} (y_\varepsilon - q_0)^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Gamma_\varepsilon} u_\varepsilon^2 \,\mathrm{d}x'\,\mathrm{d}t \longrightarrow \inf, \tag{14.1}$$

$$\left.\begin{aligned} y'_\varepsilon - \Delta_x\, y_\varepsilon + y_\varepsilon &= f_\varepsilon && \text{in } (0,T)\times\Omega_\varepsilon,\\ \partial_\nu y_\varepsilon &= -\varepsilon\, k_0 y_\varepsilon && \text{on } (0,T)\times S_\varepsilon,\\ y_\varepsilon &= u_\varepsilon && \text{on } (0,T)\times\Gamma_\varepsilon,\\ \partial_\nu y_\varepsilon &= 0 && \text{on } (0,T)\times\partial\Omega_\varepsilon\setminus(\Gamma_\varepsilon\cup S_\varepsilon),\\ y_\varepsilon(0,x) &= y_\varepsilon^0 && \text{a.e. } x\in\Omega_\varepsilon \end{aligned}\right\} \tag{14.2}$$

as a small parameter ε tends to 0. Here, $\Omega_\varepsilon \subset \mathbb{R}^n$ denotes a thick multistructure for which the following representation holds true (see Fig. 14.1 for a 3D example)

$$\Omega_\varepsilon = (B\times(0,c)) \cup \left(\bigcup_{\mathbf{k}\in\theta_\varepsilon} (\varepsilon C + \varepsilon\mathbf{k}) \times (-d, 0] \right) = \Omega^+ \cup \left(\bigcup_{\mathbf{k}\in\theta_\varepsilon} G_\varepsilon^{\mathbf{k}} \right),$$

where $B = (0,a)^{n-1}$ and C are bounded open smooth domains in $\mathbb{R}^{n-1}$ $(n \geq 2)$, $C \subset\subset (0,1)^{n-1}$, $\theta_\varepsilon = \{\mathbf{k} = (k_1, k_2, \ldots, k_{n-1}) \in \mathbb{N}^{n-1} \ : \ \varepsilon C + \varepsilon\mathbf{k} \subset\subset B\}$,

$$\begin{gathered} \Omega = B\times(-d,c), \quad G_\varepsilon^{\mathbf{k}} = \{(x', x_n) \ : \ x' \in \varepsilon C + \varepsilon\mathbf{k},\ -d < x_n \leq 0\},\\ \Sigma = B\times\{0\}, \quad \Omega^+ = B\times(0,c), \quad \Omega^- = B\times(-d,0), \\ \Gamma_0 = B\times\{-d\}, \quad \Omega_\varepsilon^- = \Omega_\varepsilon \cap \Omega^-, \end{gathered} \tag{14.3}$$

Γ_ε is the union of the lower bases

$$\Gamma_\varepsilon^{\mathbf{k}} = \{(x', x_n) \ : \ x' \in \varepsilon\cdot C + \varepsilon\mathbf{k},\ x_n = -d\}$$

of the thin cylinders $G_\varepsilon^{\mathbf{k}}$ when $\mathbf{k} \in \theta_\varepsilon$ (i.e., $\Gamma_\varepsilon = \Gamma_0 \cap \partial\Omega_\varepsilon$), S_ε is the union of their boundaries along the axis Ox_n,

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_14,
© Springer Science+Business Media, LLC 2011

$$S_\varepsilon^{\mathbf{k}} = \{(x', x_n) \ : \ x' \in \varepsilon \cdot \partial C + \varepsilon \mathbf{k}, \ -d < x_n < 0\},$$

k_0 is a positive constant, $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative, and

$$q_0 \in L^2(0,T; L^2(\Omega^+)), \ y_\varepsilon^0 \in L^2(\Omega_\varepsilon), \quad \text{and} \quad f_\varepsilon \in L^2(0,T; L^2(\Omega))$$

are given functions. Throughout this chapter, we suppose that $\varepsilon = a/N$, where N is a large positive integer. For this kind of domains and boundary value problems in Ω_ε, we refer to Brizzi and Chalot [35] and Mel'nyk and Nazarov [187].

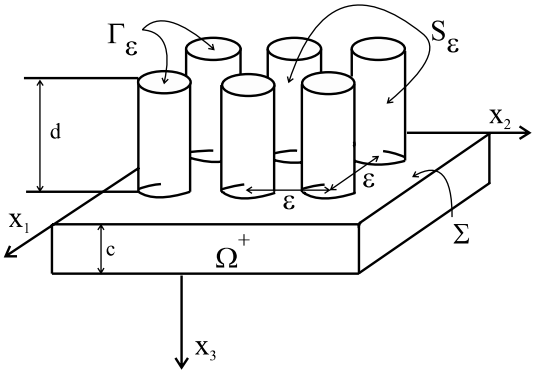


Fig. 14.1. Thick multistructure Ω_ε

We consider the OCP (14.1)–(14.2) assuming that there are two different classes of admissible boundary controls U_ε^a (the so-called regular controls) and U_ε^b (the so-called contrast controls) which are realized via the Dirichlet boundary conditions posed on the lower bases Γ_ε of the thin cylinders $G_\varepsilon^{\mathbf{k}}$, where

$$u_\varepsilon \in U_\varepsilon^a = \left\{u|_{\Gamma_\varepsilon} \ : \ u \in L^2(0,T; H^1(\Gamma_0)), \ \|u\|_{L^2(0,T; H^1(\Gamma_0))} \le \mathbf{C_0}\right\}, \quad (14.4)$$

$$u_\varepsilon \in U_\varepsilon^b = \left\{u \in L^2(0,T; H^1(\Gamma_\varepsilon)), \quad \|u\|_{L^2(0,T; L^2(\Gamma_\varepsilon))} \le \mathbf{C_0}\right\}. \quad (14.5)$$

We denote problems (14.1), (14.2), (14.4) and (14.1), (14.2), (14.5) by $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$, respectively. It is well known that the computational calculation of optimal solutions to these problems is very complicated due to singularities of the thick junctions Ω_ε. Therefore, we study the asymptotic behavior of these problems as the parameter ε tends to 0. We want to obtain appropriate asymptotic limits for the OCPs $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$ as the parameter ε tends to 0 and to show that their optimal solutions can be used as suboptimal controls to the corresponding problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$.

For comparison, we note that the asymptotic analysis of an optimal boundary control problem for linear elliptic equations in the thick multistructures with U_ε^a-admissible boundary controls was given in Chap. 12. At the same

time the OCPs for linear parabolic and hyperbolic equations in the same domains with unconstrained distributed L^2-controls were studied in [88, 100]. However, the characteristic feature of these OCPs is the fact that each of these problems has a unique optimal solution for every ε.

The OCPs $\mathbb{P}^a_\varepsilon$ and $\mathbb{P}^b_\varepsilon$ we consider are such that the existence of an optimal solution for the control problem $\mathbb{P}^b_\varepsilon$ has to be assumed. This is not typical for the majority of investigations in this field. We show that the result of homogenization for these problems as $\varepsilon \to 0$ (i.e., when the number of attached thin cylinders increases without bound and their thicknesses vanish) essentially depends on the classes of admissible controls, namely let $|C|$ be the $(n-1)$-dimensional Lebesgue measure of the set C, let $\widetilde{v}_\varepsilon$ be the zero-extension to Ω of a function v defined on Ω_ε, and let χ_{Ω^+} and χ_{Ω^-} be the characteristic functions of the sets Ω^+ and Ω^-, respectively. Having assumed

$$\widetilde{y}^0_\varepsilon \rightharpoonup (|C|\chi_{\Omega^-} + \chi_{\Omega^+})\, y^0 \quad \text{in} \quad L^2(\Omega) \text{ as } \quad \varepsilon \to 0, \tag{14.6}$$

$$\widetilde{f}_\varepsilon \rightharpoonup (|C|\chi_{\Omega^-} + \chi_{\Omega^+})\, f_0 \quad \text{in} \quad L^2(0,T;\, L^2(\Omega)) \text{ as } \quad \varepsilon \to 0, \tag{14.7}$$

we prove that for the $\mathbb{P}^a_\varepsilon$ problem there exists a unique variational limit $(\mathbb{P}^a_{hom})$ as $\varepsilon \to 0$ that can be represented in the form

$$\left.\begin{aligned} (y^+)' - \Delta_x y^+ + y^+ &= f_0 \ \text{ in } \ (0,T)\times\Omega^+,\\ (y^-)' - \partial^2_{x_n} y^- + \frac{|C| + k_0|\partial C|_H}{|C|} y^- &= f_0 \ \text{ in } \ (0,T)\times\Omega^-,\\ \partial_\nu y^+ &= 0 \ \text{ in } \ (0,T)\times\partial\Omega^+ \setminus \Sigma,\\ y^- &= u \ \text{ on } \ (0,T)\times\Gamma_0,\\ y^+ = y^-, \quad \partial_{x_n} y^+ &= |C|\,\partial_{x_n} y^- \ \text{ on } \ (0,T)\times\Sigma,\\ y(0,x) &= y^0(x) \ \text{ a.e. } \ x\in\Omega, \end{aligned}\right\} \tag{14.8}$$

$$u \in U_a = \left\{ u \in L^2(0,T;H^1(\Gamma_0)) \,:\, \|u\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C_0} \right\}, \tag{14.9}$$

$$\begin{aligned} I_a(u,y^+,y^-) = &\int_0^T \int_{\Omega^+} (y^+ - q_0)^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ |C| \int_0^T \int_{\Gamma_0} u^2 \,\mathrm{d}x'\,\mathrm{d}t \longrightarrow \inf. \end{aligned} \tag{14.10}$$

Whereas for the OCP $\mathbb{P}^b_\varepsilon$, the limit problem $\left(\mathbb{P}^b_{hom}\right)$ has another analytical representation

$$\left.\begin{aligned} (y^+)' - \Delta_x y^+ + y^+ &= f_0 \ \text{ in } \ (0,T)\times\Omega^+,\\ (y^-)' - \partial^2_{x_n} y^- + \frac{|C| + k_0|\partial C|_H}{|C|} y^- &= f_0 \ \text{ in } \ (0,T)\times\Omega^-,\\ \partial_\nu y^+ &= 0 \ \text{ in } \ (0,T)\times\partial\Omega^+ \setminus \Sigma,\\ v_0^- &= |C|^{-1} u \ \text{ on } \ (0,T)\times\Gamma_0,\\ y^+ = y^-, \quad \partial_{x_n} y^+ &= |C|\,\partial_{x_n} y^- \ \text{ on } \ (0,T)\times\Sigma,\\ y(0,x) &= y^0(x) \ \text{ a.e. } \ x\in\Omega, \end{aligned}\right\} \tag{14.11}$$

$$u \in U_b = \left\{ u \in L^2((0,T) \times \Gamma_0) \,:\, \|u\|_{L^2((0,T)\times\Gamma_0)} \le \sqrt{|C|}\mathbf{C_0} \right\}, \qquad (14.12)$$

$$I_b(u, y^+, y^-) = \int_0^T \int_{\Omega^+} (y^+ - q_0)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|C|} \int_0^T \int_{\Gamma_0} u^2 \,\mathrm{d}x'\,\mathrm{d}t \longrightarrow \inf. \qquad (14.13)$$

Here, by v^+ and v^- we denote the restrictions of a function $v : (0,T) \times \Omega \to \mathbb{R}$ to the sets $(0,T) \times \Omega^+$ and $(0,T) \times \Omega^-$, respectively. Hence, the suboptimal controls for the original problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$ will be different.

14.1 On solvability of the original optimal control problems

We begin this section with the description of the geometry of the set S_ε in terms of a singular measure on $\mathbb{R}^n$ as we did in Sect. 13.2. Let μ_0 be a periodic finite positive Borel measure on $\mathbb{R}^{n-1}$ with the torus of periodicity $\square = [0,1)^{n-1}$. We assume that the Borel measure μ_0 is the probability measure, concentrated and uniformly distributed on the set ∂C, so $\int_\square \mathrm{d}\mu_0 = 1$.

Now, we set $\square_n = \square \times [0,1) = [0,1)^n$ and consider the measure $\mathrm{d}\mu = \mathrm{d}\mu_0 \times \mathrm{d}x_n$ on $\square_n$. It is easy to see that this measure concentrated on the set $\partial C \times [0,1)$ and for any smooth function g, we have

$$\int_{\square_n} g \,\mathrm{d}\mu = \int_0^1 \int_\square g \,\mathrm{d}x_n \,\mathrm{d}\mu_0 \;=\; \left[\mathcal{H}^{n-1}(\partial C \times [0,1))\right]^{-1} \int_{\partial C \times [0,1)} g \,\mathrm{d}\mathcal{H}^{n-1}.$$

However, $\mathcal{H}^{n-1}(\partial C \times [0,1)) = \mathcal{H}^{n-2}(\partial C)$. Then, using in the sequel the notation $|\partial C|_H = \mathcal{H}^{n-2}(\partial C)$, the previous relation can be rewritten in the form

$$\int_{\square_n} g \,\mathrm{d}\mu = \int_0^1 \int_\square g \,\mathrm{d}x_n \,\mathrm{d}\mu_0 \;=\; |\partial C|_H^{-1} \int_{\partial C \times (0,1)} g \,\mathrm{d}\mathcal{H}^{n-1}. \qquad (14.14)$$

Let Λ be any Borel set of $\mathbb{R}^n$. We introduce the so-called "scaling" measure μ_ε by the rule $\mu_\varepsilon(\Lambda) = \varepsilon^n \mu(\varepsilon^{-1}\Lambda)$. This measure has period ε. Since $\mu(\varepsilon\square_n) = \varepsilon\,\mu_0(\varepsilon\square)$ by the definition of μ, it follows that

$$\mu_\varepsilon(\varepsilon\square_n) = \varepsilon^n \int_0^\varepsilon \int_{\varepsilon\square} \mathrm{d}\mu_0(x'/\varepsilon)\,\mathrm{d}(x_n/\varepsilon) = \varepsilon^n \int_0^1 \int_\square \mathrm{d}\mu_0\,\mathrm{d}x_n = \varepsilon^n.$$

Hence, the measure μ_ε weakly converges to the Lebesgue measure on $\mathbb{R}^n$ as $\varepsilon \to 0$, that is,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mu_\varepsilon = \int_{\mathbb{R}^n} \varphi \,\mathrm{d}x \quad \text{for all functions} \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

It is clear that $y_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon))$ is the weak solution of the above-mentioned problem whenever (see Lions [169])

$$\begin{aligned}\int_0^T \int_{\Omega_\varepsilon} \Big(- y_\varepsilon \varphi\, \psi' \,+\, \nabla y_\varepsilon \cdot \nabla\varphi\, \psi \,+\, y_\varepsilon\, \varphi\, \psi \Big)\, \mathrm{d}x\, \mathrm{d}t \\ + \, k_0\, \varepsilon \int_0^T \int_{S_\varepsilon} y_\varepsilon \varphi\, \psi\, \mathrm{d}x'\, \mathrm{d}t \\ = \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon\, \varphi\, \psi\, \mathrm{d}x\, \mathrm{d}t, \;\; \forall\, \varphi \in C_0^\infty(\mathbb{R}^n;\Gamma_\varepsilon), \;\; \forall\, \psi \in C_0^\infty(0,T),\end{aligned} \tag{14.15}$$

where by $C_0^\infty(\mathbb{R}^n;\Gamma_\varepsilon)$ we denote the set of all functions of $C_0^\infty(\mathbb{R}^n)$ such that $\varphi|_{\Gamma_\varepsilon} = 0$.

Let us consider the last term on the left-hand side of identity (14.15). Using the notations introduced above, we may write

$$\begin{aligned} k_0\, \varepsilon \int_0^T \int_{S_\varepsilon} y_\varepsilon \varphi \psi\, \mathrm{d}x'\, \mathrm{d}t \\ = k_0\, \varepsilon \int_0^T \Big(\sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\partial C + \mathbf{k}_j)} \int_{-d}^0 y_\varepsilon \varphi\, d\mathcal{H}^{n-2}\, \mathrm{d}x_n \Big) \psi\, \mathrm{d}t \\ = k_0\, \varepsilon |\partial C|_H \int_0^T \Big(\sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square + \mathbf{k}_j)} \int_{-d}^0 \widehat{y}_\varepsilon \varphi\, \varepsilon^{n-2}\, d\mu_0(x'/\varepsilon)\, \mathrm{d}x_n \Big) \psi\, \mathrm{d}t \\ = k_0\, |\partial C|_H \int_0^T \Big(\sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square + \mathbf{k}_j)} \int_{-d}^0 \widehat{y}_\varepsilon \varphi\; \varepsilon^n\, \mathrm{d}\mu_0(x'/\varepsilon)\, \mathrm{d}(x_n/\varepsilon) \Big) \psi\, \mathrm{d}t \\ = k_0\, |\partial C|_H \int_0^T \Big(\sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square + \mathbf{k}_j)} \int_{-d}^0 \widehat{y}_\varepsilon \varphi\, \mathrm{d}\mu_\varepsilon \Big) \psi\, \mathrm{d}t \\ = k_0\, |\partial C|_H \int_0^T \int_{\Omega^-} \widehat{y}_\varepsilon \varphi \psi\, \mathrm{d}\mu_\varepsilon\, \mathrm{d}t. \end{aligned}$$

Here, by $\widehat{y}_\varepsilon$ we denote a function of $L^2(0,T;L^2(\Omega^-,d\mu_\varepsilon))$ taking the same values as y_ε on the set S_ε. Note that the integral $\displaystyle\int_{\Omega^-} \widehat{y}_\varepsilon \varphi\, \mathrm{d}\mu_\varepsilon$ is well defined for every function $\varphi \in C_0^\infty(\mathbb{R}^n;\Gamma_\varepsilon)$. Indeed, since the set Ω^- is bounded and $\widehat{y}_\varepsilon\, d\mu_\varepsilon$ is a Radon measure, it follows that $\displaystyle\int_{\Omega^-} \widehat{y}_\varepsilon \varphi\, \mathrm{d}\mu_\varepsilon$ is a linear continuous functional on $C_0^\infty(\mathbb{R}^n;\Gamma_\varepsilon)$.

Let $\mathbf{X}_{\mu_\varepsilon}$ be the vector space of functions $y_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon))$ such that $y_\varepsilon \in L^2(0,T;L^2(\Omega^-,d\mu_\varepsilon))$, that is, for any function $y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}$, the integral $\int_0^T \int_{\Omega^-} y_\varepsilon^2\, \mathrm{d}\mu_\varepsilon$ is well defined. It is easy to see that $\mathbf{X}_{\mu_\varepsilon}$ is a Hilbert space with respect to the following scalar product

$$(y_\varepsilon, v_\varepsilon)_{\mathbf{X}_{\mu_\varepsilon}} = \int_0^T \int_{\Omega_\varepsilon} (\nabla y_\varepsilon \cdot \nabla v_\varepsilon + y_\varepsilon v_\varepsilon)\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_{\Omega^-} y_\varepsilon v_\varepsilon\, \mathrm{d}\mu_\varepsilon\, \mathrm{d}t.$$

As a result of this motivation, we give the following variational formulation of the initial boundary value problem (14.2).

Definition 14.1. *We say that a function $y_\varepsilon = y_\varepsilon(u_\varepsilon)$ is a weak solution of the parabolic problem (14.2) for a given function $u_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon))$ if*

$$\int_0^T \int_{\Omega_\varepsilon} (-y_\varepsilon \varphi \psi' + \nabla y_\varepsilon \cdot \nabla \varphi \psi + y_\varepsilon \varphi \psi)\, \mathrm{d}x\, \mathrm{d}t + k_0\, |\partial C|_H \int_0^T \int_{\Omega^-} y_\varepsilon \varphi \psi\, \mathrm{d}\mu_\varepsilon\, \mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon \varphi \psi\, \mathrm{d}x\, \mathrm{d}t, \quad (14.16)$$

$$y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon},\ y_\varepsilon(0,x) = y_\varepsilon^0 \ \ a.e.\ \ x \in \Omega_\varepsilon,\ y_\varepsilon|_{\Gamma_\varepsilon} = u_\varepsilon \ \ a.e.\ \ t \in (0,T) \quad (14.17)$$

holds for every $\varphi \in C_0^\infty(\mathbb{R}^n;\Gamma_\varepsilon)$ and $\psi \in C_0^\infty(0,T)$.

Then using the standard Hilbert space method, we have the following result.

Proposition 14.2. *For any given function $u_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon))$, the problem (14.2) admits a unique weak solution in the sense of Definition 14.1 such that*

$$y_\varepsilon' \in L^2(0,T;(H^1(\Omega_\varepsilon))'),$$
$$\|y_\varepsilon\|_{\mathbf{X}_{\mu_\varepsilon}} \le c\left(\|f_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(0,T;H^1(\Gamma_\varepsilon))}\right), \quad \forall\, \varepsilon > 0, \quad (14.18)$$

where the constant $c > 0$ is independent of ε (see Lions [169]).

Now, we can return to the question on solvability of the OCPs $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$. For this we rewrite the original problems as follows:

$$(\mathbb{P}_\varepsilon^a)\ :\ \left\langle \inf_{(u_\varepsilon,y_\varepsilon)\in \Xi_\varepsilon^a} I_\varepsilon(u_\varepsilon,y_\varepsilon) \right\rangle, \qquad (\mathbb{P}_\varepsilon^b)\ :\ \left\langle \inf_{(u_\varepsilon,y_\varepsilon)\in \Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon,y_\varepsilon) \right\rangle, \quad (14.19)$$

where by Ξ_ε^a and Ξ_ε^b we denote the sets of admissible pairs for the corresponding control problems (i.e., $(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^i$ $(i = 1, 2)$) if

$$\Xi_\varepsilon^i = \left\{ \begin{array}{r} u_\varepsilon \in U_\varepsilon^i, \quad y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}, \quad y_\varepsilon|_{\Gamma_\varepsilon} = u_\varepsilon, \\ y_\varepsilon(0,x) = y_\varepsilon^0 \quad \text{a.e.}\ \ x \in \Omega_\varepsilon, \\ \displaystyle\int_0^T \int_{\Omega_\varepsilon} (-y_\varepsilon \varphi \psi' + \nabla y_\varepsilon \cdot \nabla \varphi \psi + y_\varepsilon \varphi \psi)\, \mathrm{d}x\, \mathrm{d}t \\ \displaystyle + k_0\, |\partial C|_H \int_0^T \int_{\Omega^-} y_\varepsilon \varphi \psi\, \mathrm{d}\mu_\varepsilon\, \mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon \varphi \psi\, \mathrm{d}x\, \mathrm{d}t, \\ \forall\, \varphi \in C_0^\infty(\mathbb{R}^n;\Gamma_\varepsilon),\ \forall\, \psi \in C_0^\infty(0,T) \end{array} \right\}. \quad (14.20)$$

Using the direct method of the Calculus of Variations, it can be easily shown that $\mathbb{P}^a_\varepsilon$ has a unique solution for every value $\varepsilon > 0$:

$$I_\varepsilon(u^a_\varepsilon, y^a_\varepsilon) = \inf_{(u_\varepsilon, y_\varepsilon) \in \Xi^a_\varepsilon} I_\varepsilon(u_\varepsilon, y_\varepsilon).$$

As for the $\mathbb{P}^b_\varepsilon$-problem, we observe that its set of admissible pairs Ξ^b_ε is convex and closed in $L^2(0,T;H^1(\Omega_\varepsilon)) \times \mathbf{X}_{\mu_\varepsilon}$ and the cost functional (14.1) is strictly convex and lower semicontinuous with respect to the weak topology of $L^2(0,T;H^1(\Omega_\varepsilon)) \times L^2(0,T;H^1(\Omega_\varepsilon))$. Hence, we cannot expect solvability of this problem, in general, that is, the existence of an optimal solution $(u^b_\varepsilon, y^b_\varepsilon) \in \Xi^b_\varepsilon$ for the $\mathbb{P}^b_\varepsilon$-problem must be taken as an assumption. However, if the $\mathbb{P}^b_\varepsilon$-problem is solvable, then its solution is unique.

Let τ^a_ε be the product of the weak topologies of

$$L^2(0,T;H^1(\Gamma_\varepsilon)) \quad \text{and} \quad L^2(0,T;H^1(\Omega_\varepsilon))$$

and let τ^b_ε be the product of the weak topologies of

$$L^2(0,T;L^2(\Gamma_\varepsilon)) \quad \text{and} \quad L^2(0,T;H^1(\Omega_\varepsilon)).$$

Let us denote by $\mathrm{cl}_{\tau^b_\varepsilon}\Xi^b_\varepsilon$ the closure of the set Ξ^b_ε with respect to the τ^b_ε topology and consider the following constrained minimization problem (the so-called τ^b_ε-relaxed problem for the OCP $\mathbb{P}^b_\varepsilon$):

$$\left\langle \inf_{(u_\varepsilon, y_\varepsilon) \in \mathrm{cl}_{\tau^b_\varepsilon}\Xi^b_\varepsilon} I_\varepsilon(u_\varepsilon, y_\varepsilon) \right\rangle.$$

It is clear that this problem is solvable for every ε. Indeed, $\mathrm{cl}_{\tau^b_\varepsilon}\Xi^b_\varepsilon$ is a convex, closed and bounded subset of $L^2(0,T;L^2(\Gamma_\varepsilon)) \times \mathbf{X}_{\mu_\varepsilon}$, and

$$I_\varepsilon : L^2(0,T;L^2(\Gamma_\varepsilon)) \times \mathbf{X}_{\mu_\varepsilon} \to \mathbb{R}$$

is a strictly convex τ^b_ε-lower semicontinuous functional. This means that this problem has a unique solution $(u^*_\varepsilon, y^*_\varepsilon) \in \mathrm{cl}_{\tau^b_\varepsilon}\Xi^b_\varepsilon$.

Theorem 14.3. *If $(u^b_\varepsilon, y^b_\varepsilon)$ is an optimal pair for the $\mathbb{P}^b_\varepsilon$ problem, then $(u^b_\varepsilon, y^b_\varepsilon)$ is the unique solution of the τ^b_ε-relaxed problem.*

Proof. Let ε be any fixed value (we recall that $\varepsilon = a/N$). Since $\Xi^b_\varepsilon \subset \mathrm{cl}_{\tau^b_\varepsilon}\Xi^b_\varepsilon$, we have

$$\inf_{(u_\varepsilon, y_\varepsilon) \in \mathrm{cl}_{\tau^b_\varepsilon}\Xi^b_\varepsilon} I_\varepsilon(u_\varepsilon, y_\varepsilon) \le \inf_{(u_\varepsilon, y_\varepsilon) \in \Xi^b_\varepsilon} I_\varepsilon(u_\varepsilon, y_\varepsilon).$$

Let $(u^*_\varepsilon, y^*_\varepsilon)$ be a solution of the τ^b_ε-relaxed problem. Assume that

$$I_\varepsilon(u^*_\varepsilon, y^*_\varepsilon) < \inf_{(u_\varepsilon, y_\varepsilon) \in \Xi^b_\varepsilon} I_\varepsilon(u_\varepsilon, y_\varepsilon) = I_\varepsilon(u^b_\varepsilon, y^b_\varepsilon) =: \alpha. \tag{14.21}$$

Thus, $(u_\varepsilon^*, y_\varepsilon^*) \in \mathrm{cl}_{\tau_\varepsilon^b} \Xi_\varepsilon^b \setminus \Xi_\varepsilon^b$. At the same time, there exists a sequence of admissible pairs $\{(u_{\varepsilon,k}, y_{\varepsilon,k}) \subset \Xi_\varepsilon^b : \ k \in \mathbb{N}\}$ such that

$$(u_{\varepsilon,k}, y_{\varepsilon,k}) \xrightarrow{\tau_\varepsilon^b} (u_\varepsilon^*, y_\varepsilon^*).$$

Obviously,

$$I_\varepsilon(u_{\varepsilon,k}, y_{\varepsilon,k}) \geq \inf_{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon, y_\varepsilon) = \alpha, \quad \forall\, k \in \mathbb{N}. \tag{14.22}$$

By virtue of the τ_ε^b-lower semicontinuity property of the cost functional I_ε, we have

$$\liminf_{k \to \infty} I_\varepsilon(u_{\varepsilon,k}, y_{\varepsilon,k}) \geq I_\varepsilon(u_\varepsilon^*, y_\varepsilon^*).$$

Then taking into account relation (14.22), we conclude that $I_\varepsilon(u_\varepsilon^*, y_\varepsilon^*) \geq \alpha$. However, this is in contradiction with inequality (14.21). Thus,

$$\alpha = I_\varepsilon(u_\varepsilon^*, y_\varepsilon^*) = \inf_{(u_\varepsilon, y_\varepsilon) \in \mathrm{cl}_{\tau_\varepsilon^b} \Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon, y_\varepsilon),$$

that is, $(u_\varepsilon^*, y_\varepsilon^*) \equiv (u_\varepsilon^b, y_\varepsilon^b)$. $\square$

14.2 Formalism of convergence in variable Banach spaces

It is clear that $\Xi_\varepsilon^a \subset \Xi_\varepsilon^b$ and these inclusions are strict for every fixed ε. So, the problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$ are drastically different from the control theory point of view. It means that the following inequality holds for every $\varepsilon > 0$:

$$I_\varepsilon(u_\varepsilon^a, y_\varepsilon^a) = \min_{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^a} I_\varepsilon(u_\varepsilon, y_\varepsilon) > \min_{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon, y_\varepsilon).$$

Hence, in the "limit" as ε tends to 0, we can obtain one homogenized problem for the (a) case, and another one for the (b) case.

To study the asymptotic behavior of the problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$, we adopt the concept of the variational convergence of constrained minimization problems (see Sect. 8.4). Then this procedure can be reduced to the limit analysis of the sequences

$$\left\{ \left\langle \inf_{(u,y) \in \Xi_\varepsilon^a} I_\varepsilon(u, y) \right\rangle \ : \ \varepsilon \to 0 \right\}, \quad \left\{ \left\langle \inf_{(u,y) \in \Xi_\varepsilon^b} I_\varepsilon(u, y) \right\rangle \ : \ \varepsilon \to 0 \right\}, \tag{14.23}$$

where the cost functionals $I_\varepsilon : \Xi_\varepsilon^i \to \mathbb{R}$, $i = a, b$, and the corresponding sets of admissible pairs are defined in (14.1) and (14.20), respectively.

Note that because of the specific construction of the domains Ω_ε, we have a rather delicate situation with the limit passage in (14.19) as $\varepsilon \to 0$. Indeed, each of the admissible pairs $(u_\varepsilon, y_\varepsilon)$ belongs to the corresponding space

$$\mathbb{Y}_\varepsilon := L^2(0, T; H^1(\Gamma_\varepsilon)) \times \mathbf{X}_{\mu_\varepsilon} \tag{14.24}$$

and this fact is as common for the $\mathbb{P}_\varepsilon^a$-problem as for the $\mathbb{P}_\varepsilon^b$-problem. Therefore, we focus our attention in this section on working up the convergence formalism in such spaces.

14.2.1 The convergence concept for the $\mathbb{P}_{\varepsilon}^{a}$-problems

Let $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ be a sequence of pairs such that $u_\varepsilon \in U_\varepsilon^a$, $y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}$ $\forall\, \varepsilon > 0$, and

$$\limsup_{\varepsilon>0} \|y_\varepsilon\|^2_{\mathbf{X}_{\mu_\varepsilon}} = \limsup_{\varepsilon>0} \left[\int_0^T \int_{\Omega_\varepsilon} \left(|\nabla y_\varepsilon|^2 \mathrm{d}x + y_\varepsilon^2 \right) \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_{\Omega^-} y_\varepsilon^2 \, \mathrm{d}\mu_\varepsilon \right] < +\infty.$$

It is clear that any sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^a\}_{\varepsilon>0}$ satisfies these assumptions.

As the definition of the sets U_ε^a indicates (see (14.4)), for every $\varepsilon > 0$ and for every control function $u_\varepsilon \in U_\varepsilon^a$, there exists an extension operator

$$P_\varepsilon : L^2(0,T; H^1(\Gamma_\varepsilon)) \to L^2(0,T; H^1(\Gamma_0))$$

such that $\|P_\varepsilon(u_\varepsilon)\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C_0}$. However, the weak limits of any two weakly convergent sequences

$$\{P_\varepsilon^{(1)}(u_\varepsilon)\}_{\varepsilon>0} \quad \text{and} \quad \{P_\varepsilon^{(2)}(u_\varepsilon)\}_{\varepsilon>0}$$

are the same. Indeed, let us assume that

$$P_\varepsilon^{(1)}(u_\varepsilon) \rightharpoonup u_1^* \quad \text{and} \quad P_\varepsilon^{(2)}(u_\varepsilon) \rightharpoonup u_2 \quad \text{in} \quad L^2(0,T; H^1(\Gamma_0)).$$

Let $\chi_{\Gamma_\varepsilon}$ be the characteristic function of the set Γ_ε. Since $\chi_{\Gamma_\varepsilon}$ is a $\square$-periodic function, it follows that $\chi_{\Gamma_\varepsilon} \to |C|$ weakly-$*$ in $L^2(B)$ as $\varepsilon \to 0$. Then passing to the limit in the integral identity

$$\int_0^T \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} P_\varepsilon^{(1)}(u_\varepsilon) \varphi(x) \psi(t)\, \mathrm{d}x\, \mathrm{d}t = \int_0^T \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} P_\varepsilon^{(2)}(u_\varepsilon) \varphi(x) \psi(t)\, \mathrm{d}x\, \mathrm{d}t, \quad \forall\, \psi \in C_0^\infty(0,T),\ \forall\, \varphi \in H^1(\Gamma_0)$$

as ε tends to 0, we conclude

$$|C| \int_0^T \int_{\Gamma_0} u_1^* \varphi(x) \psi(t)\, \mathrm{d}x\, \mathrm{d}t = |C| \int_0^T \int_{\Gamma_0} u_2^* \varphi(x) \psi(t)\, \mathrm{d}x\, \mathrm{d}t, \quad \forall\, \psi \in C_0^\infty(0,T), \forall\, \varphi \in H^1(\Gamma_0).$$

Hence, $u_1^* = u_2^*$ and we are done.

In view of this, we give the following

Definition 14.4. *We say that a sequence of controls*

$$\left\{ u_\varepsilon \in L^2(0,T; H^1(\Gamma_\varepsilon)) \right\}_{\varepsilon>0}$$

is weakly convergent to a function u^* with respect to the space $L^2(0,T; H^1(\Gamma_0))$ if some sequence of its images $\{P_\varepsilon(u_\varepsilon)\}_{\varepsilon>0} \subset L^2(0,T; H^1(\Gamma_0))$ converges to u^* weakly in $L^2(0,T; H^1(\Gamma_0))$.

As a consequence, we have the following obvious result.

Lemma 14.5. *Any sequence of admissible controls* $\{u_\varepsilon \in U_\varepsilon^a\}_{\varepsilon>0}$ *is relatively compact with respect to the weak convergence introduced above. Moreover, its weak limit* u^* *belongs to the set*

$$U_a = \left\{u \in L^2(0,T;H^1(\Gamma_0)) \mid \|u\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C_0}\right\}.$$

Now, we give the convergence formalism for the sequences of the type $\{y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}\}$. By analogy with Brizzi and Chalot [35], we extend each of the functions y_ε by 0 onto the whole domain Ω, namely

$$\widetilde{y}_\varepsilon(x) := \begin{cases} y_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon, \\ 0 & \text{if } x \in \Omega \setminus \Omega_\varepsilon, \end{cases} \tag{14.25}$$

and introduce the functions $y_\varepsilon^+(x) = y_\varepsilon(x)$ if $x \in \Omega^+$ and $\widetilde{y}_\varepsilon^-(x) = \widetilde{y}_\varepsilon(x)$ if $x \in \Omega^-$. Thanks to the rectilinear boundaries of S_ε with respect to x_n, we have

$$\partial_{x_n}(\widetilde{y}_\varepsilon^-) = \widetilde{\partial_{x_n}(y_\varepsilon^-)} \quad \text{on} \quad \Omega^-. \tag{14.26}$$

This means that $\widetilde{y}_\varepsilon^- \in L^2(0,T;W_2^{(\mathbf{0},1)}(\Omega^-))$, where $W_2^{(\mathbf{0},1)}(\Omega^-)$ is the anisotropic Sobolev space $\{v \in L^2(\Omega^-) : \partial_{x_n} v \in L^2(\Omega^-)\}$.

Let χ_C be the $\square$-periodic characteristic function of the set C. It is easy to see that $\chi_C(\cdot/\varepsilon) \overset{*}{\rightharpoonup} |C|$ in $L^\infty(B)$ as $\varepsilon \to 0$, where $B = (0,a)^{n-1}$ and $|C|$ is the $(n-1)$-dimensional Lebesgue measure of C. We recall also that (see Brizzi and Chalot [35])

$$\chi_{\Omega_\varepsilon^-} \overset{*}{\rightharpoonup} |C| \quad \text{in} \quad L^\infty(\Omega^-) \quad \text{as} \quad \varepsilon \to 0, \tag{14.27}$$

$$\chi_{\Omega_\varepsilon \cap \Sigma} \overset{*}{\rightharpoonup} |C| \quad \text{in} \quad L^\infty(\Sigma) \quad \text{as} \quad \varepsilon \to 0, \tag{14.28}$$

$$\chi_{\Gamma_\varepsilon} \overset{*}{\rightharpoonup} |C| \quad \text{in} \quad L^\infty(\Gamma_0) \quad \text{as} \quad \varepsilon \to 0, \tag{14.29}$$

Definition 14.6. *We say that a sequence* $\{y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}\}_{\varepsilon>0}$ *is weakly convergent to a function* $y_* = (y_*^+, y_*^-)$ *(with respect to the space* $L^2(0,T;H^1(\Omega^+) \times W_2^{(\mathbf{0},1)}(\Omega^-))$*) as* ε *tends to* 0 *(in symbols,* $y_\varepsilon \rightsquigarrow y_* = (y_*^+, y_*^-)$*) if the following hold:*

(*a*) $y_\varepsilon^+ \rightharpoonup y_*^+$ *in* $L^2(0,T;H^1(\Omega^+))$;
(*b*) $\widetilde{y}_\varepsilon^- \rightharpoonup |C|\, y_*^-$ *in* $L^2(0,T;W_2^{(\mathbf{0},1)}(\Omega^-))$.

We prove the following compactness property.

Proposition 14.7. *Let* $\{y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}\}_{\varepsilon>0}$ *be a bounded sequence. Then there exist a subsequence* $\{y_{\varepsilon'}\}_{\varepsilon'>0}$ *and a function*

$$y_0 = (y_0^+, y_0^-) \in L^2(0,T;H^1(\Omega^+)) \times L^2(0,T;W_2^{(\mathbf{0},1)}(\Omega^-))$$

such that $y_{\varepsilon'} \rightsquigarrow y_0 = (y_0^+, y_0^-)$.

Proof. In accordance with the initial assumptions, there exists a constant $\mathbf{C} > 0$ independent of ε such that $\|y_\varepsilon\|_{\mathbf{X}_{\mu_\varepsilon}} \le \mathbf{C}$. Hence,

$$\|y_\varepsilon^+\|_{L^2(0,T;H^1(\Omega^+))} + \|\widetilde{y}_\varepsilon^-\|_{L^2(0,T;W_2^{(\mathbf{0},1)}(\Omega^-))} + \|y_\varepsilon\|_{L^2(0,T;L^2(\Omega^-,\mathrm{d}\mu_\varepsilon))} \le \mathbf{C}.$$

Therefore, there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ (still denoted by ε) and elements

$$y_0^+ \in L^2(0,T;H^1(\Omega^+)), \quad y_0^- \in L^2(0,T;L^2(\Omega^-)),$$

and $y^* \in L^2(0,T;L^2(\Omega^-))$ such that

$$\left.\begin{aligned} y_\varepsilon^+ &\rightharpoonup y_0^+ && \text{in } L^2(0,T;H^1(\Omega^+)),\\ \widetilde{y}_\varepsilon^- &\rightharpoonup v =: |C| y_0^- && \text{in } L^2(0,T;L^2(\Omega^-)),\\ y_\varepsilon &\rightharpoonup y^* && \text{in } L^2(0,T;L^2(\Omega^-,\mathrm{d}\mu_\varepsilon)),\\ \partial_{x_n}\widetilde{y}_\varepsilon^- &\rightharpoonup |C|\,\partial_{x_n} y_0^- && \text{in } L^2(0,T;L^2(\Omega^-)). \end{aligned}\right\} \tag{14.30}$$

Remark 14.8. Here, we have used the fact that the bounded sequence $\{y_\varepsilon\}_{\varepsilon>0}$ is relatively compact with respect to weak convergence in

$$\left\{L^2(0,T;L^2(\Omega^-,\mathrm{d}\mu_\varepsilon))\right\}.$$

Indeed, since $\limsup_{\varepsilon\to 0} \displaystyle\int_0^T \int_{\Omega^-} (y_\varepsilon)^2 \,\mathrm{d}\mu_\varepsilon \,\mathrm{d}t < +\infty$, there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ (still denoted by ε) and an element $y^* \in L^2(0,T;L^2(\Omega^-))$ such that (see Proposition 6.15)

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\Omega^-} \varphi\,\psi\, y_\varepsilon \,\mathrm{d}\mu_\varepsilon \,\mathrm{d}t = \int_0^T \int_{\Omega^-} \varphi\,\psi\, y^* \,\mathrm{d}x\,\mathrm{d}t$$

for any functions $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R})$.

Note also that the last limit in (14.30) is a consequence of (14.26). Moreover, by analogy with Brizzi and Chalot [35], one can easily prove the relation

$$y_0^+ = y_0^- \quad \text{a.e. on } (0,T)\times\Sigma, \tag{14.31}$$

that is, in this case the traces of the limit functions (y_0^+, y_0^-) coincide on Σ.

Obviously, to establish $y_\varepsilon \rightsquigarrow (y_0^+, y_0^-)$, it suffices to prove that

$$y^* = y_0^- \quad \text{a.e. on } (0,T)\times\Omega^-. \tag{14.32}$$

For this, we introduce some periodic finite Borel measure ν on $\mathbb{R}^n$. Let $\square_n = [0,1)^n$ be the cell of periodicity for ν. Assume that ν is the probability measure, concentrated and uniformly distributed on the set $C\times[0,1)$, so $\int_{\square_n} \mathrm{d}\nu = 1$. It is easy to see that for any smooth function, the equality

$$\int_{\square_n} f\,\mathrm{d}\nu \;=\; [\mathcal{L}_n(C\times[0,1))]^{-1}\int_{C\times[0,1)} f\,\mathrm{d}x = |C|^{-1}\int_{C\times[0,1)} f\,\mathrm{d}x \quad (14.33)$$

is valid. Now, define a scaling measure ν_ε by the relation $\nu_\varepsilon(A)=\varepsilon^n\nu(\varepsilon^{-1}A)$, where A is an arbitrary Borel set in $\mathbb{R}^n$ and $\varepsilon^{-1}A=\{\varepsilon^{-1}x, x\in A\}$. Then the measure ν_ε is ε-periodic and $\nu_\varepsilon(\varepsilon\square)=\varepsilon^n\int_{\square_n}\mathrm{d}\nu=\varepsilon^n$. Therefore, this measure weakly converges to the Lebesgue measure as $\varepsilon\to 0$, that is,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\varphi\,\mathrm{d}\nu_\varepsilon=\int_{\mathbb{R}^n}\varphi\,\mathrm{d}x\quad\text{for all}\quad\varphi\in C_0^\infty(\mathbb{R}^n).$$

It means that the weak limits of both sequences

$$\{y_\varepsilon^-\in L^2(0,T;L^2(\Omega^-,\mathrm{d}\mu_\varepsilon))\}\quad\text{and}\quad\{y_\varepsilon^-\in L^2(0,T;L^2(\Omega^-,\mathrm{d}\nu_\varepsilon))\}$$

in the sense of Remark 14.8 have to be the same, namely

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^-}\varphi\,\psi\,y_\varepsilon^-\,\mathrm{d}\mu_\varepsilon\,\mathrm{d}t&=\int_0^T\int_{\Omega^-}\varphi\,\psi\,y^*\,\mathrm{d}x\,\mathrm{d}t\\&=\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^-}\varphi\,\psi\,y_\varepsilon^-\,\mathrm{d}\nu_\varepsilon\,\mathrm{d}t.\end{aligned}\quad(14.34)$$

At the same time, for every function $y_\varepsilon\in\mathbf{X}_{\mu_\varepsilon}$ and every fixed ε, the set Ω^- can be covered by a system of cubes with edges ε. We denote these cubes by the symbols $\varepsilon(\square+\mathbf{k}_j)$. Then in accordance with the definition of the measure ν_ε, we may write

$$\begin{aligned}\int_0^T\int_{\Omega^-}\widetilde{y}_\varepsilon^-\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t&=|C|\int_0^T\sum_j\int_{\varepsilon(\square+\mathbf{k}_j)}y_\varepsilon^-\varphi\,\psi\,\varepsilon^n\,\mathrm{d}\nu(x/\varepsilon)\,\mathrm{d}t\\&=|C|\int_0^T\int_{\Omega^-}y_\varepsilon^-\varphi\,\psi\,\mathrm{d}\nu_\varepsilon\,\mathrm{d}t,\end{aligned}\quad(14.35)$$

where $\varphi\in C_0^\infty(\mathbb{R}^n)$ and $\psi\in C_0^\infty(\mathbb{R})$. Now, using (14.35) and (14.34) and taking into account the fact that $\widetilde{y}_\varepsilon^-\rightharpoonup|C|y_0^-$ in $L^2(0,T;L^2(\Omega^-))$ as $\varepsilon\to 0$, we obtain

$$\begin{aligned}\int_0^T\int_{\Omega^-}|C|y_0^-\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t&=\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^-}\widetilde{y}_\varepsilon^-\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t\\&=\lim_{\varepsilon\to 0}|C|\int_0^T\int_{\Omega^-}y_\varepsilon^-\varphi\,\psi\,\mathrm{d}\nu_\varepsilon\,\mathrm{d}t\\&=|C|\int_0^T\int_{\Omega^-}y^*\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $\varphi\in C_0^\infty(\mathbb{R}^n)$ and $\psi\in C_0^\infty(\mathbb{R})$. Hence, $y^*=y_0^-$, and we obtain the required result. □

Definition 14.9. *We say that a sequence* $\{(u_\varepsilon, y_\varepsilon) \in \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ *is* w^a*-convergent to a triplet* (u, y^+, y^-) *as* ε *tends to* 0

$$\left(\text{in symbols, } (u_\varepsilon, y_\varepsilon) \xrightarrow{w^a} (u, y^+, y^-)\right)$$

if $u_\varepsilon \to u$ *in the sense of Definition 14.4 and* $y_\varepsilon \rightsquigarrow (y^+, y^-)$ *in the sense of Definition 14.6 (here, the space* $\mathbb{Y}_\varepsilon$ *is defined in (14.24)).*

As follows from the result obtained above and estimate (14.18), the following statement holds.

Proposition 14.10. *Let* $\{(u_\varepsilon, y_\varepsilon) \in \Xi^a_\varepsilon\}_{\varepsilon>0}$ *be any sequence of admissible pairs for the* $\widehat{\mathbb{P}}^a_\varepsilon$ *problem. Then there exist a subsequence* $\{(u_{\varepsilon'}, y_{\varepsilon'})\}_{\varepsilon'>0}$ *and a triplet*

$$(u, y^+, y^-) \in \mathbb{Y}^a_0 := L^2(0,T; H^1(\Gamma_0)) \times L^2(0,T; H^1(\Omega^+)) \times L^2(0,T; W_2^{(\mathbf{0},1)}(\Omega^-)) \tag{14.36}$$

such that $u \in U_a$ *and* $(u_{\varepsilon'}, y_{\varepsilon'}) \xrightarrow{w^a} (u, y^+, y^-)$*, where*

$$U_a = \left\{u \in L^2(0,T; H^1(\Gamma_0)) \mid \|u\|_{L^2(0,T;H^1(\Gamma_0))} \le \mathbf{C_0}\right\}. \tag{14.37}$$

14.2.2 The convergence concept for the $\mathbb{P}^b_\varepsilon$-problems

Let $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ be any sequence of admissible pairs for the $\mathbb{P}^b_\varepsilon$ problems. Since we cannot assert in this case the existence of extension operators $P_\varepsilon : U^b_\varepsilon \to L^2(0,T; H^1(\Gamma_0))$ that would be uniformly bounded with respect to ε, it follows that we have to give another convergence concept in the variable space (14.24). Let us denote by $\widetilde{u}_\varepsilon$ the extension by 0 of a function $u_\varepsilon \in L^2(0,T; H^1(\Gamma_\varepsilon))$ onto Γ_0. Then $\widetilde{u}_\varepsilon \in L^2(0,T; L^2(\Gamma_0))$.

Definition 14.11. *A sequence*

$$\left\{(u_\varepsilon, y_\varepsilon) \in L^2(0,T; H^1(\Gamma_\varepsilon)) \times \mathbf{X}_{\mu_\varepsilon}\right\}_{\varepsilon>0}$$

is called w^b*-convergent to a triplet* (u, y^+, y^-)

$$\left(\text{in symbols, } (u_\varepsilon, y_\varepsilon) \xrightarrow{w^b} (u, y^+, y^-)\right)$$

as ε *tends to* 0 *if* $\widetilde{u}_\varepsilon \rightharpoonup u$ *in* $L^2(0,T; L^2(\Gamma_0))$ *and* $y_\varepsilon \rightsquigarrow (y^+, y^-)$ *in the sense of Definition 14.9.*

Then, taking the definition of the sets U^b_ε, estimate (14.18), and Proposition 14.7 into account, we have the following obvious result.

Proposition 14.12. *Let* $\{(u_\varepsilon, y_\varepsilon) \in \Xi^b_\varepsilon\}_{\varepsilon>0}$ *be a sequence of admissible pairs for the* $\mathbb{P}^b_\varepsilon$ *problems such that* $\sup_{\varepsilon>0} \|y_\varepsilon\|_{\mathbf{X}_{\mu_\varepsilon}} < +\infty$. *Then there exist a subsequence* $\{(u_{\varepsilon'}, y_{\varepsilon'})\}_{\varepsilon'>0}$ *and a triplet*

$$(u, y^+, y^-) \in \mathbb{Y}^b_0 := L^2(0,T; L^2(\Gamma_0)) \times L^2(0,T; H^1(\Omega^+)) \times L^2(0,T; W_2^{(\mathbf{0},1)}(\Omega^-)) \quad (14.38)$$

for which $(u_{\varepsilon'}, y_{\varepsilon'}) \xrightarrow{w^b} (u, y^+, y^-)$ *as* $\varepsilon' \to 0$.

Let us denote by τ^a the topology associated with w^a-convergence in $\mathbb{Y}^a_0$ and by τ^b the topology associated with w^b-convergence in $\mathbb{Y}^b_0$. Then, as follows from Propositions 14.10 and 14.12, these topologies can be taken as the most natural ones for limit analysis of the OCPs $\mathbb{P}^a_\varepsilon$ and $\mathbb{P}^b_\varepsilon$, respectively.

14.3 Definition of the limit problems and their properties

As follows from the previous sections, each of the sets of admissible solutions Ξ^i_ε $(i = a, b)$ belongs to the corresponding Banach space $(\Xi^i_\varepsilon \subset \mathbb{Y}_\varepsilon)$. We recall the convergence concept of such sets using the w^i-sequential version of the set convergence in Kuratowski's sense (see Sect. 7.4). Hereinafter $i = a, b$.

Definition 14.13. *We say that a set* $\Xi_i \subset \mathbb{Y}^i_0$ *is the sequential* w^i*-limit in the Kuratowski's sense (or* $K(w^i)$*-limit) of the sequence* $\{\Xi^i_\varepsilon \subset \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ *if the following conditions are satisfied:*

(i) For every triplet $(u, y^+, y^-) \in \Xi_i$, *there exists a sequence* $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ w^i*-converging to* (u, y^+, y^-) *and a positive value* $\varepsilon_0 > 0$ *such that* $(u_\varepsilon, y_\varepsilon) \in \Xi^i_\varepsilon$ *for every* $\varepsilon \in (0, \varepsilon_0)$.

(ii) For every sequence of admissible pairs $\{(u_k, y_k) \in \Xi^i_{\varepsilon_k}\}_{k\in\mathbb{N}}$ *such that* $\varepsilon_k \longrightarrow 0$ *and* $(u_k, y_k) \xrightarrow{w^i} (u, y^+, y^-)$ *as* $k \to \infty$, *the triplet* (u, y^+, y^-) *belongs to* Ξ_i.

Let us show that the sets of admissible pairs for the $\mathbb{P}^a_\varepsilon$ problems possess the compactness property with respect to the $K(w^a)$-convergence.

Theorem 14.14. *For the sequence of sets* $\{\Xi^a_\varepsilon\}_{\varepsilon>0}$, *there exist a subsequence* $\{\Xi^a_{\varepsilon'}\}_{\varepsilon'>0}$ *and a set* $\Xi_a \subset \mathbb{Y}^a_0$ *such that* $K(w^a)\text{-}\lim_{\varepsilon'\to 0} \Xi^a_{\varepsilon'} = \Xi_a$.

Proof. We begin with the obvious fact that the w^a-convergence of any sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon) \in \Xi^a_\varepsilon\}_{\varepsilon>0}$ is equivalent to the weak convergence of its image $\{(P_\varepsilon(u_\varepsilon),\ y^+_\varepsilon,\ \widetilde{y}^-_\varepsilon)\}_{\varepsilon>0}$ in the space

$$\mathbb{Y}^a_0 = L^2(0,T; H^1(\Gamma_0)) \times L^2(0,T; H^1(\Omega^+)) \times L^2(0,T; W_2^{(\mathbf{0},1)}(\Omega^-)).$$

Since the space $\mathbb{Y}_0^a$ is separable and reflexive, there exists a metric d such that for any sequence $\{p_k = (w_k, y_k, v_k)\}_{k\in\mathbb{N}}$ in $\mathbb{Y}_0^a$, the following conditions are equivalent (see, e.g., Dunford and Schwartz [99]):

(j) $\{p_k\} \longrightarrow p = (w, y, v)$ weakly in $\mathbb{Y}_0^a$.
(jj) $\{p_k\}$ is bounded in $\mathbb{Y}_0^a$ and $d(p_k, p) \to 0$ as $k \to \infty$.

Let η be the topology associated with the metric d on $\mathbb{Y}_0^a$ and let $\{\widetilde{\Xi}_\varepsilon^a\}_{\varepsilon>0}$ be the images sequence of the sets Ξ_ε^a in $\mathbb{Y}_0^a$, that is,

$$\widetilde{\Xi}_\varepsilon^a = \left\{ \left(P_\varepsilon(u_\varepsilon),\, y_\varepsilon^+,\, \widetilde{y}_\varepsilon^-\right) \;:\; (u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^a \right\}.$$

Since the η topology has a countable base, then by the Kuratowski compactness theorem (see Dal Maso [78]), there exists a subsequence of $\left\{\widetilde{\Xi}_\varepsilon^a\right\}_{\varepsilon>0}$ still denoted by $\left\{\widetilde{\Xi}_\varepsilon^a\right\}_{\varepsilon>0}$ that $K(\eta)$-converges to a set $A \subset \mathbb{Y}_0^a$. Now, we prove that the set A coincides with the $K(w^a)$-limit of the family $\{\Xi_\varepsilon^a\}_{\varepsilon>0}$. With this aim, it is enough to show that

$$\Xi_a \subseteq A, \tag{14.39}$$

$$A \subseteq \Xi_a, \tag{14.40}$$

where by Ξ_a we denoted the $K(w^a)$-limit of the sequence $\{\Xi_\varepsilon^a\}_{\varepsilon>0}$ in the sense of Definition 14.13.

First, let us verify inclusion (14.39). Let (u, y^+, y^-) be any triplet in $\mathbb{Y}_0^a$ for which one can find a sequence $\{(u_k, y_k)\}_{k\in\mathbb{N}}$, w^a-converging to (u, y^+, y^-) and a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $(u_k, y_k) \in \Xi_{\varepsilon_k}^a$ for every $k \in \mathbb{N}$. Then $(u, y^+, y^-) \in \Xi_0^a$ by Definition 14.13. Let $\{P_k\}$ be any sequence of the extension operators $P_k : L^2(0, T; H^1(\Gamma_\varepsilon)) \to L^2(0, T; H^1(\Gamma_0))$. Then

$$\left\{(P_k u_k, y_k^+, \widetilde{y}_k^-)\right\} \rightharpoonup (u, y^+, y^-) \quad \text{in} \quad \mathbb{Y}_0^a,$$
$$(P_k u_k, y_k^+, \widetilde{y}_k^-) \in \widetilde{\Xi}_{\varepsilon_k}^a \quad \text{for every} \quad k \in \mathbb{N}.$$

Therefore, the equivalence between conditions (j) and (jj) yields η-convergence of $\left\{(P_k u_k, y_k^+, \widetilde{y}_k^-)\right\}$ to (u, y^+, y^-). Hence, $(u, y^+, y^-) \in A$ by definition of Kuratowski's limit. So, inclusion (14.39) is proved.

Now we verify (14.40). Let (u, y^+, y^-) be any triplet of A. Then there exists a sequence $\{(v_\varepsilon, p_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ η-converging to (u, y^+, y^-) such that $(v_\varepsilon, p_\varepsilon, q_\varepsilon) \in \widetilde{\Xi}_\varepsilon$ for ε small enough. It follows that each pair $(p_\varepsilon, q_\varepsilon)$ can be represented as $p_\varepsilon = y_\varepsilon^+$ and $q_\varepsilon = \widetilde{y}_\varepsilon^-$, where y_ε is a weak solution of the boundary value problem (14.2) with $u_\varepsilon = v_\varepsilon|_{\Gamma_\varepsilon}$. However, the realization of the condition $(v_\varepsilon, p_\varepsilon, q_\varepsilon) \in \widetilde{\Xi}_\varepsilon$ implies that the pair $(u_\varepsilon, y_\varepsilon)$ is admissible (i.e., $(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^a$). Since the sequence of functions $\{v_\varepsilon\}$ is bounded in $L^2(0, T; H^1(\Gamma_0))$ ($v_\varepsilon \in U_a$ by definition of the sets $\widetilde{\Xi}_\varepsilon$), we get that the sequence of corresponding solutions $\{(p_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^2(0, T; H^1(\Omega^+)) \times L^2(0, T; W_2^{(\mathbf{0},1)}(\Omega^-))$ as well. Hence, the equivalence between conditions (j) and (jj) yields the

weak convergence of the sequence $\{(v_\varepsilon, p_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ to (u, y^+, y^-). However, in view of Definition 14.9, it is equivalent to the w^a-convergence of its prototype $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ to (u, y^+, y^-). Thus, $(u, y^+, y^-) \in \Xi_a$ by Definition 14.13. The theorem is proved. □

Definition 14.15. *We say that $\mathbb{P}^b_\varepsilon$ satisfies the property $(\mathcal{N})$ if for any*

$$u \in U_b = \left\{ v \in L^2((0,T) \times \Gamma_0) \ : \ \|v\|_{L^2((0,T)\times\Gamma_0)} \le \sqrt{|C|}\mathbf{C_0} \right\},$$

there exists a sequence $\left\{u_\varepsilon \in U^b_\varepsilon\right\}_{\varepsilon>0}$ such that

$$\widetilde{u}_\varepsilon \rightharpoonup u \quad in \quad L^2((0,T) \times \Gamma_0) \quad and \quad \sup_{\varepsilon>0} \|u_\varepsilon\|_{L^2(0,T;H^1(\Gamma_\varepsilon))} < +\infty.$$

Then the following compactness property for the sets Ξ^b_ε with respect to the $K(w^b)$-convergence takes place.

Theorem 14.16. *If $\mathbb{P}^b_\varepsilon$ possesses the $(\mathcal{N})$-property, then the sequence of the sets $\left\{\Xi^b_\varepsilon \in \mathbb{Y}_\varepsilon\right\}_{\varepsilon>0}$ has a subsequence (still denoted by ε) for which there exists a nonempty set $\Xi_b \subset \mathbb{Y}^b_0$ that is the $K(w^b)$-limit of $\left\{\Xi^b_\varepsilon\right\}_{\varepsilon>0}$ with respect to the space $\mathbb{Y}^b_0$.*

Remark 14.17. It is well known (see Dal Maso [78]) that the Kuratowski limit A_0 of a sequence of subsets $\{A_n\}_{n\in\mathbb{N}}$ in a topological space (X, τ) does not change if we replace the sets A_n by their τ-closures, that is,

$$K(\tau)\text{-}\lim_{n\to\infty} A_n = A_0 = K(\tau)\text{-}\lim_{n\to\infty} \mathrm{cl}_\tau A_n$$

(τ^b_ε denotes the product of the weak topologies of the spaces $L^2(0,T;L^2(\Gamma_\varepsilon))$ and $L^2(0,T;H^1(\Omega_\varepsilon))$). Thus, the $K(w^b)$-limit of $\{\Xi^b_\varepsilon\}_{\varepsilon>0}$ coincides with the $K(w^b)$-limit of τ^b_ε-closures $\left\{\mathrm{cl}_{\tau^b_\varepsilon}\Xi^b_\varepsilon \subset L^2(0,T;L^2(\Gamma_\varepsilon)) \times \mathbf{X}_{\mu_\varepsilon}\right\}$.

Let us turn back to the main object of this section, namely to the sequences of constrained minimization problems (14.23). Using the concept of variational convergence, we give the definition of the "appropriate limits" for these sequences.

Definition 14.18. *We say that the minimization problem*

$$\left\langle \inf_{(u,y^+,y^-)\in\,\Xi_i} I_i(u, y^+, y^-) \right\rangle \quad (i = a, b), \tag{14.41}$$

where $\Xi_i \subset \mathbb{Y}^i_0$, is the variational w^i-limit of the sequence (14.23) with respect to the w^i-convergence if the following hold:

(i) Ξ_i is the $K(w^i)$-limit of the sets $\left\{\Xi^i_\varepsilon\right\}$.

(ii) *For any triplet* $(u, y^+, y^-) \in \Xi_i$ *and for any sequence* $\left\{(u_k, y_k) \in \Xi^i_{\varepsilon_k}\right\}$ *such that* $\varepsilon_k \longrightarrow 0$ *and* $(u_k, y_k) \overset{w^i}{\Longrightarrow} (u, y^+, y^-)$ *as* $k \to \infty$, *we have*

$$I_i(u, y^+, y^-) \le \liminf_{k\to\infty} I_{\varepsilon_k}(u_k, y_k). \tag{14.42}$$

(iii) *For every triplet* $(u, y^+, y^-) \in \Xi_i$, *there exist a positive constant* ε_0 *and a sequence* $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ *such that*

$$\begin{aligned} &(u_\varepsilon, y_\varepsilon) \in \Xi^i_\varepsilon \quad \text{for every } \varepsilon \le \varepsilon_0, \quad (u_\varepsilon, y_\varepsilon) \overset{w^i}{\Longrightarrow} (u, y^+, y^-), \\ &I_i(u, y^+, y^-) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon). \end{aligned} \tag{14.43}$$

Using the same arguments as in the proof of Theorem 8.25 and taking into account the $(\mathcal{N})$-property, one can establish the following results for the variational w^a- and w^b-limits.

Theorem 14.19. *Assume that the constrained minimization problem*

$$\left\langle \inf_{(u,y^+,y^-)\in\,\Xi_a} I_a(u, y^+, y^-) \right\rangle \tag{14.44}$$

is the variational w^a*-limit of the corresponding sequence (14.23) and that this problem has a unique solution* $(u^a, (y^a)^+, (y^a)^-)$ *in* Ξ_a. *Let* $\{(u^a_\varepsilon, y^a_\varepsilon) \in \Xi^a_\varepsilon\}_{\varepsilon>0}$ *be a sequence of the optimal pairs for the* $\mathbb{P}^a_\varepsilon$ *problem. Then*

$$(u^a_\varepsilon, y^a_\varepsilon) \overset{w^a}{\longrightarrow} (u^a, (y^a)^+, (y^a)^-) \quad \text{as} \quad \varepsilon \to 0 \tag{14.45}$$

and, furthermore,

$$\inf_{(u,\,y^+,\,y^-)\in\,\Xi_a} I_a(u, y^+, y^-) = I_a\left(u^a, (y^a)^+, (y^a)^-\right) = \lim_{\varepsilon\to 0} I_\varepsilon(u^a_\varepsilon, y^a_\varepsilon). \tag{14.46}$$

Theorem 14.20. *Assume that the constrained minimization problem*

$$\left\langle \inf_{(u,y^+,y^-)\in\,\Xi_b} I_b(u, y^+, y^-) \right\rangle \tag{14.47}$$

is the variational w^b*-limit of the corresponding sequence (14.23) and that this problem has a unique solution* $(u^b, (y^b)^+, (y^b)^-) \in \Xi_b \subset \mathbb{Y}^b_0$.

Let $\left\{(u^b_\varepsilon, y^b_\varepsilon) \in \Xi^b_\varepsilon\right\}_{\varepsilon>0}$ *be a sequence of optimal pairs of* $\mathbb{P}^b_\varepsilon$ *problems such that* $\sup_{\varepsilon>0} \|y^b_\varepsilon\|_{\mathbf{X}_{\mu_\varepsilon}} < +\infty$. *Then* $(u^b_\varepsilon, y^b_\varepsilon) \overset{w^b}{\longrightarrow} (u^b, (y^b)^+, (y^b)^-)$ *as* $\varepsilon \to 0$ *and, furthermore,*

$$\inf_{(u,\,y^+,\,y^-)\in\,\Xi_b} I_b(u, y^+, y^-) = I_b\left(u^b, (y^b)^+, (y^b)^-\right) = \lim_{\varepsilon\to 0} I_\varepsilon(u^b_\varepsilon, y^b_\varepsilon).$$

Definition 14.21. *We say that there exists a limit problem for the family of optimal control problems* $\{\mathbb{P}^i_\varepsilon\}_{\varepsilon>0}$ $(i = a, b)$ *with respect to the* w^i*-convergence, if for the corresponding sequence of constrained minimization problems (14.23), there exists a variational limit which can be represented in the form of some optimal control problem.*

14.4 Analytical representation of the limit sets of admissible solutions

The main objects of our consideration in this section are the sequences of the sets of admissible pairs $\{\Xi_e^a \subset \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ and $\{\Xi_e^b \subset \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ and their Kuratowski limits with respect to w^a- and w^b-convergence, respectively. In view of Theorems 14.14 and 14.16, we may always suppose that for those sequences, there exist sets Ξ_a and Ξ_b such that

$$\Xi_a = K(w^a)\text{-}\lim_{\varepsilon\to 0} \Xi_e^a, \quad \Xi_b = K(w^b)\text{-}\lim_{\varepsilon\to 0} \Xi_e^b.$$

To formulate our next results, we introduce the space

$$\mathcal{V}(\Omega) = \left\{ y \in L^2(\Omega) \ : \ \frac{\partial y}{\partial x_n} \in L^2(\Omega^-), y \in H^1(\Omega^+) \right\}$$

and endow it with the scalar product

$$(y,v)_{\mathcal{V}(\Omega)} = \int_{\Omega^+} \nabla y \cdot \nabla v \,\mathrm{d}x + \int_{\Omega^+} y\, v \,\mathrm{d}x + |C| \int_{\Omega^-} \partial_{x_n} y \partial_{x_n} v \,\mathrm{d}x + (|C| + k_0 |\partial C|_H) \int_{\Omega^-} y\, v \,\mathrm{d}x.$$

By analogy with Corbo Esposito et al. [74], it can easily be shown that $\mathcal{V}(\Omega)$ is a Hilbert space and $H^1(\Omega)$ is dense in $\mathcal{V}(\Omega)$. Moreover, as follows from (14.30) and (14.31), for any function $y_* = (y_*^+, y_*^-)$, which is a weak limit in the sense of Definition 14.6, we have the following:

(i) $y_* = (y_*^+, y_*^-) \in L^2(0,T;\mathcal{V}(\Omega))$;
(ii) $y_*^+(\cdot,\cdot) = y_*^-(\cdot,\cdot)$ almost everywhere on $(0,T)\times\Sigma$.

Moreover, it should be stressed here that any function of $\mathcal{V}(\Omega)$ has a trace on any hyperplane L in Ω^- such that $L = \{(x', x_n) \in \Omega^- \ : \ x_n = \text{const}\}$.

14.4.1 Recovery of the set Ξ_a

The crucial point in the study of the $K(w^a)$-limit properties for the sequence of admissible pairs is the following result.

Lemma 14.22. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be any sequence of admissible controls for the $\mathbb{P}_\varepsilon^a$ problems which is weakly convergent to a function u_0 in $L^2(0,T;H^1(\Gamma_0))$. Let $\{y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}\}$ be the corresponding solutions of the problem (14.2). Then*

$$(u_\varepsilon, y_\varepsilon) \xrightarrow{w^a} (u_0, v_0^+, v_0^-) \quad as \quad \varepsilon \to 0,$$

where

$$v_0(x) = \begin{cases} v_0^+(x) & \text{if } x \in \Omega^+, \\ v_0^-(x) & \text{if } x \in \Omega^- \end{cases} \tag{14.48}$$

is the unique weak solution in $L^2(0,T;\mathcal{V}(\Omega))$ *of the limit problem*

$$\left.\begin{aligned} (v_0^+)' - \Delta_x v_0^+ + v_0^+ &= f_0 \quad \text{in} \quad (0,T)\times\Omega^+, \\ (v_0^-)' - \partial_{x_n}^2 v_0^- + \frac{|C| + k_0|\partial C|_H}{|C|} v_0^- &= f_0 \quad \text{in} \quad (0,T)\times\Omega^-, \\ \partial_\nu v_0^+ &= 0 \quad \text{in} \quad (0,T)\times\partial\Omega^+ \setminus \Sigma, \\ v_0^- &= u_0 \quad \text{on} \quad (0,T)\times\Gamma_0, \\ v_0^+ = v_0^-, \quad \partial_{x_n} v_0^+ &= |C|\,\partial_{x_n} v_0^- \quad \text{on} \quad (0,T)\times\Sigma, \\ v_0(0,x) &= y^0(x) \quad \text{a.e.} \quad x \in \Omega. \end{aligned}\right\} \tag{14.49}$$

Remark 14.23. Here, the weak formulation of the problem (14.49) means that

$$\left.\begin{aligned} &v_0 \in L^2(0,T;\mathcal{V}(\Omega)), \\ &-\int_0^T \int_\Omega (\chi_{\Omega^+} + |C|\chi_{\Omega^-})\, v_0\, \varphi\, \psi'\, \mathrm{d}x\, \mathrm{d}t + \int_0^T (v_0,\varphi)_{\mathcal{V}(\Omega)}\, \psi\, \mathrm{d}t \\ &\qquad = \int_0^T \int_\Omega (\chi_{\Omega^+} + |C|\chi_{\Omega^-})\, f_0\, \varphi\, \psi\, \mathrm{d}x\, \mathrm{d}t, \\ &\qquad \forall\, \varphi \in \mathcal{V}(\Omega;\Gamma_0), \quad \forall\, \psi \in C_0^\infty(0,T), \\ &v_0^- = u_0, \ \text{on} \ \ (0,T)\times\Gamma_0, \qquad v_0(0,x) = y^0(x) \quad \text{a.e.} \ \ x \in \Omega, \end{aligned}\right\} \tag{14.50}$$

where

$$\mathcal{V}(\Omega;\Gamma_0) = \big\{ v \in L^2(\Omega) \ : \ \partial_{x_n} v \in L^2(\Omega^-),\ v \in H^1(\Omega^+),\ v = 0 \ \text{ a.e. on } \ \Gamma_0 \big\}.$$

Moreover, in this case, we have $v_0' \in L^2(0,T;(\mathcal{V}(\Omega))')$ (see Lions [169, p. 107]).

Proof. From Proposition 14.7, it follows that there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and a triplet $(u_0, v_0^+, v_0^-) \in \mathbb{Y}_0^a$ such that

$$(u_\varepsilon, y_\varepsilon) \xrightarrow{w^a} (u_0, v_0^+, v_0^-) \ \text{ as } \ \varepsilon \to 0.$$

Similar to the proof of Proposition 14.7 (see relations (14.30) and (14.31)), we can show that

$$\left.\begin{aligned} y_\varepsilon^+ &\rightharpoonup v_0^+ && \text{in } L^2(0,T;H^1(\Omega^+)), \\ \widetilde{y}_\varepsilon^- &\rightharpoonup |C|\, v_0^- && \text{in } L^2(0,T;L^2(\Omega^-)), \\ v_0^+ = v_0^-,\ \partial_{x_n} v_0^+ &= |C|\partial_{x_n} v_0^- && \text{a.e. on } \ (0,T)\times\Sigma, \\ y_\varepsilon^- &\rightharpoonup v_0^- && \text{in } \{L^2(0,T;L^2(\Omega^-,\mathrm{d}\mu_\varepsilon))\}. \end{aligned}\right\} \tag{14.51}$$

Moreover, there exist functions $\gamma_i \in L^2(0,T;L^2(\Omega^-))$ $(i=1,\dots,n-1)$ such that

$$\widetilde{\nabla_x y_\varepsilon^-} \rightharpoonup \left(\gamma_1,\dots,\gamma_{n-1},|C|\partial v_0^-/\partial x_n\right) \quad \text{in} \quad \left[L^2(0,T;L^2(\Omega^-))\right]^n. \tag{14.52}$$

Let us prove that the function v_0^- satisfies the boundary condition

$$v_0^- = u_0 \quad \text{almost everywhere on} \quad (0,T)\times\Gamma_0. \tag{14.53}$$

We note that any function $f \in \mathcal{V}(\Omega)$ has a trace $f|_{\Gamma_0} \in L^2(\Gamma_0)$ (see Brizzi & Chalot [35]), so (14.53) makes sense. It is easy to see that the following statements hold:

$$\widetilde{y}_\varepsilon^- = \chi_{\Gamma_\varepsilon} P_\varepsilon(u_\varepsilon) \quad \text{a.e. on} \quad (0,T)\times\Gamma_0,\ \forall\,\varepsilon>0, \tag{14.54}$$

$$\chi_{\Gamma_\varepsilon} P_\varepsilon(u_\varepsilon) \rightharpoonup |C|u_0 \quad \text{in} \quad L^2((0,T)\times\Gamma_0) \tag{14.55}$$

(as a product of strongly and weakly convergent sequences). Then from the integral identity

$$\begin{aligned}\int_0^T\int_{\Gamma_0} \widetilde{y}_\varepsilon^-\,\varphi\,\psi\,\mathrm{d}x'\,\mathrm{d}t \\ = -\int_0^T\int_{\Omega^-} \partial\widetilde{y}_\varepsilon^-/\partial x_n\,\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\Omega^-} \widetilde{y}_\varepsilon^-\,\partial\psi/\partial x_n\,\varphi\,\mathrm{d}x\,\mathrm{d}t,\\ \forall\varphi\in C_0^\infty(\mathbb{R}^n;\partial\Omega^-\setminus\Gamma_0),\ \forall\,\psi\in C_0^\infty(\mathbb{R}),\end{aligned} \tag{14.56}$$

where

$$C_0^\infty(\mathbb{R}^n;\partial\Omega^-\setminus\Gamma_0) = \{\varphi\in C_0^\infty(\mathbb{R}^n)\ :\ \varphi=0\ \text{ on }\partial\Omega^-\setminus\Gamma_0\},$$

we immediately get $(\widetilde{y}_\varepsilon^-)|_{\Gamma_0} \longrightarrow |C|\ \left(v_0^-\right)\big|_{\Gamma_0}$ weakly in $L^2((0,T)\times\Gamma_0)$. Thus, passing to the limit in (14.54) as $\varepsilon\to 0$, we obtain the required relation (14.53).

Now, let us show that the function v_0 is the unique weak solution of problem (14.49). With this aim, we rewrite the integral identity (14.16) as

$$\begin{aligned}-\int_0^T\int_{\Omega^+} y_\varepsilon^+\varphi\,\psi'\,\mathrm{d}x\,\mathrm{d}t \\ -\int_0^T\int_{\Omega^-}\widetilde{y}_\varepsilon^-\varphi\,\psi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega^+}\nabla y_\varepsilon^+\cdot\nabla\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t \\ +\int_0^T\int_{\Omega^-}\widetilde{\nabla y_\varepsilon^-}\cdot\nabla\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega^+} y_\varepsilon^+\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t \\ +\int_0^T\int_{\Omega^-}\widetilde{y}_\varepsilon^-\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t + k_0|\partial C|_H\int_0^T\int_{\Omega^-} y_\varepsilon^-\,\varphi\,\psi\,\mathrm{d}\mu_\varepsilon\,\mathrm{d}t \\ = \int_0^T\int_{\Omega^+} f_\varepsilon\,\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega^-}\chi_{\Omega_\varepsilon^-}\,f_\varepsilon\,\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{14.57}$$

Passing to the limit in (14.57) as $\varepsilon \to 0$ and using the properties (14.27), and (14.51), and (14.52), we get

$$
\begin{aligned}
&-\int_0^T \int_{\Omega^+} v_0^+ \varphi\, \psi'\, \mathrm{d}x\, \mathrm{d}t - |C| \int_0^T \int_{\Omega^-} v_0^- \varphi\, \psi'\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega^+} \nabla v_0^+ \cdot \nabla \varphi\, \psi\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_{\Omega^-} \sum_{i=1}^{n-1} \gamma_i\, (\partial \varphi / \partial x_i)\, \psi\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + |C| \int_0^T \int_{\Omega^-} (\partial v_0^- / \partial x_n)\, (\partial \varphi / \partial x_n)\, \psi\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega^+} v_0^+ \varphi\, \psi\, \mathrm{d}x\, \mathrm{d}t + |C| \int_0^T \int_{\Omega^-} v_0^- \varphi\, \psi\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + k_0 |\partial C|_H \int_0^T \int_{\Omega^-} v_0^- \varphi\, \psi\, \mathrm{d}x\, \mathrm{d}t \\
&\quad = \int_0^T \int_{\Omega^+} f_0\, \varphi\, \psi\, \mathrm{d}x\, \mathrm{d}t + |C| \int_0^T \int_{\Omega^-} f_0\, \varphi\, \psi\, \mathrm{d}x\, \mathrm{d}t, \\
&\qquad\qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_0),\ \forall\, \psi \in C_0^\infty(0,T). \qquad (14.58)
\end{aligned}
$$

Let us fix $i \in \{1, \ldots, n-1\}$ and let w_ε^i be a sequence in $W^{1,\infty}(\Omega^-)$ satisfying the following conditions:

$$
w_\varepsilon^i \longrightarrow x_i \quad \text{strongly in } L^\infty(\Omega^-), \qquad (14.59)
$$

$$
D\, w_\varepsilon^i = 0 \quad \text{a.e. on } \Omega_\varepsilon^- \qquad (14.60)
$$

for every $\varepsilon > 0$. The existence of such a sequence is proved in Brizzi and Chalot [35] and Corbo Esposito et al. [74]. Let us prove that $\gamma_i = 0$ a.e. on $(0,T) \times \Omega^-$. Take the test functions $\varphi = w_\varepsilon^i\, \phi$ and $\varphi = x_i\, \phi$ with $\phi \in C_0^\infty(\Omega^-)$ in (14.57). Then, by virtue of (14.60), we have

$$
\begin{aligned}
&-\int_0^T \int_{\Omega^-} y_\varepsilon^-\, \phi\, w_\varepsilon^i\, \psi'\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_{\Omega^-} \widetilde{\nabla y_\varepsilon^-} \cdot \nabla \phi\, w_\varepsilon^i\, \psi\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega^-} \widetilde{y_\varepsilon^-}\, \phi\, w_\varepsilon^i\, \psi\, \mathrm{d}x\, \mathrm{d}t + k_0\, |\partial C|_H \int_0^T \int_{\Omega^-} y_\varepsilon^-\, \phi\, w_\varepsilon^i\, \psi\, \mathrm{d}\mu_\varepsilon\, \mathrm{d}t \\
&\quad = \int_0^T \int_{\Omega^-} \chi_{\Omega_\varepsilon^-}\, f_\varepsilon\, \phi\, w_\varepsilon^i\, \psi\, \mathrm{d}x\, \mathrm{d}t, \qquad (14.61)
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^T \int_{\Omega^-} y_\varepsilon^-\, \phi\, x_i\, \psi'\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_{\Omega^-} \widetilde{\nabla y_\varepsilon^-} \cdot \nabla(\phi\, x_i)\, \psi\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega^-} \widetilde{y_\varepsilon^-}\, \phi\, x_i\, \psi\, \mathrm{d}x\, \mathrm{d}t + k_0\, |\partial C|_H \int_0^T \int_{\Omega^-} y_\varepsilon^-\, \phi\, x_i\, \psi\, \mathrm{d}\mu_\varepsilon\, \mathrm{d}t \\
&\quad = \int_0^T \int_{\Omega^-} \chi_{\Omega_\varepsilon^-}\, f_\varepsilon\, \phi\, x_i\, \psi\, \mathrm{d}x\, \mathrm{d}t \qquad (14.62)
\end{aligned}
$$

for every $\varepsilon > 0$, $\phi \in C_0^\infty(\Omega^-)$, and $\psi \in C_0^\infty(0,T)$. Hence, passing to the limit in (14.61) and (14.62) as $\varepsilon \to 0$ and using the properties (14.27), (14.51), (14.52), and (14.59) and Proposition 14.7, we obtain

$$
\begin{aligned}
-|C| \int_0^T \int_{\Omega^-} v_0^- \phi x_i \, \psi' \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\Omega^-} \sum_{k=1}^{n-1} \gamma_k \, (\partial \phi / \partial x_k) \, x_i \, \psi \, \mathrm{d}x \, \mathrm{d}t \\
+ \, |C| \int_0^T \int_{\Omega^-} (\partial v_0^- / \partial x_n) \, (\partial \phi / \partial x_n) \, x_i \, \psi \, \mathrm{d}x \, \mathrm{d}t \\
+ \, |C| \int_0^T \int_{\Omega^-} v_0^- \phi \, x_i \, \psi \, \mathrm{d}x \, \mathrm{d}t + \, k_0 \, |\partial C|_H \int_0^T \int_{\Omega^-} v_0^- \, \phi \, x_i \, \psi \, \mathrm{d}x \, \mathrm{d}t \\
= \, |C| \int_0^T \int_{\Omega^-} f_0 \, \phi \, x_i \, \psi \, \mathrm{d}x \, \mathrm{d}t, \qquad (14.63)
\end{aligned}
$$

$$
\begin{aligned}
-|C| \int_0^T \int_{\Omega^-} v_0^- \phi x_i \, \psi' \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\Omega^-} \sum_{k=1}^{n-1} \gamma_k \, (\partial (\phi \, x_i) / \partial x_k) \, \psi \, \mathrm{d}x \, \mathrm{d}t \\
+ \, |C| \int_0^T \int_{\Omega^-} (\partial v_0^- / \partial x_n) \, (\partial \phi / \partial x_n) \, x_i \, \psi \, \mathrm{d}x \, \mathrm{d}t \\
+ \, |C| \int_0^T \int_{\Omega^-} v_0^- \phi \, x_i \, \psi \, \mathrm{d}x \, \mathrm{d}t + \, k_0 \, |\partial C|_H \int_0^T \int_{\Omega^-} v_0^- \, \phi \, x_i \, \psi \, \mathrm{d}x \, \mathrm{d}t \\
= \, |C| \int_0^T \int_{\Omega^-} f_0 \, \phi \, x_i \, \psi \, \mathrm{d}x \, \mathrm{d}t. \qquad (14.64)
\end{aligned}
$$

Comparing (14.63) with (14.64), we conclude that

$$
\int_0^T \int_{\Omega^-} \gamma_k \, \phi \, \psi \, \mathrm{d}x \, \mathrm{d}t \, = \, 0,
$$

$\forall \, k \in \{1, \ldots, n-1\}$, $\forall \phi \in C_0^\infty(\Omega^-)$, and $\forall \, \psi \in C_0^\infty(0,T)$. Thus, $\gamma_i = 0$ a.e. on $(0,T) \times \Omega^-$, and we obtain the required result. As for the function v_0, we have the identity

$$
\begin{aligned}
& - \int_0^T \int_{\Omega^+} v_0^+ \varphi \, \psi' \, \mathrm{d}x \, \mathrm{d}t - |C| \int_0^T \int_{\Omega^-} v_0^- \varphi \, \psi' \, \mathrm{d}x \, \mathrm{d}t \\
& + \int_0^T \int_{\Omega^+} \nabla v_0^+ \cdot \nabla \varphi \, \psi \, \mathrm{d}x \, \mathrm{d}t + \, |C| \int_0^T \int_{\Omega^-} (\partial v_0^- / \partial x_n) \, (\partial \varphi / \partial x_n) \, \psi \, \mathrm{d}x \, \mathrm{d}t \\
& + \int_0^T \int_{\Omega^+} v_0^+ \varphi \, \psi \, \mathrm{d}x \, \mathrm{d}t \, + \, |C| \int_0^T \int_{\Omega^-} v_0^- \varphi \, \psi \, \mathrm{d}x \, \mathrm{d}t \\
& + k_0 |\partial C|_H \int_0^T \int_{\Omega^-} v_0^- \varphi \, \psi \, \mathrm{d}x \, \mathrm{d}t \\
& = \int_0^T \int_{\Omega^+} f_0 \, \varphi \, \psi \, \mathrm{d}x \, \mathrm{d}t, + |C| \int_0^T \int_{\Omega^-} f_0 \, \varphi \, \psi \, \mathrm{d}x \, \mathrm{d}t \\
& \qquad \forall \, \varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_0), \ \forall \, \psi \in C_0^\infty(0,T). \qquad (14.65)
\end{aligned}
$$

However, using the fact that $C_0^\infty(\mathbb{R}^n;\Gamma_0)$ is dense in

$$H^1(\Omega;\Gamma_0)=\{\varphi\in H^1(\Omega)\ :\ \varphi=0\ \ \text{on}\ \ \Gamma_0\}$$

and $H^1(\Omega;\Gamma_0)$ is dense in

$$\mathcal{V}(\Omega;\Gamma_0)=\left\{v\in L^2(\Omega)\ :\ \partial_{x_n}v\in L^2(\Omega^-),\ v\in H^1(\Omega^+),\ v=0\ \ \text{a.e. on}\ \ \Gamma_0\right\}$$

with the continuous injection $H^1(\Omega;\Gamma_0)\hookrightarrow\mathcal{V}(\Omega;\Gamma_0)$ (see Corbo Esposito et al. [74]), we observe that the integral identity (14.65) is valid with $\varphi\in\mathcal{V}(\Omega;\Gamma_0)$. Hence, it can be rewritten in the form

$$-\int_0^T\int_\Omega(\chi_{\Omega^+}+|C|\chi_{\Omega^-})\,v_0\,\varphi\,\psi'\,\mathrm{d}x\,\mathrm{d}t+\int_0^T(v_0,\varphi)_{\mathcal{V}(\Omega)}\,\psi\,\mathrm{d}t$$
$$=\int_0^T\int_\Omega(\chi_{\Omega^+}+|C|\chi_{\Omega^-})\,f_0\,\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t,$$
$$\forall\,\varphi\in\mathcal{V}(\Omega;\Gamma_0),\quad\forall\,\psi\in C_0^\infty(0,T).\quad(14.66)$$

In addition, taking the initial supposition (14.6) into account and using the approach of De Maio et al. [88], one can prove the relation $v_0(0,x)=y^0(x)$ a.e. on Ω. Following the standard Hilbert space method and the arguments in [88], we can state that the function v_0 is a unique weak solution of the problem (14.49) in the sense of Remark 14.23. However, due to the uniqueness of the solution to the problem (14.49), the above reasoning holds for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. Thus, the lemma is proved. □

We are now in a position to state the first important result which deals with the recovery problem of the Kuratowski $K(w^a)$-limit set Ξ_a in the analytical form.

Theorem 14.24. *For the sequence of the sets of admissible pairs for the $\mathbb{P}_\varepsilon^a$ problems $\{\Xi_\varepsilon^a\}$, there exists a nonempty $K(w^a)$-limit set $\Xi_a\subset\mathbb{Y}_0^a$ with the following structure: $(u_0,v_0^+,v_0^-)\in\Xi_a$ if*

$$\Xi_a=\left\{\begin{array}{r}u_0\in U_a,\\ (v_0^+)'-\Delta_x v_0^++v_0^+=f_0\ \ in\ \ (0,T)\times\Omega^+,\\ (v_0^-)'-\partial_{x_n}^2v_0^-+\dfrac{|C|+k_0|\partial C|_H}{|C|}v_0^-\\ =f_0\ \ in\ \ (0,T)\times\Omega^-,\\ \partial_\nu v_0^+=0\ \ in\ \ (0,T)\times\partial\Omega^+\setminus\Sigma,\\ v_0^-=u_0\ \ on\ \ (0,T)\times\Gamma_0,\\ v_0^+=v_0^-,\quad\partial_{x_n}v_0^+=|C|\,\partial_{x_n}v_0^-\ \ on\ \ (0,T)\times\Sigma,\\ v_0(0,x)=y^0(x)\ \ a.e.\ \ x\in\Omega\end{array}\right\}.\qquad(14.67)$$

Here, $U_a=\left\{u\in L^2(0,T;H^1(\Gamma_0))\ :\ \|u\|_{L^2(0,T;H^1(\Gamma_0))}\le\mathbf{C_0}\right\}$.

Proof. First, we note that in view of Theorem 14.14, the sequence of sets $\{\Xi_\varepsilon^a \subset \mathbb{Y}_\varepsilon^a\}$ is relatively compact with respect to $K(w^a)$-convergence. We show that the $K(w^a)$-limit set exists for the whole sequence and it can be represented in the form (14.67). For this, in accordance with the definition of the $K(w^a)$-limit, we have to verify conditions (i) and (ii) of Definition 14.13. From the previous lemma, we see that the set Ξ_a is nonempty. Let (u, y^+, y^-) be any triplet of the set Ξ_a. We construct the w^a-convergent to (u, y^+, y^-) sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ as follows: $u_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon))$ is the restriction of the control $u \in U_a$ on Γ_ε given above and y_ε is the corresponding to u_ε weak solution of the boundary value problem (14.2). Then, in view of Definition 14.4, we have $u_\varepsilon \to u$ weakly with respect to the space $L^2(0,T;H^1(\Gamma_0))$. Further, using Lemma 14.22, we obtain $(u_\varepsilon, y_\varepsilon) \xrightarrow{w^a} (u, v_0^+, v_0^-)$, where (u, v_0^+, v_0^-) is a solution in $L^2(0,T;\mathcal{V}(\Omega))$ of the limit problem (14.49). Since this problem has a unique solution, we immediately deduce that $(u, v_0^+, v_0^-) = (u, y^+, y^-)$; thereby, property (i) of Definition 14.13 is valid.

The second condition of this definition is an evident consequence of Lemma 14.22 and the lower semicontinuity of the norm in $L^2((0,T)\times\Gamma_0)$ with respect to the weak convergence. This concludes the proof. □

14.4.2 Recovery of the set Ξ_b

To establish the structure of the Kuratowski's $K(w^b)$-limit set Ξ_b, we make use of the following result (see Sect. 7.2, Proposition 7.12).

Proposition 14.25. *For every bounded sequence*

$$\{u_\varepsilon \in L^2(0,T;L^2(\Gamma_\varepsilon))\}_{\varepsilon>0}$$

such that $\widetilde{u}_\varepsilon \longrightarrow u_*$ *weakly in* $L^2((0,T)\times\Gamma_0)$*, the following inequality holds:*

$$\liminf_{\varepsilon\to 0}\int_0^T\int_{\Gamma_\varepsilon} u_\varepsilon^2\,\mathrm{d}x'\,\mathrm{d}t \geq |C|^{-1}\int_0^T\int_{\Gamma_0} u_*^2\,\mathrm{d}x'\,\mathrm{d}t. \tag{14.68}$$

The following assertion can be viewed as an analogue of Lemma 14.22 with respect to w^b-convergence.

Lemma 14.26. *Let* $\{u_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon))\}_{\varepsilon>0}$ *be any sequence of admissible controls for the* $\mathbb{P}_\varepsilon^b$ *problems such that* $\widetilde{u}_\varepsilon \rightharpoonup u_*$ *in* $L^2((0,T)\times\Gamma_0)$*. Let* $\{y_\varepsilon\}$ *be the corresponding solutions of the parabolic problem (14.2) for which* $\sup_{\varepsilon>0}\|y_\varepsilon\|_{\mathbf{X}_{\mu_\varepsilon}} < +\infty$*. Then* $(u_\varepsilon, y_\varepsilon) \xrightarrow{w^b} (u_*, v_0^+, v_0^-)$ *as* $\varepsilon \to 0$*, where*

$$v_0(x) = \begin{cases} v_0^+(x) & \text{if } x \in \Omega^+, \\ v_0^-(x) & \text{if } x \in \Omega^-, \end{cases} \tag{14.69}$$

is the unique weak solution in $L^2(0,T;\mathcal{V}(\Omega))$ *of the limit problem*

$$\left.\begin{aligned}
(v_0^+)' - \Delta_x v_0^+ + v_0^+ &= f_0 \quad in \quad (0,T)\times\Omega^+,\\
(v_0^-)' - \partial^2_{x_n} v_0^- + \frac{|C| + k_0|\partial C|_H}{|C|} v_0^- &= f_0 \quad in \quad (0,T)\times\Omega^-,\\
\partial_\nu v_0^+ &= 0 \quad in \quad (0,T)\times\partial\Omega^+\setminus\Sigma,\\
v_0^- &= |C|^{-1}u_* \quad on \quad (0,T)\times\Gamma_0,\\
v_0^+ = v_0^-, \quad \partial_{x_n} v_0^+ &= |C|\,\partial_{x_n} v_0^- \quad on \quad (0,T)\times\Sigma,\\
v_0(0,x) &= y^0(x) \quad a.e. \quad x\in\Omega.
\end{aligned}\right\} \tag{14.70}$$

Proof. As follows from Proposition 14.12 and Lemma 14.22, we have to show that the relation

$$v_0^- = |C|^{-1}u_* \text{ on } (0,T)\times\Gamma_0 \tag{14.71}$$

holds true. For this, we note that

$$\left(\widetilde{y}_\varepsilon^-\right)\big|_{\Gamma_0} \rightharpoonup |C|\ \left(v_0^-\right)\big|_{\Gamma_0} \quad \text{in} \quad L^2((0,T)\times\Gamma_0)$$

and the statements

$$\begin{aligned}
\widetilde{y}_\varepsilon^- &= \widetilde{u}_\varepsilon \quad \text{a.e. on } (0,T)\times\Gamma_0, \forall\,\varepsilon>0,\\
\widetilde{u}_\varepsilon &\rightharpoonup u_0 \quad \text{in } L^2((0,T)\times\Gamma_0)
\end{aligned} \tag{14.72}$$

are valid. Then, the required relation (14.71) immediately follows from (14.72) after passing to the limit in (14.72) as $\varepsilon\to 0$. In order to conclude our proof, we have to follow the arguments in the proof of Lemma 14.22 closely. □

Now, we are able to prove the theorem concerning the structure of the Kuratowski $K(w^b)$-limit set Ξ_b.

Theorem 14.27. *Let $\{\Xi_\varepsilon^b\}_{\varepsilon>0}$ be the sets of admissible pairs for the $\mathbb{P}_\varepsilon^b$ problems possessing the $(\mathcal{N})$-property. Then for this sequence there exists a nonempty $K(w^b)$-limit set $\Xi_b\subset\mathbb{Y}_0^b$ which can be represented in the form $(u_0, v_0^+, v_0^-)\in\Xi_b$ if*

$$\Xi_b = \left\{\begin{aligned}
u_0 &\in U_b,\\
(v_0^+)' - \Delta_x v_0^+ + v_0^+ &= f_0 \ in \ (0,T)\times\Omega^+,\\
(v_0^-)' - \partial^2_{x_n} v_0^- + \frac{|C| + k_0|\partial C|_H}{|C|} v_0^- &\\
= f_0 \ in \ (0,T)\times\Omega^-,&\\
\partial_\nu v_0^+ = 0 \ in \ (0,T)\times\partial\Omega^+\setminus\Sigma,&\\
v_0^- = |C|^{-1}u_0 \ on \ (0,T)\times\Gamma_0,&\\
v_0^+ = v_0^- \quad \partial_{x_n} v_0^+ = |C|\,\partial_{x_n} v_0^- \ on \ (0,T)\times\Sigma,&\\
v_0(0,x) = y^0(x) \ a.e. \ x\in\Omega&
\end{aligned}\right\}. \tag{14.73}$$

Here, $U_b = \left\{u\in L^2((0,T)\times\Gamma_0) : \|u\|_{L^2((0,T)\times\Gamma_0)} \le \sqrt{|C|}\mathbf{C_0}\right\}$.

Proof. To obtain the representation (14.73), we have to verify conditions (i) and (ii) of Definition 14.13. Let (u, y^+, y^-) be any triplet of the set Ξ_b. In accordance with $(\mathcal{N})$-property of the $\mathbb{P}^b_\varepsilon$ problems, there is a w^b-convergent to (u, y^+, y^-) sequence $\{(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)\}_{\varepsilon>0}$ such that $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi^b_\varepsilon$ for every $\varepsilon > 0$. However, due to Lemma 14.26, we have $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \xrightarrow{w^b} (u, v_0^+, v_0^-)$, where $\widehat{v} = (v_0^+, v_0^-)$ is a weak solution in $L^2(0,T;\mathcal{V}(\Omega))$ of the limit problem (14.70). Since this problem has a unique weak solution (see Lions [169]), we immediately deduce that $(u, v_0^+, v_0^-) = (u, y^+, y^-)$. Thus, property (i) of Definition 14.13 holds for any triplet $(u, y^+, y^-) \in \Xi_0$.

We now verify the second property of Definition 14.13. Let $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ be a w^b-convergent sequence for which there exists a sequence $\{\varepsilon_k \to 0\}$ such that $(u_k, y_k) \in \Xi^b_{\varepsilon_k}$ for all $k \in \mathbb{N}$. Let (u, y^+, y^-) be its w^b-limit. Then by Proposition 14.25, we immediately have

$$\mathbf{C_0} \geq \liminf_{k\to\infty} \|u_k\|_{L^2((0,T)\times\Gamma_{\varepsilon_k})} \geq \sqrt{|C|^{-1}} \cdot \|u\|_{L^2((0,T)\times\Gamma_0)},$$

that is, $u \in \widehat{U}_b$. In conclusion, it remains only to apply Lemma 14.26. Thus, $(u, y^+, y^-) \in \Xi_b$, and we obtain the required result. This concludes the proof. □

14.5 Identification of the cost functionals I_a and I_b

In this section, we show that the cost functionals of the limit constrained minimization problems

$$\left\langle \inf_{(u,y^+,y^-)\in\,\Xi_a} I_a(u, y^+, y^-) \right\rangle \quad \text{and} \quad \left\langle \inf_{(u,y^+,y^-)\in\,\Xi_b} I_b(u, y^+, y^-) \right\rangle \tag{14.74}$$

can be recovered in an explicit form and their analytical representations are different. We begin with the following results.

Theorem 14.28. *For the sequence of the $\mathbb{P}^a_\varepsilon$ problems (14.23) there exists a variational w^a-limit (in the sense of Definition 14.18)*

$$\left\langle \inf_{(u,\,y^+,\,y^-)\in\,\Xi_a} I_a(u, y^+, y^-) \right\rangle, \tag{14.75}$$

where the set Ξ_a is defined in (14.67) and

$$I_a(u, y^+, y^-) = \int_0^T \int_{\Omega^+} (y^+ - q_0)^2 \,\mathrm{d}x\,\mathrm{d}t \,+\, |C| \int_0^T \int_{\Gamma_0} u^2 \,\mathrm{d}x'\,\mathrm{d}t. \tag{14.76}$$

Proof. In order to obtain relation (14.76), we verify conditions (ii) and (iii) of Definition 14.18. Let (u, y^+, y^-) be any triplet of Ξ_a and let $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ be a w^a-convergent sequence such that $(u_k, y_k) \xrightarrow{w^a} (u, y^-, y^+)$, $(u_k, y_k) \in \Xi_{\varepsilon_k}$

for every $k \in \mathbb{N}$, where $\{\varepsilon_k\}$ is a subsequence of $\{\varepsilon\}$ converging to 0. Then using Lemma 14.22, the definition of the class of admissible controls, and the properties of w^a-convergence, we get

$$\int_0^T \int_{\Omega^+} (y_\varepsilon^+)^2 \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_0^T \int_{\Omega^+} (y^+)^2 \,\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T \int_{\Gamma_{\varepsilon_k}} u_k^2 \,\mathrm{d}x'\,\mathrm{d}t = \int_0^T \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} \left(P_{\varepsilon_k} u_k\right)^2 \mathrm{d}x'\,\mathrm{d}t \quad \text{for every } k \in \mathbb{N},$$

$$\int_0^T \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} \left(P_{\varepsilon_k} u_k\right)^2 \mathrm{d}x'\,\mathrm{d}t \longrightarrow |C| \int_0^T \int_{\Gamma_0} u^2 \mathrm{d}x'\,\mathrm{d}t \quad \text{as} \quad k \to \infty$$

(as the limit of the product of weakly and strongly convergent sequences) and, therefore,

$$\begin{aligned} \liminf_{k\to\infty} I_{\varepsilon_k}(u_k, y_k) &= \int_0^T \int_{\Omega_0} (y^+ - q_0)^2 \,\mathrm{d}x \\ &\quad + |C| \int_0^T \int_{\Gamma_0} u^2 \mathrm{d}x'\,\mathrm{d}t = I_a(u, y^+, y^-), \end{aligned} \tag{14.77}$$

that is, the property (ii) of Definition 14.18 is valid.

Similarly, we can show the correctness of the "reverse" inequality (14.43). Indeed, in this case for a given triplet $(u, y^+, y^-) \in \Xi_a$, it is enough to consider as a "realized sequence" $\{(u_\varepsilon, y_\varepsilon)\}$ the following one: $u_\varepsilon \in L^2(0,T; H^1(\Gamma_\varepsilon))$ is the restriction of $u \in U_a$ on Γ_ε and y_ε is the corresponding to u_ε solution of the boundary value problem (14.2). Then, by Lemma 14.22, we have $(u_\varepsilon, y_\varepsilon) \xrightarrow{w^a} (u, y^+, y^-)$. For the conclusion of this proof, we repeat the arguments concerning the correctness of the limit passage (14.77). □

Theorem 14.29.

$$\left\langle \inf_{(u,\, y^+,\, y^-) \in\, \Xi_b} I_b(u, y^+, y^-) \right\rangle \tag{14.78}$$

is the variational w^b-limit for the sequence of the $\mathbb{P}_\varepsilon^b$ problems (14.23). Here, the set Ξ_b is defined in (14.73) and

$$I_b(u, y^+, y^-) = \int_0^T \int_{\Omega^+} (y^+ - q_0)^2 \,\mathrm{d}x \,+\, |C|^{-1} \int_0^T \int_{\Gamma_0} u^2 \,\mathrm{d}x'\,\mathrm{d}t. \tag{14.79}$$

Proof. To obtain representation (14.79), it is enough to repeat the same arguments of the proof of Theorem 14.28 and to apply Proposition 14.25 and Lemma 14.26. In this case, for any sequence $\left\{(u_k, y_k) \in \Xi_{\varepsilon_k}^b\right\}_{k\in\mathbb{N}}$ w^b-converging to (u, y^+, y^-), we have

$$\liminf_{k\to\infty} I_{\varepsilon_k}(u_k, y_k) \geq \int_0^T \int_{\Omega^+} |\nabla y^+|^2 \,\mathrm{d}x\,\mathrm{d}t + |C|^{-1} \int_0^T \int_{\Gamma_0} u^2 \,\mathrm{d}x'\,\mathrm{d}t = I_b(u, y^+, y^-).$$

To verify the correctness of inequality (14.43), for arbitrary triplets

$$(u, y^+, y^-) \in \Xi_b,$$

we have to construct a special "realizing sequence" $\{(u_\varepsilon, y_\varepsilon)\}$ satisfying condition (iii) of Definition 14.18. With this aim, we construct a w^b-convergent to (u, y^+, y^-) sequence $\left\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^b\right\}_{\varepsilon>0}$ as follows. Let $\{\overline{u}_\varepsilon \in L^2((0,T)\times\Gamma_0)\}$ be any sequence such that

$$\overline{u}_\varepsilon \rightharpoonup u \quad \text{in } L^2((0,T)\times\Gamma_0), \quad \|\overline{u}_\varepsilon\|_{L^2((0,T)\times\Gamma_0)} < \sqrt{|C|}\cdot \mathbf{C_0}$$

for every $\varepsilon > 0$. Since the weak topology of $L^2((0,T)\times\Gamma_0)$ is metrizable on the set

$$\widehat{U}_b = \left\{u \in L^2((0,T)\times\Gamma_0) \,:\, \|u\|_{L^2((0,T)\times\Gamma_0)} \leq \sqrt{|C|}\cdot \mathbf{C_0}\right\},$$

one can construct a sequence $\{w_\varepsilon \in L^2((0,T)\times\Gamma_0)\}_{\varepsilon>0}$ satisfying the following condition: Each element w_ε is a convex envelope of a finite number of the elements $\{\overline{u}_\varepsilon\}_{\varepsilon>0}$ and $w_\varepsilon \longrightarrow u$ strongly in $L^2((0,T)\times\Gamma_0)$. Note that in this case we have

$$\|w_\varepsilon\|_{L^2((0,T)\times\Gamma_0)} < \sqrt{|C|}\cdot \mathbf{C_0} \quad \text{for every } \ \varepsilon > 0.$$

Thus, a weakly to u convergent sequence $\{u_\varepsilon\}_{\varepsilon>0}$ can be taken in the following form: $u_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon))$ are elements such that

$$\|u_\varepsilon - |C|^{-1} w_\varepsilon\|_{L^2((0,T)\times\Gamma_\varepsilon)} < \varepsilon^2.$$

In view of the $(\mathcal{N})$-property, we may assume that the sequence of norms $\left\{\|u_\varepsilon\|_{L^2(0,T;H^1(\Gamma_\varepsilon))}\right\}$ is uniformly bounded. Since

$$|C|^{-1}\chi_{\Gamma_\varepsilon} w_\varepsilon \rightharpoonup u \ \text{ in } \ L^2((0,T)\times\Gamma_0)$$

(as the limit of the product of weakly and strongly convergent sequences) and

$$\| |C|^{-1} w_\varepsilon\|_{L^2((0,T)\times\Gamma_\varepsilon)} = \sqrt{\int_0^T \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} \frac{w_\varepsilon^2}{|C|^{-2}} \,\mathrm{d}x'\,\mathrm{d}t} \to \sqrt{\int_0^T \int_{\Gamma_0} \frac{u^2}{|C|^{-1}} \,\mathrm{d}x'\,\mathrm{d}t} < \mathbf{C_0},$$

it follows that $\widetilde{u}_\varepsilon \rightharpoonup u$ in $L^2((0,T)\times\Gamma_0)$ and $u_\varepsilon \in U_\varepsilon^b$ for ε sufficiently small.

We may assume that the elements w_ε have the representation $w_\varepsilon = |C|^{-1}\widehat{w}_\varepsilon$, where the sequence $\{\widehat{w}_\varepsilon\}$ is constructed as

$$L^2((0,T)\times\Gamma_0) \ni \widehat{w}_\varepsilon \to u \quad \text{strongly in } L^2((0,T)\times\Gamma_0),$$
$$\|\widehat{w}_\varepsilon\|_{L^2((0,T)\times\Gamma_0)} < \sqrt{|C|}\cdot \mathbf{C_0}, \quad \forall\ \varepsilon > 0.$$

Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\|w_\varepsilon\|^2_{L^2((0,T)\times\Gamma_\varepsilon)} &= \lim_{\varepsilon\to 0}\||C|^{-1}\chi_{\Gamma_\varepsilon}\widehat{w}_\varepsilon\|^2_{L^2((0,T)\times\Gamma_0)}\\ &= |C|^{-2}\lim_{\varepsilon\to 0}\int_0^T\int_{\Gamma_0}\chi_{\Gamma_\varepsilon}\widehat{w}^2_\varepsilon\,\mathrm{d}x'\,\mathrm{d}t\\ &= |C|^{-1}\|u\|^2_{L^2((0,T)\times\Gamma_0)},\end{aligned}$$

that is, for the realizing sequence of the Dirichlet boundary controls $\{u_\varepsilon\}$ we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Gamma_\varepsilon} u^2_\varepsilon\,\mathrm{d}x'\,\mathrm{d}t = |C|^{-1}\int_0^T\int_{\Gamma_0} u^2\,\mathrm{d}x'\,\mathrm{d}t$$

and $u_\varepsilon \in U^b_\varepsilon$ for all $\varepsilon > 0$.

In order to conclude, we take y_ε as solutions of the boundary value problem (14.2) corresponding to u_ε. Then by the $(\mathcal{N})$-property and Lemma 14.26, we have $(u_\varepsilon, y_\varepsilon) \xrightarrow{w^b} (u, y^+, y^-)$ and, therefore,

$$\begin{aligned}&\limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon)\\ &\quad= \lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^+}(y^+_\varepsilon - q_0)^2\,\mathrm{d}x\,\mathrm{d}t + \lim_{\varepsilon\to 0}\int_0^T\int_{\Gamma_\varepsilon} u^2_\varepsilon\,\mathrm{d}x'\,\mathrm{d}t\\ &\quad\ge \int_0^T\int_{\Omega^+}(y^+ - q_0)^2\,\mathrm{d}x\,\mathrm{d}t + |C|^{-1}\int_0^T\int_{\Gamma_0} u^2\,\mathrm{d}x'\,\mathrm{d}t.\end{aligned}$$

This concludes the proof. □

Thus, in accordance with Definition 14.21 and the results obtained above, we may infer the following: Each of the constrained minimization problems (14.74) can be recovered in the form of some OCPs, namely $(\mathbb{P}^a_{hom})$ (see (14.1), (14.2), and (14.4)) and $(\mathbb{P}^b_{hom})$ (see (14.1), (14.2), and (14.5)). Hence, there exist limiting OCPs to the problems $\mathbb{P}^a_\varepsilon$ and $\mathbb{P}^b_\varepsilon$ as ε tends to 0. However, the corresponding limit problems have different mathematical descriptions and these differences appear not only in the state equation and boundary conditions, but also in the control constrains and limit cost functionals. In fact, the reason for this gap phenomenon is the choice of different topologies for the limit analysis of the original control problem (14.2), (14.4) that were associated with w^a- and w^b-convergence, respectively. It should be stressed that in our case, this choice was dictated by the characteristic properties of the control constraints.

In conclusion, we give some results concerning the variational properties of the limit problems. As was noted earlier, the problems $(\mathbb{P}^a_{hom})$ and $(\mathbb{P}^b_{hom})$ have to preserve the well-known variational property, namely both optimal pairs and minimal values of the cost functionals for the the original problems have to converge to the corresponding characteristics of the limit OCPs as ε tends to 0. To establish this result, we begin with the following evident assertion.

Proposition 14.30. *Each of the limit OCPs (see (14.8)–(14.10) and (14.11)–(14.13)) has a unique solution.*

Indeed, taking into account the weak lower semicontinuity of the cost functionals $I_i : \Xi_i \to \mathbb{R}$ $(i = a, b)$, the topological properties of their domains Ξ_i $(i = a, b)$, and applying the direct method of Calculus of Variation, we just obtain the required result.

Let us denote by $(u^a, (y^a)^+, (y^a)^-) \in \mathbb{Y}^a_0$ and $(u^b, (y^b)^+, (y^b)^-) \in \mathbb{Y}^b_0$ the optimal triplets for the $(\mathbb{P}^a_{hom})$ and $(\mathbb{P}^b_{hom})$ problems, respectively.

Lemma 14.31. *If the functions f_ε and y^0_ε satisfy conditions (14.6)–(14.7), then the sequence of optimal solutions $\{(u^a_\varepsilon, y^a_\varepsilon) \in \Xi^a_\varepsilon\}$ to the $\mathbb{P}^a_\varepsilon$ problems and the corresponding minimal values of the cost functional (14.1) possess the following properties:*

$$\lim_{\varepsilon \to 0} I_\varepsilon(u^a_\varepsilon, y^a_\varepsilon) = \lim_{\varepsilon \to 0} \inf_{(u_\varepsilon, y_\varepsilon) \in \Xi^a_\varepsilon} I_\varepsilon(u_\varepsilon, y_\varepsilon)$$

$$= \inf_{(u, y^+, y^-) \in \Xi_a} I_a(u, y^+, y^-) = I_a(u^a, (y^a)^+, (y^a)^-), \quad (14.80)$$

$$(u^a_\varepsilon, y^a_\varepsilon) \xrightarrow{w^a} (u^a, (y^a)^+, (y^a)^-). \quad (14.81)$$

Proof. It is clear that for every value of ε, the OCP (14.1)–(14.4) has a unique solution $(u^a_\varepsilon, y^a_\varepsilon) \in \Xi^a_\varepsilon$. Since the sequence $\{u^a_\varepsilon\}_{\varepsilon > 0} \subset U^a_\varepsilon$ is bounded, there exists a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$, which we again denote by $\{\varepsilon\}$, such that $u^a_\varepsilon \to u^a \in U_a$ weakly with respect to the space $L^2(0, T; H^1(\Gamma_0))$ as $\varepsilon \to 0$. Then, in view of Lemma 14.22, we have $(u^a_\varepsilon, y^a_\varepsilon) \xrightarrow{w^a} (u^a, (y^*)^+, (y^*)^-)$ as $\varepsilon \to 0$, where the triplet $(u^a, (y^*)^+, (y^*)^-)$ is the unique solution of the problem (14.8) with the Dirichlet condition $y^- = u^a$ on Γ_0. By Theorem 14.19, we immediately conclude that $(u^a, (y^*)^-, (y^*)^+)$ is an optimal solution to the limit problem (14.8)–(14.10) and property (14.80) is valid. Hence,

$$(u^a, (y^*)^-, (y^*)^+) = (u^a, (y^a)^+, (y^a)^-).$$

So, we have obtained the required result. □

Remark 14.32. It should be noted that the realization of conditions (14.80)–(14.81) does not imply strong convergence of the optimal states y^a_ε.

Lemma 14.33. *Assume that the functions f_ε and y_ε^0 satisfy conditions (14.6)–(14.7), the $\mathbb{P}_\varepsilon^b$ problem is solvable for every value $\varepsilon > 0$, and the sequence of optimal solutions $\{(u_\varepsilon^b, y_\varepsilon^b) \in \Xi_\varepsilon^b\}$ for the $\mathbb{P}_\varepsilon^b$ problems is such that*

$$\sup_{\varepsilon>0} \|y_\varepsilon^b\|_{\mathbf{X}_{\mu_\varepsilon}} < +\infty.$$

Then

$$\begin{aligned}\lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^b, y_\varepsilon^b) &= \lim_{\varepsilon\to 0} \inf_{(u_\varepsilon, y_\varepsilon)\in\Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon, y_\varepsilon) \\ &= \inf_{(u,\, y^+,\, y^-)\in\Xi_b} I_b(u, y^+, y^-) = I_b(u^b, (y^b)^+, (y^b)^-), \qquad (14.82)\end{aligned}$$

$$(u_\varepsilon^b, y_\varepsilon^b) \xrightarrow{w^b} (u^b, (y^b)^+, (y^b)^-), \qquad (14.83)$$

$$\widetilde{u}_\varepsilon^b - |C|^{-1}\chi_{\Gamma_\varepsilon} u^b \longrightarrow 0 \quad \textit{strongly in} \quad L^2((0,T)\times\Gamma_0). \qquad (14.84)$$

Proof. Using the arguments of Lemma 14.31 and Theorem 14.27, it can be easily checked that conditions (14.82) and (14.83) hold. To conclude the proof, it remains to verify assertion (14.84). However, in view of (14.29) and the fact that

$$\|u_\varepsilon^b\|^2_{L^2(0,T;L^2(\Gamma_\varepsilon))} \longrightarrow |C|^{-1}\|u^b\|^2_{L^2((0,T)\times\Gamma_0)}$$

(see (14.82)), we get

$$\begin{aligned}&\int_0^T \int_{\Gamma_0} \left(\widetilde{u}_\varepsilon^b - |C|^{-1}\chi_{\Gamma_e} u^b\right)^2 \mathrm{d}x'\,\mathrm{d}t \\ &\quad = \int_0^T \int_{\Gamma_e} \left(u_e^b\right)^2 \mathrm{d}x'\,\mathrm{d}t \\ &\quad - 2|C|^{-1}\int_0^T \int_{\Gamma_0} u^b \widetilde{u}_\varepsilon^b \,\mathrm{d}x'\,\mathrm{d}t + |C|^{-2}\int_0^T \int_{\Gamma_0} \chi_{\Gamma_e}(u^b)^2 \,\mathrm{d}x'\,\mathrm{d}t \\ &\quad \longrightarrow |C|^{-1}\int_0^T \int_{\Gamma_0} (u^b)^2 \,\mathrm{d}x'\,\mathrm{d}t - 2|C|^{-1}\int_0^T \int_{\Gamma_0} (u^b)^2 \,\mathrm{d}x'\,\mathrm{d}t \\ &\quad + |C|^{-1}\int_0^T \int_{\Gamma_0} (u^b)^2 \,\mathrm{d}x'\,\mathrm{d}t = 0 \quad \text{as} \quad \varepsilon \to 0,\end{aligned}$$

which yields (14.84). This completes the proof. □

To conclude this chapter, we would like to point out a possible application of the results obtained above. It concerns the approximation of the optimal solutions to the original problems for ε small enough. Since the computational calculation of the solutions of these problems is very complicated, it is natural to define suboptimal solutions which have to guarantee the closeness of the corresponding value of the cost functional $I_\varepsilon(u_\varepsilon^{sub}, y_\varepsilon^{sub})$ to its minimum for ε small enough (see Definition 11.30).

As follows from Proposition 14.30, each of the limit OCPs (14.8)–(14.10) and (14.11)–(14.13) has a unique solution. Then Lemma 14.31 and the a priori estimate (14.18) immediately lead us to the following result.

Theorem 14.34. *Let* $(u^a, (y^a)^+, (y^a)^-) \in \mathbb{Y}_0^a$ *be an optimal solution for the limit* $(\mathbb{P}_{hom}^a)$*-problem. Then the sequence* $\left\{ u^a|_{\Gamma_\varepsilon} \right\}_{\varepsilon>0}$ *is asymptotically suboptimal for the* $\mathbb{P}_\varepsilon^a$ *problem in the sense of Definition 11.30.*

Using the idea of Sect. 13.7, a similar result can be established for the $\mathbb{P}_\varepsilon^b$ problem.

In conclusion, we would like to note that the asymptotic analysis of OCPs may essentially depend on the differential properties of its solutions. Choosing different topologies on the space of "control-state", the corresponding limit OCPs may have drastically different mathematical descriptions. So, the choice of such topologies is a very important and nontrivial matter when dealing with the questions of asymptotic behavior of the OCPs. In the homogenization theory of boundary value problems, this fact is called the Lavrentieff phenomenon [263].

15

Boundary Velocity Suboptimal Control of Incompressible Flow in Cylindrically Perforated Domains

Optimal control problems (OCPs) for the Navier–Stokes equations have been the subject of extensive study in recent years. A systematic mathematical and numerical analysis of OCPs of different types (e.g., having Dirichlet, Neumann, and distributed controls) for the steady-state Navier–Stokes system was given by Abergel and Temam [1], Fursikov, Gunzburger, and Hou [111, 112], Hou and Ravindran [123], Gunzburger, Hou, and Svobodny [116], Ivanenko and Mel'nik [126], and Zgurovsky and Mel'nik [253]. Dirichlet controls (i.e., boundary velocity controls or boundary mass flux control) are common in applications [237]. However, as is shown in [124], even though the admissible controls are smooth, the optimality systems for optimal Dirichlet control problems involve a boundary Laplacian or a boundary biharmonic equation. This circumstance makes the numerical resolution of the optimality systems, and hence the numerical calculation of an optimal control for such systems, very complicated. So, much effort has been made for the development of penalty, approximation, and relaxation methods for solving optimal Dirichlet control problems (see [1, 12, 123]).

These problems are especially complicated in perforated domains and in domains with quickly oscillating boundaries (see [222]). So, we are concerned with the development of approximate methods for the solution of optimal Dirichlet control problems for Navier–Stokes equations. The approach we propose gives the possibility to replace the original OCP with some limit problem defined in a more simple domain. We show that an optimal control for the limit problem can be taken as a suboptimal control to the original one.

We now turn to a more detailed description of the main object of our study. Let $\widetilde{\Omega} \subset \mathbb{R}^2$ be a bounded connected open domain with a simple connected smooth boundary $\partial\widetilde{\Omega}$, and let ε be a small positive parameter. To define a cylindrically perforated flow domain Ω_ε in $\mathbb{R}^3$, we introduce the sets: $\widetilde{Y} = [-1/2, 1/2)^2$; Q is a compact subset of $\widetilde{Y}$ such that $0 \in \partial Q$,

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_15,
© Springer Science+Business Media, LLC 2011

$$\Theta_\varepsilon = \{\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2 : \varepsilon(r_\varepsilon Q + \mathbf{k}) \subset\subset \widetilde{\Omega}\}, \tag{15.1}$$

$$\widetilde{T}_\varepsilon^{\mathbf{k}} = \varepsilon(r_\varepsilon Q + \mathbf{k}), \mathbf{k} \in \Theta_\varepsilon, \quad \widetilde{T}_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \widetilde{T}_\varepsilon^{\mathbf{k}} = \bigcup_{j=1}^{J_\varepsilon} \widetilde{T}_j^{\mathbf{k}}, \tag{15.2}$$

$$T_\varepsilon = \widetilde{T}_\varepsilon \times [0, \ell], \quad \widetilde{\Omega}_\varepsilon = \widetilde{\Omega} \setminus \widetilde{T}_\varepsilon, \quad \Omega = \widetilde{\Omega} \times (0, \ell), \tag{15.3}$$

$$\Gamma^1 = \widetilde{\Omega} \times \{0\}, \quad \Gamma^2 = \widetilde{\Omega} \times \{\ell\}, \quad \Gamma^3 = \partial\widetilde{\Omega} \times [0, \ell], \tag{15.4}$$

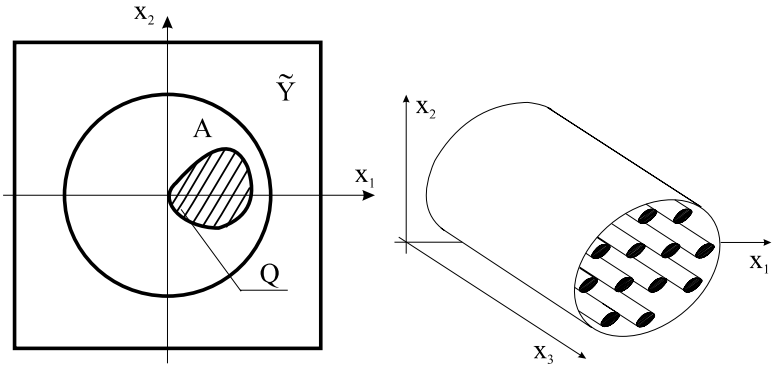
where the parameter r_ε denotes the cross-size of the thin cylinders $T_\varepsilon^{\mathbf{k}} = \widetilde{T}_\varepsilon^{\mathbf{k}} \times [0, \ell]$ and satisfies the conditions $0 < r_\varepsilon \leq 1$ and $\lim_{\varepsilon\to 0} r_\varepsilon = 0$.

Then the domain Ω_ε is defined by removing the cylinders $T_\varepsilon^{\mathbf{k}}$ from Ω, that is (see Fig. 15.1),

$$\Omega_\varepsilon = \Omega \setminus \left[\bigcup_{\mathbf{k}\in\Theta_\varepsilon} T_\varepsilon^{\mathbf{k}}\right] = \Omega \setminus \left[\bigcup_{\mathbf{k}\in\Theta_\varepsilon} \varepsilon(r_\varepsilon Q + \mathbf{k}) \times [0, \ell]\right]. \tag{15.5}$$

We use the following decomposition for the boundary of this domain: $\partial\Omega_\varepsilon = \Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^2 \cup \Gamma^3 \cup \partial T_\varepsilon$, where

$$\Gamma_\varepsilon^1 = \widetilde{\Omega}_\varepsilon \times \{0\}, \quad \Gamma_\varepsilon^2 = \widetilde{\Omega}_\varepsilon \times \{\ell\}. \tag{15.6}$$



(a) The cell of perforation (b) The domain

Fig. 15.1. An example of a cylindrically perforated domain

To characterize the different possible cross-sizes of thin cylinders which can be considered ("critical", smaller, and larger cylinders), we define a ratio σ_ε between the current size of the cross-sections and the critical one:

$$\sigma_\varepsilon = \varepsilon^2 \left(\log 1/r_\varepsilon\right). \tag{15.7}$$

If the limit of σ_ε, as ε tends to 0, is positive and finite, then the cross-size of the cylinders is called critical. If $\lim_{\varepsilon\to 0}\sigma_\varepsilon = +\infty$, the cross-size of cylinders is smaller, and if $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$, the cross-size is larger (see Cioranescu and Murat [65] and Allaire [5]).

The OCP we consider is to minimize the vorticity of viscous, incompressible flow by choosing an appropriate boundary velocity on the "vertical" sides of thin cylinders

$$T_\varepsilon^{\mathbf{k}} = \{(x_1,x_2,x_3) : (x_1,x_2) \in \varepsilon(r_\varepsilon Q + \mathbf{k}), 0 \le x_3 \le \ell\}\,, \quad \forall\, \mathbf{k} \in \Theta_\varepsilon.$$

Precisely, we study the following OCP: Find a boundary velocity field $\overline{\boldsymbol{\alpha}}_\varepsilon = \left(\alpha_{\mathbf{k}_1}, \alpha_{\mathbf{k}_2}, \ldots, \alpha_{\mathbf{k}_{J_\varepsilon}}\right)$ and a corresponding velocity–pressure pair $(\mathbf{y}_\varepsilon, p_\varepsilon)$ such that the functional

$$\mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon) = \lambda \int_{\Omega_\varepsilon} |\nabla\, \mathbf{y}_\varepsilon|^2 \,\mathrm{d}x + \frac{\beta\varepsilon}{r_\varepsilon} \sum_{j=1}^{J_\varepsilon} \int_{\partial T_\varepsilon^{\mathbf{k}_j}} \left|\alpha_{\mathbf{k}_j}\right|^2 \mathrm{d}\mathcal{H}^2 \tag{15.8}$$

is minimized subject to the steady-state Navier–Stokes equations

$$-\nu\triangle \mathbf{y}_\varepsilon + (\mathbf{y}_\varepsilon \cdot \nabla)\mathbf{y}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}_\varepsilon \quad \text{in } \ \Omega_\varepsilon, \tag{15.9}$$

$$\operatorname{div} \mathbf{y}_\varepsilon = 0 \quad \text{in } \ \Omega_\varepsilon, \tag{15.10}$$

$$\mathbf{y}_\varepsilon|_{\Gamma_\varepsilon^1} = \mathbf{y}_\varepsilon^1,\ \mathbf{y}_\varepsilon|_{\Gamma_\varepsilon^2} = \mathbf{y}_\varepsilon^2,\ \mathbf{y}_\varepsilon|_{\Gamma^3} = 0, \tag{15.11}$$

$$\mathbf{y}_\varepsilon\big|_{\partial T_\varepsilon^{\mathbf{k}_j}} = \alpha_{\mathbf{k}_j}, \quad \forall\, j = 1,\ldots,J_\varepsilon. \tag{15.12}$$

Here, ν denotes the constant viscosity; $\mathbf{y}_\varepsilon$ and p_ε denote the velocity field and the pressure field, respectively; $\mathbf{f}_\varepsilon$ is a prescribed forcing term; $\mathbf{y}_\varepsilon^1$ and $\mathbf{y}_\varepsilon^2$ are given boundary velocities on the lower and upper boundaries Γ_ε^1 and Γ_ε^2, respectively, and $\overline{\boldsymbol{\alpha}}_\varepsilon$ is the boundary velocity – the control field. Because of the divergence-free condition on $\mathbf{y}_\varepsilon$, the vector-valued functions $\mathbf{y}_\varepsilon^1$, $\mathbf{y}_\varepsilon^2$, and $\boldsymbol{\alpha}_{\mathbf{k}_j}$ must necessarily satisfy the relations

$$\left.\begin{aligned} &\int_{\Gamma_\varepsilon^1} \mathbf{y}_\varepsilon^1 \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^2 = 0,\ \int_{\Gamma_\varepsilon^2} \mathbf{y}_\varepsilon^2 \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^2 = 0, \\ &\int_{\partial T_\varepsilon^{\mathbf{k}_j}} \alpha_{\mathbf{k}_j} \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^2 = 0,\ \forall\, j = 1,\ldots,J_\varepsilon. \end{aligned}\right\} \tag{15.13}$$

Throughout this chapter, we assume that there are functions

$$\mathbf{y}^*{}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{sol}(\Omega_\varepsilon),\ \mathbf{f} \in \mathbf{L}^2(\Omega), \ \text{ and } \ \mathbf{y}^* \in \mathbf{H}^2(\Omega)$$

such that

$$\mathbf{y}_\varepsilon^*|_{\Gamma_\varepsilon^1} = \mathbf{y}_\varepsilon^1,\ \mathbf{y}_\varepsilon^*|_{\Gamma_\varepsilon^2} = \mathbf{y}_\varepsilon^2,\ \mathbf{y}_\varepsilon^*|_{\Gamma^3} = 0, \text{ and } \mathbf{y}_\varepsilon^* \rightharpoonup \mathbf{y}^* \ \text{ in } \ \mathbf{H}^2(\Omega), \tag{15.14}$$

$$\mathbf{f}_\varepsilon \rightharpoonup \mathbf{f} \ \text{ in } \ \mathbf{L}^2(\Omega). \tag{15.15}$$

We also say that a boundary velocity field $\overline{\boldsymbol{\alpha}}_\varepsilon = \left(\alpha_{\mathbf{k}_1}, \ldots, \alpha_{\mathbf{k}_{J_\varepsilon}}\right)$ is admissible if there exists a function $\mathbf{u} \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega)$ (the so-called prototype of the boundary control $\overline{\boldsymbol{\alpha}}_\varepsilon$) such that $\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \leq \gamma$ (for a given value $\gamma > 0$) and

$$\mathbf{u}|_{\Gamma^1_\varepsilon} = \mathbf{y}^1_\varepsilon, \ \ \mathbf{u}|_{\Gamma^2_\varepsilon} = \mathbf{y}^2_\varepsilon, \ \ \mathbf{u}|_{\Gamma^3} = 0, \ \ \mathbf{u}|_{\partial T_\varepsilon^{\mathbf{k}_j}} = \alpha_{\mathbf{k}_j}, \quad \forall j = 1, \ldots, J_\varepsilon. \tag{15.16}$$

The constants λ and β in the functional (15.8) are two positive parameters that adjust the relative weights of the two terms in the functional. Such a choice of the cost functional is motivated by the fact that irrotational flows have no local flow recirculations. On the other hand, for both physical and mathematical reasons, the size of the control should be constrained. So, following the representation

$$\begin{aligned} |\operatorname{curl} \mathbf{y}_\varepsilon|^2 &= \operatorname{curl} \mathbf{y}_\varepsilon : \operatorname{curl} \mathbf{y}_\varepsilon \\ &= |\nabla \mathbf{y}_\varepsilon|^2 + \left|{}^t(\nabla \mathbf{y}_\varepsilon)\right|^2 - \nabla \mathbf{y}_\varepsilon : {}^t(\nabla \mathbf{y}_\varepsilon) - {}^t(\nabla \mathbf{y}_\varepsilon) : \nabla \mathbf{y}_\varepsilon, \end{aligned} \tag{15.17}$$

we hope that minimizing the functional (15.8) will lead to a reduction in flow recirculations.

Our main result is: The boundary velocity field

$$\overline{\boldsymbol{\alpha}}^{sub}_\varepsilon = \left(\alpha^{sub}_{\mathbf{k}_1}, \alpha^{sub}_{\mathbf{k}_2}, \ldots, \alpha^{sub}_{\mathbf{k}_{J_\varepsilon}}\right) = \Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}$$

can be taken as the suboptimal control for the problem (15.8)–(15.12), where $\Lambda_\varepsilon : \mathbf{H}^1_{sol}(\Omega) \mapsto \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ is some linear bounded operator (see (15.103) and (15.104)) and $\mathbf{u}^0$ is a solution to one of the following problems:

1. In the case when $C_0 = \lim_{\varepsilon \to 0} \varepsilon^2 (\log 1/r_\varepsilon) = +\infty$, the functional

$$\int_\Omega |\mathbf{u}(x)|^2 \, \mathrm{d}x \tag{15.18}$$

is minimized subject to the constraints

$$\left\{ \mathbf{u}(x) \in \mathbf{H}^2(\Omega) \ \middle| \ \begin{array}{l} \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \leq \gamma, \quad \nabla \cdot \mathbf{u} = 0 \ \text{ on } \ \Omega, \\ \mathbf{u}|_{\Gamma^1} = \mathbf{y}^*|_{\Gamma^1}, \ \ \mathbf{u}|_{\Gamma^2} = \mathbf{y}^*|_{\Gamma^2}, \ \ \mathbf{u}|_{\Gamma^3} = 0, \\ \mathbf{y}^* \cdot \mathbf{n} = 0 \ \text{ on } \ \Gamma^1 \cup \Gamma^2 \end{array} \right\}. \tag{15.19}$$

2. When $0 < C_0 < +\infty$, the cost functional

$$\begin{aligned} \mathcal{J}_0(\mathbf{u}, \mathbf{y}) &= \lambda \int_\Omega |\nabla \mathbf{y}|^2 \, \mathrm{d}x \\ &\quad + \frac{2\pi\lambda}{C_0} \int_\Omega |\mathbf{y} - \mathbf{u}|^2 \, \mathrm{d}x + \beta |\partial Q|_H \int_\Omega |\mathbf{u}|^2 \, \mathrm{d}x \end{aligned} \tag{15.20}$$

is minimized subject to the boundary value problem for a Brinkman-type law

$$-\nu\triangle\mathbf{y}+\frac{2\pi\nu}{C_0}(\mathbf{y}-\mathbf{u})+(\mathbf{y}\cdot\nabla)\mathbf{y}+\nabla p=\mathbf{f}\quad\text{ in }\ \Omega, \tag{15.21}$$

$$\operatorname{div}\mathbf{y}=0\quad\text{ in }\ \Omega,\quad \mathbf{y}|_{\partial\Omega}=\mathbf{u}|_{\partial\Omega}, \tag{15.22}$$

$$\mathbf{u}|_{\Gamma^1}=\mathbf{y}^*|_{\Gamma^1}\,,\ \mathbf{u}|_{\Gamma^2}=\mathbf{y}^*|_{\Gamma^2}\,,\ \mathbf{u}|_{\Gamma^3}=0, \tag{15.23}$$

$$p\in L^2_0(\Omega),\ \mathbf{u}\in\mathbf{H}^2(\Omega),\ \mathbf{y}-\mathbf{u}\in\mathbf{H}^1_{0,sol}(\Omega),\ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}\le\gamma. \tag{15.24}$$

15.1 Preliminaries and notation

Throughout this chapter, we suppose that the boundary $\partial\widetilde{\Omega}$ is of class C^∞, so $\Omega=\widetilde{\Omega}\times(0,l)$ is a measurable set in the sense of Jordan; the small parameter ε varies in a strictly decreasing sequence of positive numbers which converges to 0; Q is a compact subset of Y with Lipschitz boundary ∂Q, $\operatorname{int}Q$ is a strongly connected set, $Q\subset\{x=(x_1,\dots,x_n)\in\mathbb{R}^n\ :\ x_1\ge 0\}$, and its boundary ∂Q contains the origin; $A=B(\mathbf{0},r_0)$ is an open ball centered at the origin with a radius $r_0<1/2$, so that $A\subset\subset\widetilde{Y}$ and $Q\subset\subset A$ (see Fig. 15.1); C or C_i (where i is any subscript) denotes a constant independent of ε. For any subset $E\subset\mathbb{R}^n$, we denote by $|E|$ its n-dimensional Lebesgue measure $\mathcal{L}_n(E)$, whereas $|\partial E|_H$ denotes the $(n-1)$-dimensional Hausdorff measure of the manifold ∂E on $\mathbb{R}^n$. For the definition of fractional ordered Sobolev spaces $H^l(\partial\Omega)$ (l noninteger), see [2]. Let

$$L^2_0(\Omega)=\left\{q\in L^2(\Omega)\ :\ \int_\Omega q\,\mathrm{d}x=0\right\}$$

with the norm

$$\|p\,\|^2_{L^2_0(\Omega)}=\int_\Omega\left(p(x)-\int_\Omega p(x)\,\mathrm{d}x\right)^2\,\mathrm{d}x.$$

Let $\mathcal{M}_b(\Omega)$ be the space of bounded Borel measures on Ω. Let $\mathcal{M}^+_0(\Omega)$ be the cone of all non-negative Borel measures μ on Ω such that $\mu(B)=0$ for every set $B\subseteq\Omega$ with $\operatorname{cap}(B,\Omega)=0$ and $\mu(B)=\inf\{\mu(U)\ :\ U\text{ quasi-open}, B\subseteq U\}$ for every Borel set $B\subseteq\Omega$. The space $\mathcal{D}'(\Omega)$ of distributions on Ω is the dual of the space $C^\infty_0(\Omega)$. For $m\ge 0$, we introduce the subspaces $H^m_0(\Omega)$ of the Sobolev spaces $H^m(\Omega)$ as the closure of $C^\infty_0(\Omega)$ in $H^m(\Omega)$ and the dual spaces $H^{-m}(\Omega)=(H^m_0(\Omega))^*$. The duality pairing between a Sobolev space $H^s(\Omega)$ ($s>0$) and its dual space is denoted by $\langle\cdot,\cdot\rangle_{H^{-s}(\Omega);H^s(\Omega)}$. The trace spaces $H^l(\partial\Omega)$ are the restriction to the boundary of $H^{l+1/2}(\Omega)$ (see [237]). The vector-valued counterparts of these spaces are denoted by boldface symbols (e.g., $\mathbf{L}^r(\Omega)$, $\mathbf{H}^m(\Omega)$, $\mathbf{H}^l(\partial\Omega)$, $\mathbf{H}^m_0(\Omega)$, and $\mathbf{C}^\infty_0(\Omega)$). If $\mathbf{u}$ is a vector-valued function from $\mathbb{R}^N$ to $\mathbb{R}^N$, then we have the following:

1. The gradient of $\mathbf{u}$ is an $N\times N$ tensor: $\nabla\mathbf{u}=(\partial u_i/\partial x_j)_{1\le i,j\le N}$.
2. The inner product of two $N\times N$ tensors $A=(a_{ij})$ and $B=(b_{ij})$ is denoted by $A:B=\operatorname{tr}({}^tAB)=\sum_{1\le i,j\le N}a_{ij}b_{ij}$.

We will also use the spaces of solenoidal vector fields

$$\begin{aligned}
\mathbf{C}^\infty_{0,sol}(\Omega) &= \left\{ \mathbf{y} \in \mathbf{C}^\infty_0(\Omega) \ : \ \operatorname{div} \mathbf{y} = \textstyle\sum_{i=1}^3 \dfrac{\partial y_i}{\partial x_i} = 0 \ \text{ in } \ \Omega \right\}, \\
\mathbf{H}^m_{sol}(\Omega) &= \left\{ \mathbf{y} \in \mathbf{H}^m(\Omega) \ : \ \nabla \cdot \mathbf{y} = 0, \ \displaystyle\int_{\partial\Omega} \mathbf{y} \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^2 = 0 \right\}, \\
\mathbf{H}^m_{0,sol}(\Omega) &= \text{ the closure of } \ \mathbf{C}^\infty_{0,sol}(\Omega) \ \text{ in } \ \mathbf{H}^m(\Omega)\text{-norm},
\end{aligned}$$

where when $m = 0$, $\displaystyle\int_{\partial\Omega} \mathbf{y} \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^2$ is the $\big(H^{-1/2}(\partial\Omega); H^{1/2}(\partial\Omega)\big)$ duality pairing between the function $(\mathbf{y} \cdot \mathbf{n}) \in H^{-1/2}(\partial\Omega)$ and the constant scalar function $1 \in H^{1/2}(\partial\Omega)$. The norm on $\mathbf{H}^m_{sol}(\Omega)$ and $\mathbf{H}^m_{0,sol}(\Omega)$ is chosen to be that of $\mathbf{H}^m(\Omega)$. We use the standard definition of the divergence operator:

$$\langle \operatorname{div} \mathbf{y}, \varphi \rangle_{H^{-1}(\Omega); H^1_0(\Omega)} = -\int_\Omega \mathbf{y} \cdot \nabla \varphi \, \mathrm{d}x, \quad \forall \, \varphi \in H^1_0(\Omega). \tag{15.25}$$

Let $\mathbf{H}(\operatorname{div}, \Omega) = \big\{ \mathbf{y} \in \mathbf{L}^2(\Omega) \ : \ \operatorname{div} \mathbf{y} \in L^2(\Omega) \big\}$. We will need the following lemma on integration by parts for functions in the space $\mathbf{H}(\operatorname{div}, \Omega)$ (for the proof, see [237]).

Lemma 15.1. *Let* $\mathbf{w} \in \mathbf{H}(\operatorname{div}, \Omega)$. *Then* $(\mathbf{w} \cdot \mathbf{n})|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ *and*

$$\begin{aligned}
&\langle \mathbf{w} \cdot \mathbf{n}, v \rangle_{H^{-1/2}(\partial\Omega); H^{1/2}(\partial\Omega)} \\
&\qquad = \int_\Omega v \operatorname{div} \mathbf{w} \, \mathrm{d}x + \int_\Omega \mathbf{w} \cdot \nabla v \, \mathrm{d}x, \quad \forall \, v \in H^1(\Omega).
\end{aligned} \tag{15.26}$$

To end this section, we define the standard bilinear and trilinear forms associated with the Navier–Stokes equations:

$$\begin{aligned}
a_\varepsilon(\mathbf{y}, \mathbf{v}) &= \int_{\Omega_\varepsilon} (\nabla \mathbf{y}) : (\nabla \mathbf{v}) \, \mathrm{d}x, \quad \forall \, \mathbf{y}, \mathbf{v} \in \mathbf{H}^1(\Omega_\varepsilon), \\
b_\varepsilon(\mathbf{y}, q) &= -\int_{\Omega_\varepsilon} q \operatorname{div} \mathbf{y} \, \mathrm{d}x, \quad \forall \, \mathbf{y} \in \mathbf{H}^1(\Omega_\varepsilon), \forall \, q \in L^2(\Omega_\varepsilon), \\
c_\varepsilon(\mathbf{y}, \mathbf{v}, \mathbf{w}) &= \int_{\Omega_\varepsilon} (\mathbf{y} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, \mathrm{d}x, \quad \mathbf{y}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega_\varepsilon).
\end{aligned}$$

15.2 Admissible controls and regularity of solutions to the boundary value problem for Navier–Stokes equations

We devote this section to the study of the boundary value problem (15.9)–(15.12). For this, we give the definition of a solution for the Navier–Stokes equations with inhomogeneous Dirichlet boundary conditions. Throughout, we assume that $\mathbf{f}_\varepsilon \in \mathbf{L}^2(\Omega)$ and $\sup_{\varepsilon>0} \|\mathbf{f}_\varepsilon\|_{\mathbf{L}^2(\Omega)} < +\infty$. We also assume that there are functions $\mathbf{y}^*{}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ and $\mathbf{y}^* \in \mathbf{H}^2(\Omega)$ satisfying (15.14).

Definition 15.2. *A pair* $(\mathbf{y}_\varepsilon, p_\varepsilon) \in \mathbf{H}^1(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon)$ *is said to be a solution of the Navier–Stokes equations (15.9)–(15.12) if and only if*

$$\nu a_\varepsilon(\mathbf{y}_\varepsilon, \mathbf{v}) + c_\varepsilon(\mathbf{y}_\varepsilon, \mathbf{y}_\varepsilon, \mathbf{v}) + b_\varepsilon(\mathbf{v}, p_\varepsilon) = \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{v}\, \mathrm{d}x, \ \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega_\varepsilon), \quad (15.27)$$

$$b_\varepsilon(\mathbf{y}_\varepsilon, q) = 0, \quad \forall q \in L^2_0(\Omega_\varepsilon), \quad (15.28)$$

and

$$\mathbf{y}_\varepsilon|_{\Gamma^1_\varepsilon} = \mathbf{y}^1_\varepsilon, \ \mathbf{y}_\varepsilon|_{\Gamma^2_\varepsilon} = \mathbf{y}^2_\varepsilon, \ \mathbf{y}_\varepsilon|_{\Gamma^3_\varepsilon} = 0, \ \mathbf{y}_\varepsilon|_{\partial T^{\mathbf{k}_j}_\varepsilon} = \alpha_{\mathbf{k}_j}, \ \forall j = 1, \dots, J_\varepsilon. \quad (15.29)$$

A proof of the existence of a solution in the sense of Definition 15.2 can be found in [237].

Let $\gamma > 0$ be a given value. We say that a boundary velocity field $\overline{\boldsymbol{\alpha}}_\varepsilon = \left(\alpha_{\mathbf{k}_1}, \dots, \alpha_{\mathbf{k}_{J_\varepsilon}}\right)$ is admissible if there exists a function $\mathbf{u} \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega)$ (the so-called prototype of the boundary control $\overline{\boldsymbol{\alpha}}_\varepsilon$) such that $\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma$ and conditions (15.16) are satisfied. Such a choice of admissible controls is motivated by the fact that the trace space $\mathbf{H}^{1/2}(\partial\Omega_\varepsilon)$ is the restriction to the boundary of $\mathbf{H}^1(\Omega_\varepsilon)$. Hence, in view of the initial supposition, for a fixed $\overline{\boldsymbol{\alpha}}_\varepsilon$ there exists a function $\mathbf{u} \in \mathbf{H}^1(\Omega)$ satisfying conditions (15.16). Then, due to (15.13) and the Stokes formula (15.26), we have $\int_{\Omega_\varepsilon} \operatorname{div} \mathbf{u}\, \mathrm{d}x = 0$.

We denote by $\mathbf{U}_\varepsilon$ the set of all admissible controls for a fixed ε

$$\left\{ \overline{\boldsymbol{\alpha}}_\varepsilon = \left(\alpha_{\mathbf{k}_1}, \alpha_{\mathbf{k}_2}, \dots, \alpha_{\mathbf{k}_{J_\varepsilon}}\right) \;\middle|\; \begin{array}{l} \alpha_{\mathbf{k}_j} = \mathbf{u}|_{\partial T^{\mathbf{k}_j}_\varepsilon}, \ \forall j = 1, \dots, J_\varepsilon, \\ \mathbf{u} \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega), \\ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma, \\ \mathbf{u}|_{\Gamma^1_\varepsilon} = \mathbf{y}^1_\varepsilon, \ \mathbf{u}|_{\Gamma^2_\varepsilon} = \mathbf{y}^2_\varepsilon, \ \mathbf{u}|_{\Gamma^3_\varepsilon} = 0 \end{array} \right\}. \quad (15.30)$$

Definition 15.3. *We say that a triplet* $(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)$ *is admissible to the OCP (15.8)–(15.12) if* $\overline{\boldsymbol{\alpha}}_\varepsilon \in \mathbf{U}_\varepsilon$ *and the pair* $(\mathbf{y}_\varepsilon, p_\varepsilon)$ *is a corresponding solution of the variational problem (15.27)–(15.29).*

At the end of this section, we cite the following results (their proof can be found in Duvaut and Lions [102], Fursikov [111], and Temam [237]) that will be useful in the sequel.

Theorem 15.4. *Let* $\overline{\boldsymbol{\alpha}}_\varepsilon$ *be an admissible control (*$\overline{\boldsymbol{\alpha}}_\varepsilon \in \mathbf{U}_\varepsilon$*) and let* $\mathbf{u}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega)$ *be its prototype. Then there exists a corresponding velocity–pressure pair* $(\mathbf{y}_\varepsilon, p_\varepsilon) \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon)$ *satisfying the boundary value problem (15.9)–(15.12) in the following variational sense:*

$$\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon), \quad (15.31)$$

$$\nu a_\varepsilon(\mathbf{y}_\varepsilon, \mathbf{v}) + c_\varepsilon(\mathbf{y}_\varepsilon, \mathbf{y}_\varepsilon, \mathbf{v}) = \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{v}\, \mathrm{d}x, \quad \forall \mathbf{v} \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon), \quad (15.32)$$

$$\nabla p_\varepsilon = \nu \triangle \mathbf{y}_\varepsilon - (\mathbf{y}_\varepsilon \cdot \nabla)\mathbf{y}_\varepsilon + \mathbf{f}_\varepsilon \quad \text{in} \ \ \mathcal{D}'(\Omega_\varepsilon). \quad (15.33)$$

Proposition 15.5. *Let $\mathbf{u}_\varepsilon \in \mathbf{U}_\varepsilon$ be a prototype of some admissible control. Assume $\mathbf{y}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ satisfies conditions (15.31)–(15.32). Then there exists a continuous positive function $B : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (independent of ε) such that the following estimate holds true:*

$$\|\mathbf{y}_\varepsilon\|_{\mathbf{H}^1(\Omega_\varepsilon)} \le B\left(\|\mathbf{f}_\varepsilon\|_{\mathbf{L}^2(\Omega)}, \|\mathbf{u}_\varepsilon\|_{\mathbf{H}^1(\Omega)}, \|\mathbf{u}_\varepsilon\|^2_{\mathbf{H}^1(\Omega)}\right). \tag{15.34}$$

15.3 On solvability of the optimal boundary control problem

The optimal Dirichlet control problem we consider can be precisely stated as follows: Find a triplet $(\overline{\boldsymbol{\alpha}}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon) \in \mathbf{U}_\varepsilon \times \mathbf{H}^1_{sol}(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon)$ such that the cost functional (15.8) is minimized subject to the relations (15.27)–(15.29). We denote this problem by $(\mathbb{P}_\varepsilon)$. We also define the set of admissible solutions for $(\mathbb{P}_\varepsilon)$ by

$$\Xi_\varepsilon = \left\{ (\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \;\middle|\; \begin{array}{l} \overline{\boldsymbol{\alpha}}_\varepsilon = \left(\alpha_{\mathbf{k}_1}, \alpha_{\mathbf{k}_2}, \ldots, \alpha_{\mathbf{k}_{\mathrm{J}_\varepsilon}}\right) \in \mathbf{U}_\varepsilon, \\ \mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon), \\ (\mathbf{y}_\varepsilon, p_\varepsilon) \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon), \\ (\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \text{ satisfies (15.27)–(15.29)} \end{array} \right\}. \tag{15.35}$$

We are now in a position to prove the existence of a solution to $(\mathbb{P}_\varepsilon)$.

Theorem 15.6. *The OCP $(\mathbb{P}_\varepsilon)$ has a solution if and only if this problem is regular, that is, $\Xi_\varepsilon \neq \emptyset$ for every fixed $\varepsilon > 0$.*

Proof. It is clear that we need to prove only sufficient conditions of this statement. Let us fix $\varepsilon > 0$. Since $\mathbf{U}_\varepsilon \neq \emptyset$, it obviously follows from Theorem 15.4 that the set Ξ_ε is nonempty as well. Hence, we may choose a minimizing sequence $\{(\overline{\boldsymbol{\alpha}}_{\varepsilon,m}, \mathbf{y}_{\varepsilon,m}, p_{\varepsilon,m}) \in \Xi_\varepsilon\}$ such that

$$\lim_{m\to\infty} \mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_{\varepsilon,m}, \mathbf{y}_{\varepsilon,m}) = \inf_{(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)\in\Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon).$$

The boundedness of $\{\mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_{\varepsilon,m}, \mathbf{y}_{\varepsilon,m})\}$ and implicit control constraints imply the boundedness of control prototypes $\left\{\|\mathbf{u}_{\varepsilon,m}\|_{\mathbf{H}^2(\Omega)}\right\}$ and $\left\{\|\nabla \mathbf{y}_{\varepsilon,m}\|_{\mathbf{L}^2(\Omega_\varepsilon)}\right\}$. Then using (15.34), we see that the set $\left\{\|\mathbf{y}_{\varepsilon,m}\|_{\mathbf{H}^1(\Omega_\varepsilon)}\right\}$ is also bounded independent of m. Hence, we may extract subsequences (still denoted by $\mathbf{u}_{\varepsilon,m}$ and $\mathbf{y}_{\varepsilon,m}$, respectively) such that $\mathbf{u}_{\varepsilon,m} \rightharpoonup \mathbf{u}_\varepsilon$ in $\mathbf{H}^2(\Omega)$ and $\mathbf{y}_{\varepsilon,m} \rightharpoonup \mathbf{y}_\varepsilon$ in $\mathbf{H}^1(\Omega)$ for some $(\mathbf{u}_\varepsilon, \mathbf{y}_\varepsilon) \in \mathbf{H}^2(\Omega)\times\mathbf{H}^1(\Omega)$ as $m \to \infty$. Since $\mathbf{u}_{\varepsilon,m} \in \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ for every $m \in \mathbb{N}$, $\|\mathbf{u}_{\varepsilon,m}\|_{\mathbf{H}^2(\Omega)} \le \gamma$, and $\mathbf{u}_{\varepsilon,m} \to \mathbf{u}_\varepsilon$ in $\mathbf{H}^1(\Omega)$, it follows that $\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{sol}(\Omega_\varepsilon)$, $\|\mathbf{u}_\varepsilon\|_{\mathbf{H}^2(\Omega)} \le \gamma$, that is, $\mathbf{u}_\varepsilon$ is an admissible control prototype ($\mathbf{u}_\varepsilon \in \mathbf{U}_\varepsilon$). On the other hand, since the trace of $\mathbf{H}^2(\Omega)$ equals $\mathbf{H}^{3/2}(\partial T_\varepsilon)$ and the space $\mathbf{H}^{3/2}(\partial T_\varepsilon)$ is compactly imbedded into $\mathbf{C}(\partial T_\varepsilon)$, we have that

$$\mathbf{u}_{\varepsilon,m} \rightharpoonup \mathbf{u}_\varepsilon \quad \text{in} \quad \mathbf{H}^2(\Omega)$$

implies

$$\mathbf{u}_{\varepsilon,m}|_{\partial T_\varepsilon^{\mathbf{k}_j}} \equiv \alpha_{\mathbf{k}_j,m} \to \alpha_{\mathbf{k}_j} \equiv \mathbf{u}_\varepsilon|_{\partial T_\varepsilon^{\mathbf{k}_j}} \quad \text{in} \quad \mathbf{C}(\Omega)$$

as $m \to \infty$ for all $j = 1, \ldots, J_\varepsilon$.

Moreover, due to the imbedding results for Sobolev spaces, we also get the strong convergence $\mathbf{y}_{\varepsilon,m} \to \mathbf{y}_\varepsilon$ in $\mathbf{L}^4(\Omega_\varepsilon)$ as $m \to \infty$. Then, using standard techniques in proving the existence of a solution to the steady-state Navier–Stokes equations, we may pass to the limit in the relation

$$\mathbf{y}_{\varepsilon,m} - \mathbf{u}_{\varepsilon,m} \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon),$$
$$\nu a_\varepsilon(\mathbf{y}_{\varepsilon,m}, \mathbf{v}) + c_\varepsilon(\mathbf{y}_{\varepsilon,m}, \mathbf{y}_{\varepsilon,m}, \mathbf{v}) = \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{v}\, \mathrm{d}x, \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon),$$

as $m \to \infty$ to conclude that $\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon)$ and $\mathbf{y}_\varepsilon$ satisfies (15.32). As a result, applying Theorem 15.4, we get that $\mathbf{y}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ and there exists a corresponding pressure $p_\varepsilon \in L^2_0(\Omega_\varepsilon)$ such that relations (15.27)–(15.28) hold true, that is, the triplet $(\mathbf{u}_\varepsilon|_{\partial T_\varepsilon}, \mathbf{y}_\varepsilon, p_\varepsilon)$ is admissible to the problem $(\mathbb{P}_\varepsilon)$.

Finally, using the compact imbedding results and the sequential weak lower semicontinuity of the cost functional $\mathcal{J}_\varepsilon(\cdot,\cdot)$ with respect to the product of the weak topologies of $\mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega_\varepsilon)$, we obtain

$$\mathcal{J}_\varepsilon(\mathbf{u}_\varepsilon|_{\partial T_\varepsilon}, \mathbf{y}_\varepsilon) \le \liminf_{m\to\infty} \mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_{\varepsilon,m}, \mathbf{y}_{\varepsilon,m}) = \inf_{(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon).$$

Hence, $(\mathbf{u}_\varepsilon|_{\partial T_\varepsilon}, \mathbf{y}_\varepsilon)$ is an optimal triplet for the problem $(\mathbb{P}_\varepsilon)$. □

Since the boundary value problem (15.9)–(15.12) may have a nonunique solution for a fixed boundary control, in what follows we define the binary relation $\langle L; \Xi_\varepsilon \rangle$ on each of the sets Ξ_ε by the rule $(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)\, L\, (\widehat{\overline{\boldsymbol{\alpha}}}_\varepsilon, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon)$ if and only if $\overline{\boldsymbol{\alpha}}_\varepsilon = \widehat{\overline{\boldsymbol{\alpha}}}_\varepsilon$. It is easily seen that $\langle L; \Xi_\varepsilon \rangle$ is an equivalence relation. So, hereinafter we will not distinguish the triplets belonging to the same class of equivalence.

15.4 Reformulation of the problem $(\mathbb{P}_\varepsilon)$

We begin this section with the description of the geometry of the perforated domain Ω_ε. We describe the class of admissible solutions to the OCP (15.8)–(15.12) in terms of singular periodic Borel measures on $\mathbb{R}^3$, using the approach of Zhikov, and Bouchitté and Fragala (see [26, 256]).

Let us denote by Q^r the homothetic contraction of the set Q with a factor of r^{-1}. In what follows, it is assumed that $0 < r \le 1$. Let η_0^r be the normalized periodic Borel measure on $\mathbb{R}^2$ with the periodicity cell $\widetilde{Y}$ such that η_0^r is concentrated and uniformly distributed on the set ∂Q^r, and is proportional to the 1D Hausdorff measure. It is clear that in this case, $\eta_0^r(\widetilde{Y} \setminus \partial Q^r) = 0$.

Now, we consider the measure

$$\mathrm{d}\eta^r = \mathrm{d}\eta_0^r \times \mathrm{d}x_3 \quad \text{on} \quad Y = [-1/2, 1/2)^2 \times [0, 1).$$

It is easy to see that this measure concentrated on the set $\partial Q^r \times [0, 1)$, and for any smooth function g, we have

$$\int_Y g \,\mathrm{d}\eta^r = \int_0^1 \int_{\widetilde{Y}} g \,\mathrm{d}x_3 \,\mathrm{d}\eta_0^r = \left[\mathcal{H}^2(\partial Q^r \times [0, 1))\right]^{-1} \int_{\partial Q^r \times [0,1)} g \,\mathrm{d}\mathcal{H}^2.$$

However, as follows from the properties of the Hausdorff measure, we have

$$\mathcal{H}^2(\partial Q^r \times [0, 1)) = \mathcal{H}^1(\partial Q^r) = r\mathcal{H}^1(\partial Q)$$

(see [106]). Using the notation $|\partial Q|_H = \mathcal{H}^1(\partial Q)$, the previous relation can be rewritten in the form

$$r \int_Y g \,\mathrm{d}\eta^r = r \int_0^1 \int_{\widetilde{Y}} g \,\mathrm{d}x_3 \,\mathrm{d}\eta_0^r = |\partial Q|_H^{-1} \int_{\partial Q \times (0,1)} g \,\mathrm{d}\mathcal{H}^2. \tag{15.36}$$

Thus, $|\partial Q|_H$ is the 1D Hausdorff measure of the set ∂Q.

We introduce the scaling measure η_ε^r by setting $\eta_\varepsilon^r(B) = \varepsilon^3 \eta^r(\varepsilon^{-1}B)$ for every Borel set $B \subset \mathbb{R}^3$. The parameters r and ε are related by the rule $r(\varepsilon) = r_\varepsilon$, where $0 < r_\varepsilon \le 1$ and $\lim_{\varepsilon \to 0} r_\varepsilon = 0$. Then

$$\int_{\varepsilon Y} \mathrm{d}\eta_\varepsilon^r = \varepsilon^3 \int_Y \mathrm{d}\eta^r = \varepsilon^3.$$

It means that the measure η_ε^r weakly converges to the Lebesgue measure: $\mathrm{d}\eta_\varepsilon^r \rightharpoonup \mathrm{d}x$, that is, for every $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} \varphi \,\mathrm{d}\eta_\varepsilon^r = \int_{\mathbb{R}^3} \varphi \,\mathrm{d}x. \tag{15.37}$$

By analogy, it can be shown that the scaling measure $\eta_{0,\varepsilon}^r$ on $\mathbb{R}^2$ (by definition, $\eta_{0,\varepsilon}^r(B) = \varepsilon^2 \eta_0^r(\varepsilon^{-1}B)$ for every Borel set $B \subset \mathbb{R}^2$) weakly converges to the 2D Lebesgue measure.

Remark 15.7. It is easy to see that the scaling measure η_ε^r belongs to the class $\mathcal{M}_0^+(\Omega)$. Note that if $\eta \in \mathcal{M}_0^+(\Omega)$, then the functions of $H^1(\Omega)$ are defined η-almost everywhere and are η-measurable on Ω. Hence, the space $\mathbf{H}^1(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$ is well defined (see Sect. 2.6).

We note that, in view of relation (15.36), the term

$$\frac{\beta\varepsilon}{r_\varepsilon} \sum_{j=1}^{J_\varepsilon} \int_{\partial T_\varepsilon^{\mathbf{k}_j}} \left|\alpha_{\mathbf{k}_j}\right|^2 \mathrm{d}\mathcal{H}^2$$

can be rewritten in the equivalent form

$$\frac{\beta\varepsilon}{r_\varepsilon}\sum_{j=1}^{J_\varepsilon}\int_{\partial T_\varepsilon^{\mathbf{k}_j}}\left|\alpha_{\mathbf{k}_j}\right|^2\,\mathrm{d}\mathcal{H}^2$$

$$=\beta\varepsilon^2|\partial Q|_H\sum_{j=1}^{J_\varepsilon}\int_0^1\int_{\varepsilon(\widetilde{Y}+\mathbf{k}_j)}|\mathbf{u}_\varepsilon|^2\,\mathrm{d}\eta_0^{r(\varepsilon)}(x/\varepsilon)\,\mathrm{d}x_3$$

$$=\beta\varepsilon^3|\partial Q|_H\sum_{j=1}^{J_\varepsilon}\int_{\varepsilon(Y+\mathbf{k}_j)}|\mathbf{u}_\varepsilon|^2\,\mathrm{d}\eta^{r(\varepsilon)}(x/\varepsilon)$$

$$=\beta|\partial Q|_H\int_\Omega|\mathbf{u}_\varepsilon|^2\,\mathrm{d}\eta_\varepsilon^r\,, \tag{15.38}$$

where $\mathbf{u}_\varepsilon$ is a prototype of the control function $\overline{\boldsymbol{\alpha}}_\varepsilon=\left(\alpha_{\mathbf{k}_1},\alpha_{\mathbf{k}_2},\ldots,\alpha_{\mathbf{k}_{J_\varepsilon}}\right)$. As a result, the original cost functional (15.8) takes the form

$$\mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon,\mathbf{y}_\varepsilon)=\lambda\int_\Omega\chi_\varepsilon|\nabla\breve{\mathbf{y}}_\varepsilon|^2\,\mathrm{d}x+\beta|\partial Q|_H\int_\Omega|\mathbf{u}_\varepsilon|^2\,\mathrm{d}\eta_\varepsilon^{r(\varepsilon)}. \tag{15.39}$$

Here, χ_ε is the characteristic function of the perforated domain Ω_ε, and $\breve{\mathbf{y}}_\varepsilon\in\mathbf{H}^1(\Omega)$ is "some" extension of the function $\mathbf{y}_\varepsilon$ onto the whole Ω.

Remark 15.8. Note that any admissible control $\overline{\boldsymbol{\alpha}}_\varepsilon=\left(\alpha_{\mathbf{k}_1},\alpha_{\mathbf{k}_2},\ldots,\alpha_{\mathbf{k}_{J_\varepsilon}}\right)\in\mathbf{U}_\varepsilon$ can be obviously interpreted as an element of the space $\mathbf{L}^2(\Omega,d\eta_\varepsilon^{r(\varepsilon)})$. Indeed, let $\mathbf{u}_\varepsilon\in\mathbf{H}^1_{sol}(\Omega_\varepsilon)\cap\mathbf{H}^2(\Omega)$ be a prototype of $\overline{\boldsymbol{\alpha}}_\varepsilon$. Then, using the imbedding result $\mathbf{H}^2(\Omega)\hookrightarrow\mathbf{C}(\Omega)$, we get that $\mathbf{u}_\varepsilon\in\mathbf{C}(\Omega)$. Hence, $\mathbf{u}_\varepsilon$ is a $\eta_\varepsilon^r(\varepsilon)$-measurable function such that

$$\|\mathbf{u}_\varepsilon\|^2_{\mathbf{L}^2(\Omega,\mathrm{d}\eta_\varepsilon^{r(\varepsilon)})}=\int_\Omega|\mathbf{u}_\varepsilon|^2\,\mathrm{d}\eta_\varepsilon^r\le\|\mathbf{u}_\varepsilon\|^2_{\mathbf{C}(\Omega)}\eta_\varepsilon^r(\overline{\Omega})<+\infty. \tag{15.40}$$

So, $\mathbf{u}_\varepsilon\in\mathbf{L}^2(\Omega,\mathrm{d}\eta_\varepsilon^{r(\varepsilon)})$, and we obtain the required result.

Definition 15.9. *Let $(\overline{\boldsymbol{\alpha}}_\varepsilon,\mathbf{y}_\varepsilon,p_\varepsilon)$ be any admissible solution to the problem $(\mathbb{P}_\varepsilon)$. Then we say that a triplet $(\mathbf{u}_\varepsilon,\breve{\mathbf{y}}_\varepsilon,\breve{p}_\varepsilon)\in\mathbb{X}_\varepsilon$ is a prototype to $(\overline{\boldsymbol{\alpha}}_\varepsilon,\mathbf{y}_\varepsilon,p_\varepsilon)$ if*

$$\mathbb{X}_\varepsilon=\left[\mathbf{H}^1_{sol}(\Omega_\varepsilon)\cap\mathbf{H}^2(\Omega)\cap\mathbf{L}^2(\Omega,\mathrm{d}\eta_\varepsilon^{r(\varepsilon)})\right]\times\left[\mathbf{H}^1_{sol}(\Omega_\varepsilon)\cap\mathbf{H}^1(\Omega)\right]\times L^2_0(\Omega),$$

$\mathbf{u}_\varepsilon$ is a control prototype, and $(\breve{\mathbf{y}}_\varepsilon,\breve{p}_\varepsilon)$ are some extensions of the functions $(\mathbf{y}_\varepsilon,p_\varepsilon)$ on the whole Ω.

Remark 15.10. Let us recall that the perforated domain Ω_ε considered here satisfies the so-called "condition of strong connectedness" (see [182]). It means that there exist a family $\{\mathbf{P}_\varepsilon\}_{\varepsilon>0}$ of extension operators $\mathbf{P}_\varepsilon:\mathbf{H}^1(\Omega_\varepsilon)\to\mathbf{H}^1(\Omega)$ and a constant C independent of ε and $\mathbf{y}_\varepsilon$ such that

$$\|(\mathbf{P}_\varepsilon\mathbf{y}_\varepsilon)\|_{\mathbf{H}^1(\Omega)}\le C\,\|\mathbf{y}_\varepsilon\|_{\mathbf{H}^1(\Omega_\varepsilon)}\quad\text{for every}\quad\mathbf{y}_\varepsilon\in\mathbf{H}^1(\Omega_\varepsilon).$$

So, we can assume that $\breve{\mathbf{y}}_\varepsilon := \mathbf{P}_\varepsilon \mathbf{y}_\varepsilon$ for some extension operator with the above properties. The main problem is to find an a priori estimate for the pressure p_ε, which yields the existence of the corresponding prototype $\breve{p}_\varepsilon \in L^2_0(\Omega)$ with a uniformly bounded norm in $L^2_0(\Omega)$. In fact, it is not obvious how to construct such an extension of the pressure p_ε. For that purpose, following Allaire [5], we introduce an abstract framework of hypotheses on the cylindrical holes.

We denote by $\{\mathbf{e}_k\}_{k=1,2,3}$ the canonical basis in $\mathbb{R}^3$ and by $\tilde{}$ the extension by 0 onto the cylindrical holes.

Hypotheses (H1)–(H5). Let us assume that there exist functions

$$(\mathbf{w}_k^\varepsilon, q_k^\varepsilon, \mu_k) \quad (1 \le k \le 3)$$

and a linear map

$$R_\varepsilon \in \mathcal{L}\left(\mathbf{H}^1_0(\Omega); \mathbf{H}^1_0(\Omega_\varepsilon)\right)$$

such that the following hold:

(H1) $\mathbf{w}_k^\varepsilon \in \mathbf{H}^1(\Omega)$, $q_k^\varepsilon \in L^2_0(\Omega)$, $\mu_k \in \mathbf{W}^{-1,\infty}(\Omega)$.
(H2) $\nabla \cdot \mathbf{w}_k^\varepsilon = 0$ on Ω and $\mathbf{w}_k^\varepsilon = 0$ on T_ε.
(H3) $\mathbf{w}_k^\varepsilon \rightharpoonup \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$, and $q_k^\varepsilon \rightharpoonup 0$ in $L^2_0(\Omega)$.
(H4) $\forall\, \mathbf{v}_\varepsilon \in \mathbf{H}^1(\Omega)$ and $\forall\, \mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ in $\mathbf{H}^1(\Omega)$, $\mathbf{v}_\varepsilon = 0$ on the cylinders T_ε, it follows that

$$\lim_{\varepsilon \to 0} \langle \nabla q_k^\varepsilon - \nu \Delta \mathbf{w}_k^\varepsilon, \varphi\, \mathbf{v}_\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)} = \langle \mu_k, \varphi\, \mathbf{v} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)}, \quad \forall\, \varphi \in C^\infty_0(\Omega).$$

(H5) If $\mathbf{y} \in \mathbf{H}^1_0(\Omega_\varepsilon)$, then $R_\varepsilon(\chi_\varepsilon \mathbf{y}) = \mathbf{y}$ on Ω_ε; if $\nabla \cdot \mathbf{y} = 0$ on Ω, then $\nabla \cdot (R_\varepsilon\, \mathbf{y}) = 0$ on Ω_ε;

$$\|R_\varepsilon\, \mathbf{y}\|_{\mathbf{H}^1_0(\Omega_\varepsilon)} \le C \|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)}, \quad \text{and} \quad C \quad \text{does not depend on} \quad \varepsilon.$$

Remark 15.11. In this case, following Tartar's [236] idea, a linear continuous extension operator $P_\varepsilon \in \mathcal{L}(L^2_0(\Omega_\varepsilon); L^2_0(\Omega))$ can be constructed as

$$\langle \nabla [P_\varepsilon q_\varepsilon], \mathbf{w} \rangle_{H^{-1}(\Omega); H^1_0(\Omega)} = \langle \nabla q_\varepsilon, R_\varepsilon \mathbf{w} \rangle_{H^{-1}(\Omega_\varepsilon); H^1_0(\Omega_\varepsilon)}, \quad \forall\, \mathbf{w} \in \mathbf{H}^1_0(\Omega). \quad (15.41)$$

Then it can be easily proved that the following properties hold true (see [5, Theorem 1.1.8]):

(i) $P_\varepsilon q_\varepsilon = q_\varepsilon$ in $L^2_0(\Omega_\varepsilon)$;
(ii) $\|P_\varepsilon q_\varepsilon\|_{L^2_0(\Omega)} \le C \|q_\varepsilon\|_{L^2_0(\Omega_\varepsilon)}$;
(iii) $\|\nabla [P_\varepsilon q_\varepsilon]\|_{\mathbf{H}^{-1}(\Omega)} \le C \|\nabla q_\varepsilon\|_{\mathbf{H}^{-1}(\Omega_\varepsilon)}$,

where the constant C is independent of q_ε and ε.

15.5 Convergence in the variable space $\mathbb{X}_\varepsilon$

The characteristic feature of the OCP $(\widehat{\mathbb{P}}_\varepsilon)$ is the fact that for every fixed value of ε, each of the admissible triplets $(\mathbf{u}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \widehat{\Xi}_\varepsilon$ belongs to the corresponding functional space $\mathbb{X}_\varepsilon$. In view of this, we will use the convergence concept in variable spaces following definitions of Sect. 6.2.

We observe that, in fact, the characteristic function χ_ε of the perforated domain Ω_ε is a two-parametric and ε-periodic one, that is,

$$\chi_\varepsilon(x) = \chi^r_\varepsilon(x) = \chi^r\left(\frac{x}{\varepsilon}\right), \quad \chi^r(x) = \begin{cases} 1 & \text{if } x \in Y \setminus [Q^r \times [0,1)], \\ 0 & \text{if } x \in Q^r \times [0,1) = rQ \times [0,1). \end{cases}$$

It means that the Radon measure $\chi_\varepsilon(x)\mathrm{d}x$ can be viewed as a scaling measure $\mathrm{d}\eta^r_\varepsilon$ such that $\eta^r_\varepsilon(B) = \varepsilon^3\eta^r(\varepsilon^{-1}B)$ for every Borel set $B \subset \mathbb{R}^3$, where η^r is the Y-periodic Borel measure on $\mathbb{R}^3$ concentrated on $Y \setminus [Q^r \times [0,1)]$ and proportional there to the Lebesgue measure $\mathcal{L}_3$, that is,

$$\int_Y \mathrm{d}\eta^r = |\widetilde{Y} \setminus Q^r|_{\mathcal{L}_2} = |\widetilde{Y}|_{\mathcal{L}^2} - r^2\,|Q|_{\mathcal{L}_2} = 1 - r^2\,|Q|_{\mathcal{L}_2}.$$

Therefore, $\mathrm{d}\eta^r \rightharpoonup |\widetilde{Y}|_{\mathcal{L}_2}\,\mathrm{d}x = \mathrm{d}x$ as $r \to 0$, and

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi\chi_\varepsilon\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_\Omega \varphi\chi^{r(\varepsilon)}\left(\frac{x}{\varepsilon}\right)\mathrm{d}x = |Y|\int_\Omega \varphi\,\mathrm{d}x = \int_\Omega \varphi\,\mathrm{d}x \qquad (15.42)$$

for every $\varphi \in C_0^\infty(\Omega)$.

Taking the definition of strong convergence in the variable space $L^2(\Omega, \chi_\varepsilon\,\mathrm{d}x)$ and relation (15.42) into account, we get the following obvious result.

Lemma 15.12. *χ_ε converges strongly to 1 both in $L^2(\Omega)$ and in the variable space $L^2(\Omega, \chi_\varepsilon\,\mathrm{d}x)$ as $\varepsilon \to 0$.*

To introduce the convergence formalism for the sequences

$$\{(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0},$$

we begin with the following concepts.

Definition 15.13. *We say that a sequence of controls $\{\overline{\boldsymbol{\alpha}}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ w_a-converges to a function $\mathbf{a}_0$ if some sequence of its prototypes*

$$\left\{\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, \mathrm{d}\eta^{r(\varepsilon)}_\varepsilon)\right\}_{\varepsilon>0}$$

converges to $\mathbf{a}_0$ weakly in $\mathbf{H}^2(\Omega)$.

Definition 15.14. *We say that a sequence $\{\overline{\boldsymbol{\alpha}}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ w_b-converges to a function $\mathbf{b}_0 \in \mathbf{L}^2(\Omega)$ if some sequence of its prototypes*

$$\left\{\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^{r(\varepsilon)})\right\}_{\varepsilon>0}$$

converges to $\mathbf{b}_0$ *weakly in* $\mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^r)$*, that is,*

$$\limsup_{\varepsilon\to 0} \|\mathbf{u}_\varepsilon\|_{\mathbf{L}^2(\Omega,\mathrm{d}\eta_\varepsilon^r)} < +\infty \quad \textit{and} \quad \mathbf{u}_\varepsilon^r \rightharpoonup \mathbf{b}_0 \quad \textit{in} \quad \mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^r). \tag{15.43}$$

In order to relate the w_a- and w_b-limits, we prove the following result.

Lemma 15.15. *Any sequence of admissible controls* $\{\overline{\boldsymbol{\alpha}}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ *contains a subsequence for which the* w_a*- and* w_b*-limits coincide almost everywhere on the domain* Ω*.*

Proof. Let $\left\{\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^{r(\varepsilon)})\right\}_{\varepsilon>0}$ be a sequence of some control prototypes. Since this sequence is bounded in $\mathbf{H}^2(\Omega)$, we may suppose that there is an element $\mathbf{a}_0 \in \mathbf{H}^2(\Omega)$ and a subsequence of $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ (still denoted by the same index) such that $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{a}_0$ as $\varepsilon \to 0$.

By the Sobolev imbedding theorem, we have that $\mathbf{a}_0 \in \mathbf{C}(\Omega)$ and $\mathbf{u}_\varepsilon \to \mathbf{u}_0$ in $\mathbf{C}(\Omega)$. Then

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_\Omega |\mathbf{u}_\varepsilon|^2 \, \mathrm{d}\eta_\varepsilon^r \\
&\qquad \le 2 \limsup_{\varepsilon\to 0} \int_\Omega |\mathbf{u}_\varepsilon - \mathbf{a}_0|^2 \, \mathrm{d}\eta_\varepsilon^r + 2 \limsup_{\varepsilon\to 0} \int_\Omega |\mathbf{a}_0|^2 \, \mathrm{d}\eta_\varepsilon^r \\
&\qquad \le 2 \limsup_{\varepsilon\to 0} \|\mathbf{u}_\varepsilon - \mathbf{a}_0\|^2_{\mathbf{C}(\Omega)} \, \eta_\varepsilon^r\left(\overline{\Omega}\right) + 2 \|\mathbf{a}_0\|^2_{\mathbf{C}(\Omega)} \, \eta_\varepsilon^r\left(\overline{\Omega}\right) \\
&\qquad = 2 \|\mathbf{a}_0\|^2_{\mathbf{C}(\Omega)} \, \eta_\varepsilon^r\left(\overline{\Omega}\right).
\end{aligned}$$

Hence, the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^r)$, and by the compactness criterium of the weak convergence in $\mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^r)$, we may suppose that there is an element $\mathbf{b}_0 \in \mathbf{L}^2(\Omega)$ such that $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{b}_0$ in $\mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^r)$ (passing to a subsequence, if it is necessary). On the other hand, for any function $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega \varphi \, (\mathbf{a}_0 - \mathbf{b}_0) \, \mathrm{d}x \\
&\qquad \le \left| \int_\Omega \varphi \mathbf{a}_0 \, \mathrm{d}x - \int_\Omega \varphi \mathbf{a}_0 \, \mathrm{d}\eta_\varepsilon^r \right| + \left| \int_\Omega \varphi \, (\mathbf{a}_0 - \mathbf{u}_\varepsilon) \, \mathrm{d}\eta_\varepsilon^r \right| \\
&\qquad + \left| \int_\Omega \varphi \mathbf{u}_\varepsilon \, \mathrm{d}\eta_\varepsilon^r - \int_\Omega \varphi \, \mathbf{b}_0 \, \mathrm{d}x \right| \le \left| \int_\Omega \varphi \mathbf{a}_0 \, \mathrm{d}x - \int_\Omega \varphi \mathbf{a}_0 \, \mathrm{d}\eta_\varepsilon^r \right| \\
&\qquad + \|\mathbf{u}_\varepsilon - \mathbf{a}_0\|_{\mathbf{C}(\Omega)} \int_\Omega |\varphi| \, \mathrm{d}\eta_\varepsilon^r + \left| \int_\Omega \varphi \mathbf{u}_\varepsilon \, \mathrm{d}\eta_\varepsilon^r - \int_\Omega \varphi \, \mathbf{b}_0 \, \mathrm{d}x \right| \\
&\qquad = I_1 + I_2 + I_3.
\end{aligned} \tag{15.44}$$

Owing to the weak convergence $\mathrm{d}\eta_\varepsilon^r \rightharpoonup \mathrm{d}x$ and to the fact that $(\varphi \, \mathbf{a}_0) \in \mathbf{C}_0(\Omega)$, we obtain $I_1 \to 0$ as $\varepsilon \to 0$. By analogy, we also have that $I_2 \to 0$ as $\varepsilon \to 0$.

Then, taking into account (15.43), inequality (15.44) leads us to the following conclusion:

$$\int_\Omega \varphi\,(\mathbf{a}_0 - \mathbf{b}_0)\,\mathrm{d}x = 0, \quad \forall\,\varphi \in C_0^\infty(\Omega),$$

that is, $\mathbf{a}_0 = \mathbf{b}_0$ almost everywhere in Ω. The proof is complete. □

As a consequence, the following statements are readily true.

Lemma 15.16. *Let $\{\overline{\boldsymbol{\alpha}}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible controls. Then the weak limits in $\mathbf{H}^2(\Omega)$ of any weakly convergent sequences of prototypes*

$$\left\{\mathbf{u}_\varepsilon^{(1)} \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^{r(\varepsilon)})\right\}_{\varepsilon>0}$$

and

$$\left\{\mathbf{u}_\varepsilon^{(2)} \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^{r(\varepsilon)})\right\}_{\varepsilon>0}$$

are the same.

Lemma 15.17. *Any sequence of admissible controls $\{\overline{\boldsymbol{\alpha}}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ is relatively compact with respect to the w_a-convergence. Moreover, its w_a-limit $\mathbf{u}_0$ belongs to the set*

$$\mathbf{U} = \left\{\mathbf{u} \in \mathbf{H}^2(\Omega) \;:\; \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma\right\}. \tag{15.45}$$

Using the above results, we are now able to introduce the convergence concept for the admissible triplets. As follows from Remark 15.10, for any uniformly bounded sequence of functions $\left\{\mathbf{y}_\varepsilon \in \mathbf{H}^1(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ there are extension operators $\mathbf{P}_\varepsilon : \mathbf{H}^1(\Omega_\varepsilon) \to \mathbf{H}^1(\Omega)$ and a constant C independent of ε such that

$$\|\breve{\mathbf{y}}_\varepsilon\|_{\mathbf{H}^1(\Omega)} \equiv \|(\mathbf{P}_\varepsilon \mathbf{y}_\varepsilon)\|_{\mathbf{H}^1(\Omega)} \le C\,\|\mathbf{y}_\varepsilon\|_{\mathbf{H}^1(\Omega_\varepsilon)} \quad \text{for every} \quad \varepsilon.$$

Let us suppose that there are two different bounded sequences of prototypes

$$\{\breve{\mathbf{y}}_\varepsilon^{(1)} = \mathbf{P}_\varepsilon^{(1)}(\mathbf{y}_\varepsilon)\}_{\varepsilon>0} \quad \text{and} \quad \{\breve{\mathbf{y}}_\varepsilon^{(2)} = \mathbf{P}_\varepsilon^{(2)}(\mathbf{y}_\varepsilon)\}_{\varepsilon>0}$$

such that $\breve{\mathbf{y}}_\varepsilon^{(1)} \rightharpoonup \mathbf{y}_1^*$ and $\breve{\mathbf{y}}_\varepsilon^{(2)} \rightharpoonup \mathbf{y}_2^*$ in $\mathbf{H}^1(\Omega)$. Then using Lemma 15.12 and passing to the limit in the integral identity

$$\int_\Omega \chi_\varepsilon \breve{\mathbf{y}}_\varepsilon^{(1)} \cdot \varphi\,\mathrm{d}x = \int_\Omega \chi_\varepsilon \breve{\mathbf{y}}_\varepsilon^{(2)} \cdot \varphi\,\mathrm{d}x, \quad \forall\,\varphi \in \mathbf{H}^1(\Omega),$$

as ε tends to 0, we obtain

$$\int_\Omega \mathbf{y}_1^* \cdot \varphi\,\mathrm{d}x = \int_\Omega \mathbf{y}_2^* \cdot \varphi\,\mathrm{d}x, \quad \forall\,\varphi \in \mathbf{H}^1(\Omega).$$

Hence, $\mathbf{y}_1^* = \mathbf{y}_2^*$. In view of this, we give the following notion (we refer to Sects. 6.1 and 6.6 for the details).

Definition 15.18. *We say that a bounded sequence* $\{(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *is w-convergent to a triplet*

$$(\mathbf{u}, \mathbf{y}, p) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$$

in the variable space $\mathbb{X}_\varepsilon$ *as* ε *tends to* 0

$$\Big(\textit{in symbols, } (\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \overset{w}{\longrightarrow} (\mathbf{u}, \mathbf{y}, p)\Big)$$

if some bounded sequence of its prototypes $\big\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon) \in \widehat{\Xi}_\varepsilon\big\}_{\varepsilon>0}$ *converges to* $(\mathbf{u}, \mathbf{y}, p)$ *in the following sense:*

(i) $\mathbf{u}_\varepsilon \overset{w_a}{\longrightarrow} \mathbf{u}$ *in* $\mathbf{H}^2(\Omega)$*;*
(ii) $\breve{p}_\varepsilon \rightharpoonup p$ *in* $L^2_0(\Omega)$*;*
(iii) $\breve{\mathbf{y}}_\varepsilon \rightharpoonup \mathbf{y}$ *in* $\mathbf{H}^1(\Omega)$*.*

As a consequence, we have the following result.

Theorem 15.19. *Let* $\{(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *be a sequence of admissible triplets for the* $\mathbb{P}_\varepsilon$*-problems. Then there exist a subsequence* $\{(\overline{\boldsymbol{\alpha}}_{\varepsilon'}, \mathbf{y}_{\varepsilon'}, p_{\varepsilon'})\}_{\varepsilon'>0}$ *and a triplet*

$$(\mathbf{u}, \mathbf{y}, p) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$$

such that $\mathbf{u} \in \mathbf{U}$*,* $(\overline{\boldsymbol{\alpha}}_{\varepsilon'}, \mathbf{y}_{\varepsilon'}, p_{\varepsilon'}) \overset{w}{\longrightarrow} (\mathbf{u}, \mathbf{y}, p)$*, and*

$$\mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega). \tag{15.46}$$

Proof. We set $\breve{\mathbf{y}}_\varepsilon = \mathbf{u}_\varepsilon$ in $\Omega \setminus \Omega_\varepsilon$. Thanks to Lemma 15.17, Proposition 15.5, and Remark 15.10, the sequence $\{\breve{\mathbf{y}}_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $\mathbf{H}^1(\Omega)$. Let

$$\Big\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon) \in \widehat{\Xi}_\varepsilon\Big\}_{\varepsilon>0} \tag{15.47}$$

be a sequence of some prototypes, where $\breve{p}_\varepsilon = P_\varepsilon(p_\varepsilon) \in L^2_0(\Omega)$ (see Remark 15.11). Then the sequence of pressures $\big\{\breve{p} \in L^2_0(\Omega)\big\}_{\varepsilon>0}$ is also uniformly bounded. So, the sequence (15.47) is relatively compact with respect to the weak convergence in the space $\mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$. Let $(\mathbf{u}, \mathbf{y}, p) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ be its weak limit.

We aim to show that $\mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega)$. Since

$$\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon) \cap \mathbf{H}^1_0(\Omega)$$

and

$$\operatorname{div} : \mathbf{H}^1_0(\Omega_\varepsilon) \mapsto L^2(\Omega_\varepsilon)/\mathbb{R} = \left\{ g \in L^2(\Omega_\varepsilon) \ : \ \int_{\Omega_\varepsilon} g(x)\,\mathrm{d}x = 0 \right\},$$

it follows that $\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}(\operatorname{div}, \Omega_\varepsilon)$ for every ε. Therefore,

$$0 = \int_{\Omega_\varepsilon} \varphi \operatorname{div}(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon)\,dx = -\int_{\Omega_\varepsilon} (\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla\varphi\,dx, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^3), \tag{15.48}$$

$\forall\,\varepsilon > 0$ (see Lemma 15.1). However, we have

$$\int_\Omega \varphi\,\chi_\varepsilon \operatorname{div}(\check{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\,dx = -\int_\Omega \chi_\varepsilon(\check{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla\varphi\,dx, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^3).$$

Hence,

$$\operatorname{div}\,[\chi_\varepsilon(\check{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)] = \chi_\varepsilon \operatorname{div}(\check{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon),$$

and we get

$$\chi_\varepsilon(\check{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \in \mathbf{H}(\operatorname{div}, \Omega).$$

Then, due to Lemma 15.12, we can pass to the limit in (15.48). As a result, we have

$$0 = \lim_{\varepsilon\to 0}\int_\Omega \chi_\varepsilon(\check{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla\varphi\,dx = \int_\Omega (\mathbf{y}-\mathbf{u})\nabla\varphi\,dx, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^3).$$

Thus, $(\mathbf{y}-\mathbf{u}) \in H^1_{0,div}(\Omega)$, and this concludes the proof. □

Corollary 15.20. *The supposition of Theorem 15.19 and condition (15.14) imply the inclusions*

$$\begin{gathered} \mathbf{y} \in \mathbf{H}^1_{sol}(\Omega), \\ \mathbf{u} \in \mathbf{U} = \left\{\mathbf{u} \in \mathbf{H}^1_{sol}(\Omega) \cap \mathbf{H}^2(\Omega) \;:\; \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma\right\}. \end{gathered} \tag{15.49}$$

Proof. Since $\mathbf{u}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega)$ for every ε, it follows that

$$\int_{\Omega_\varepsilon} v \operatorname{div}\mathbf{u}_\varepsilon\,dx = 0 \quad \text{for every} \quad v \in H^1(\Omega).$$

On the other hand, $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ in $\mathbf{H}^2(\Omega)$ as $\varepsilon \to 0$. Hence, taking into account (15.25), we get

$$\begin{aligned} 0 = \lim_{\varepsilon\to 0}\int_\Omega \chi_\varepsilon\,\mathbf{u}_\varepsilon\cdot\nabla v\,dx &= \int_\Omega \mathbf{u}\cdot\nabla v\,dx \\ &= \langle \operatorname{div}\mathbf{u}, v\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}, \quad \forall\, v \in H^1_0(\Omega). \end{aligned}$$

Since $\operatorname{div}\mathbf{u} \in L^2(\Omega)$, this yields $\operatorname{div}\mathbf{u} = 0$. As a result, we have $\mathbf{u} \in \mathbf{H}^1_{sol}(\Omega)$. The inclusion $\mathbf{y} \in \mathbf{H}^1_{sol}(\Omega)$ immediately follows from the previous one and (15.46). □

15.6 Definition of suboptimal controls

The main question we are going to consider in this section concerns the approximation of the optimal solutions to the original problem $(\mathbb{P}_\varepsilon)$ for ε small

enough. We focus our attention on the possibility of defining suboptimal solutions which have to guarantee the closeness of the corresponding value of the cost functional $\mathcal{J}_\varepsilon(\mathbf{u}_\varepsilon^{sub}, \mathbf{y}_\varepsilon^{sub})$ to its minimum if ε is small enough. In contrast to Definition 11.30, we introduce the following concept.

Definition 15.21. *We say that a sequence of functions*

$$\left\{\overline{\boldsymbol{\alpha}}_\varepsilon^{sub} = \left(\alpha_{\mathbf{k}_1}^{sub}, \alpha_{\mathbf{k}_2}^{sub}, \dots, \alpha_{\mathbf{k}_{\mathrm{J}_\varepsilon}}^{sub}\right)\right\}_{\varepsilon>0}$$

is an asymptotically suboptimal control for the problem $(\mathbb{P}_\varepsilon)$ *if*

$$\alpha_{\mathbf{k}_j}^{sub} \in \mathbf{H}^{1/2}(\partial T_\varepsilon^{\mathbf{k}_j}), \quad \int_{\partial T_\varepsilon^{\mathbf{k}_j}} \mathbf{n} \cdot \alpha_{\mathbf{k}_j}^{sub}\, \mathrm{d}\mathcal{H}^2 = 0, \quad \forall j = 1, \dots, \mathrm{J}_\varepsilon, \tag{15.50}$$

and for every $\delta > 0$*, there is* $\varepsilon_0 > 0$ *such that*

$$\left| \inf_{(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon) - \mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon^{sub}, \mathbf{y}_\varepsilon^{sub}) \right| < \delta, \quad \forall\, \varepsilon < \varepsilon_0, \tag{15.51}$$

where $\mathbf{y}_\varepsilon^{sub} = \mathbf{y}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon^{sub})$ *denotes the corresponding solution of the boundary value problem (15.9)–(15.12).*

To construct such controls, we use the variational convergence of constrained minimization problems (see Sect. 8.4). In view of this, we study the asymptotic behavior of the problem $(\mathbb{P}_\varepsilon)$ as $\varepsilon \to 0$. We represent the $\mathbb{P}_\varepsilon$-problem for various values of ε in the form of the following sequence:

$$\left\{ \left\langle \inf_{(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon) \right\rangle ;\, \varepsilon > 0 \right\}. \tag{15.52}$$

Then the definition of an appropriate limit problem to the family (15.52) can be reduced to the analysis of the limit properties of this sequence as $\varepsilon \to 0$. To get this limit in the form of some constrained minimization problem, we introduce the following definition (for comparison, see Definition 8.24).

Definition 15.22. *We say that a minimization problem*

$$\left\langle \inf_{(\mathbf{u}, \mathbf{y}, p) \in \Xi_0} \mathcal{J}_0(\mathbf{u}, \mathbf{y}) \right\rangle \tag{15.53}$$

is the variational w-limit of the sequence (15.52) if the following conditions are satisfied:

(i) If a sequence $\{(\overline{\boldsymbol{\alpha}}_k, \mathbf{y}_k, p_k)\}_{k \in \mathbb{N}}$ *w-converges to a triplet* $(\mathbf{u}, \mathbf{y}, p)$ *as* $k \to \infty$ *and there exists a subsequence* $\{\varepsilon_k\}$ *of* $\{\varepsilon\}$ *such that* $\varepsilon_k \to 0$ *as* $k \to \infty$ *and* $(\overline{\boldsymbol{\alpha}}_k, \mathbf{y}_k, p_k) \in \Xi_{\varepsilon_k}$ *for all* k*, then*

$$(\mathbf{u}, \mathbf{y}, p) \in \Xi_0; \quad \mathcal{J}_0(\mathbf{u}, \mathbf{y}) \le \liminf_{k \to \infty} \mathcal{J}_{\varepsilon_k}(\overline{\boldsymbol{\alpha}}_k, \mathbf{y}_k). \tag{15.54}$$

(ii) For every equivalence class Ξ_0/L, there exist a triplet $(\mathbf{u},\mathbf{y},p) \in \Xi_0/L$ and a realizing sequence $\{(\overline{\boldsymbol{\alpha}}_\varepsilon,\mathbf{y}_\varepsilon,p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ such that

$$(\overline{\boldsymbol{\alpha}}_\varepsilon,\mathbf{y}_\varepsilon,p_\varepsilon) \overset{w}{\longrightarrow} (\mathbf{u},\mathbf{y},p), \text{ and } \mathcal{J}_0(\mathbf{u},\mathbf{y}) \geq \limsup_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\boldsymbol{\alpha}_\varepsilon,\mathbf{y}_\varepsilon). \quad (15.55)$$

The following theorem deals with the main property of the variational w-limit problems (for the proof, we refer to Theorem 9.26).

Theorem 15.23. *Assume that (15.53) is a weak variational w-limit of the sequence (15.52), and that this problem has a solution. Let $\left\{(\overline{\boldsymbol{\alpha}}^0_\varepsilon,\mathbf{y}^0_\varepsilon,p^0_\varepsilon) \in \Xi_\varepsilon\right\}$ be a sequence of optimal triplets for the $\mathbb{P}_\varepsilon$-problems. Then there exists a triplet $(\mathbf{u}^0,\mathbf{y}^0,p^0) \in \Xi_0$ such that*

$$(\overline{\boldsymbol{\alpha}}^0_\varepsilon,\mathbf{y}^0_\varepsilon,p^0_\varepsilon) \overset{w}{\longrightarrow} (\mathbf{u}^0,\mathbf{y}^0,p^0), \quad (15.56)$$

$$\inf_{(\mathbf{u},\mathbf{y},p)\in\,\Xi_0} \mathcal{J}_0(\mathbf{u},\mathbf{y}) = \mathcal{J}_0\left(\mathbf{u}^0,\mathbf{y}^0\right) = \lim_{\varepsilon\to 0} \inf_{(\overline{\boldsymbol{\alpha}}_\varepsilon,\mathbf{y}_\varepsilon,p_\varepsilon)\in\,\Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon,\mathbf{y}_\varepsilon). \quad (15.57)$$

Taking into account this result, we will show below that any solution of the w-limit problem (15.53) can be taken as a prototype for the construction of the suboptimal controls in the sense of Definition 15.21.

15.7 Convergence theorem

The main question of this section is the study of asymptotic behavior of the boundary value problem (15.9)–(15.12) as ε tends to 0. To begin, we give some technical lemmas. We suppose that hypotheses (H1)–(H5) are fulfilled.

Lemma 15.24. $\mu_{ij} \in \mathcal{M}_b(\Omega)$, $\forall\, i,j\ (1 \leq i,j \leq 3)$.

Proof. We will prove that for every compact set $B \subset \Omega$ of zero capacity, we have $\mu_{ij}(K) = 0$. By standard properties of Radon measures, it follows that $\mu_{ij}(D) = 0$ for any Borel set $D \subset \Omega$ of zero capacity.

Let K be a compact subset of Ω. Then for any $k \in \mathbb{N}$, there exists $\varphi_k \in C_0^\infty(\Omega)$ such that $\varphi_k \geq \chi_K$, $0 \leq \varphi_k \leq 1$, $\|\varphi_k\|_{H^1_0(\Omega)} \leq 1/k$. In view of hypothesis (H4), we have

$$\lim_{\varepsilon\to 0} \langle \nabla q_i^\varepsilon - \nu \Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_\varepsilon\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)} = 0, \quad (15.58)$$

$$\forall\, \mathbf{v}_\varepsilon \in \mathbf{H}^1_0(\Omega) \ :\ \mathbf{v}_\varepsilon \rightharpoonup 0 \ \text{ in } \ \mathbf{H}^1(\Omega) \ \text{ and } \ \mathbf{v}_\varepsilon = 0 \ \text{ on } \ T_\varepsilon.$$

Applying this to the sequence $\left\{\mathbf{v}_{\varepsilon,k} = \varphi_k \mathbf{w}_j^\varepsilon\right\}$, we obtain that for any $\delta > 0$,

$$\left|\langle \nabla q_i^\varepsilon, \mathbf{v}_{\varepsilon,k}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)}\right| + \left|\nu\, \langle -\Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_{\varepsilon,k}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)}\right| \leq \delta,$$

$\forall\, \varepsilon < \varepsilon_0(\delta)$, $k > k_0(\delta)$. Since $\mathbf{w}_j^\varepsilon$ is divergence-free (see hypothesis (H2)), it follows that

$$\langle \nabla q_i^\varepsilon, \mathbf{v}_{\varepsilon,k} \rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = -\int_\Omega q_i^\varepsilon \mathbf{w}_j^\varepsilon \cdot \nabla \varphi_k \, \mathrm{d}x.$$

Then taking into account the obvious estimates

$$\left| \langle \nabla q_i^\varepsilon, \mathbf{v}_{\varepsilon,k} \rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} \right| \le \|q_i^\varepsilon\|_{L_0^2(\Omega)} \|\mathbf{w}_j^\varepsilon\|_{\mathbf{H}^1(\mathrm{div},\Omega)} \|\varphi_k\|_{H_0^1(\Omega)} \le \frac{C}{k},$$

$$\langle -\Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_{\varepsilon,k} \rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = \int_\Omega \varphi_k \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon \, \mathrm{d}x + \int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla \varphi_k \, \mathrm{d}x,$$

and

$$\left| \int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla \varphi_k \, \mathrm{d}x \right| \le \|\mathbf{w}_i^\varepsilon\|_{\mathbf{H}^1(\Omega)} \|\mathbf{w}_j^\varepsilon\|_{\mathbf{H}^1(\Omega)} \|\nabla \varphi_k\|_{\mathbf{L}^2(\Omega)} \le \frac{C}{k} \ ,$$

we obtain

$$\int_\Omega \left| \varphi_k \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon \right| \, \mathrm{d}x \le 2\delta, \quad \forall\, \varepsilon < \varepsilon_1(\delta), \ k > k_1(\delta). \tag{15.59}$$

Due to hypothesis (H3), each of the sequences

$$\left\{ \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon \right\} \quad (1 \le i, j \le 3)$$

is bounded in $L^1(\Omega)$. So, extracting, if necessary, a subsequence, we can suppose the existence of a symmetric matrix $\mathbf{M} = \{\mu_{ij}\}_{1 \le i,j \le 3}$ of bounded Radon measures μ_{ij} such that $\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon$ converges to μ_{ij} in the weak-* sense of the space $\mathcal{M}_b(\Omega)$. Then, passing to the limit in inequality (15.59) as $\varepsilon \to 0$, we get

$$\int_\Omega \varphi_k \, \mathrm{d}\mu_{ij} \le 2\delta, \quad \forall\ k > k_1(\delta).$$

Since $\varphi_k \ge \chi_K$, this yields $\mu_{ij}(K) \le 2\delta$, $\forall\ \delta > 0$, and we obtain the required result. □

Lemma 15.25. *For any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and each i, j $(1 \le i, j \le 3)$, we have*

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon \right) \, \mathrm{d}x = \int_\Omega \varphi \, \mathrm{d}\mu_{ij}. \tag{15.60}$$

Proof. Let $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. By analogy with Casado-Diaz [55], we consider a sequence of functions $\{\varphi_k \in C_0^\infty(\Omega)\}$ satisfying the conditions

$$\sup_{k \in \mathbb{N}} \|\varphi_k\|_{L^\infty(\Omega)} < +\infty, \quad \varphi_k \to \varphi \ \text{ in } \ H_0^1(\Omega) \ \text{ and } \ \mu_{ij}\text{-a.e. on } \ \Omega$$

(the existence of such a sequence has been proved in [267]). From Lebesgue's dominated convergence theorem, we have $\varphi \in L^\infty(\Omega, \mathrm{d}\mu_{ij})$ and φ_k converges strongly to φ in $L^1(\Omega, \mathrm{d}\mu_{ij})$. Then

$$\begin{aligned}\Big|\int_\Omega \varphi\,\big|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\big|\,\mathrm{d}x - \int_\Omega \varphi\,\mathrm{d}\mu_{ij}\Big| &\\ \leq \int_\Omega \big(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\big)\,|\varphi-\varphi_k|\,\mathrm{d}x &\\ + \left|\int_\Omega \varphi_k\,\big(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\big)\,\mathrm{d}x - \int_\Omega \varphi_k\,\mathrm{d}\mu_{ij}\right| &\\ + \int_\Omega |\varphi_k-\varphi|\,\mathrm{d}\mu_{ij}.&\end{aligned}$$

Passing to the limit in this relation for a fixed k and taking into account the weak-* convergence of $\big(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\big)$ to μ_{ij} in $\mathcal{M}_b(\Omega)$, we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0}\left|\int_\Omega \varphi\,\big(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\big)\,\mathrm{d}x - \int_\Omega \varphi\,\mathrm{d}\mu_{ij}\right| &\\ \leq \limsup_{\varepsilon\to 0}\int_\Omega \big|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\big|\,|\varphi-\varphi_k|\,\mathrm{d}x + \limsup_{\varepsilon\to 0}\int_\Omega |\varphi_k-\varphi|\,\mathrm{d}\mu_{ij}.&\end{aligned}$$

Passing, now, to the limit as $k\to\infty$, we find

$$\begin{aligned}\limsup_{\varepsilon\to 0}\left|\int_\Omega \varphi\,\big(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\big)\,\mathrm{d}x - \int_\Omega \varphi\,\mathrm{d}\mu_{ij}\right| &\\ \leq \lim_{k\to\infty}\limsup_{\varepsilon\to 0}\int_\Omega \big|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\big|\,|\varphi-\varphi_k|\,\mathrm{d}x.&\end{aligned}$$

Let us show that the limit on the right-hand side is 0. We apply property (15.58) to the sequence $\big\{\mathbf{v}_{\varepsilon,k} = \pm|\varphi_k-\varphi|\mathbf{w}_j^\varepsilon\big\}$. One gets

$$\begin{aligned}\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\Big[\pm\int_\Omega |\varphi_k-\varphi|\,\big(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\big)\,\mathrm{d}x &\\ \pm\int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla|\varphi_k-\varphi|\,\mathrm{d}x\Big] = 0.&\end{aligned}$$

Since

$$\int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla|\varphi_k-\varphi|\,\mathrm{d}x \leq 2\|\mathbf{w}_i^\varepsilon\|_{\mathbf{H}^1(\Omega)}\|\mathbf{w}_j^\varepsilon\|_{\mathbf{H}^1(\Omega)}\|\nabla|\varphi_k-\varphi|\|_{\mathbf{L}^2(\Omega)}$$

and φ_k tends to φ strongly in $H_0^1(\Omega)$, it immediately follows that

$$\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\left|\int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla|\varphi_k-\varphi|\,\mathrm{d}x\right| = 0.$$

Thus,

$$\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\int_\Omega \big|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\big|\,|\varphi-\varphi_k|\,\mathrm{d}x = 0,$$

and this concludes the proof. □

Lemma 15.26. *If a sequence* $\{\mathbf{v}_\varepsilon \in \mathbf{H}_0^1(\Omega)\}$ *and* $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ *are such that* $\mathbf{v}_\varepsilon = 0$ *on* T_ε *and* $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ *in* $\mathbf{H}_0^1(\Omega)$, *then* $\mathbf{v} \in \mathbf{L}^1(\Omega, d\mu_i)$ *for each* i $(1 \le i \le 3)$.

Proof. For every $k > 0$, we define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by $T_k(s) = k$ if $s \ge k$, $T_k(s) = s$ if $-k \le s \le k$, and $T_k(s) = -k$ if $s \le -k$. We denote the vector-valued counterpart of this function with the bold symbol $\mathbf{T}_k : \mathbb{R}^3 \to \mathbb{R}^3$. Let $\{\mathbf{v}_\varepsilon \in \mathbf{H}_0^1(\Omega)\}$ and $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ be the above given functions. We wish to show the fulfillment of the relation

$$\lim_{k\to\infty} \limsup_{\varepsilon\to 0} \left| \langle \nabla q_i^\varepsilon - \nu \Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} - \int_\Omega \mathbf{T}_k(\mathbf{v})\, \mathrm{d}\mu_i \right| = 0, \quad (15.61)$$

which implies that $\int_\Omega \mathbf{T}_k(\mathbf{v})\, \mathrm{d}\mu_i$ is bounded independently of k, and hence, by the Beppo Levi's monotone convergence theorem, $\mathbf{v} \in \mathbf{L}^1(\Omega, \mathrm{d}\mu_i)$.

For every $\varepsilon > 0$ and $k \in \mathbb{R}$, we define the functions $\mathbf{v}_{\varepsilon,k}$ by the rule $\mathbf{v}_\varepsilon = \mathbf{T}_k(\mathbf{v}_\varepsilon) + \mathbf{v}_{\varepsilon,k}$ and note that

$$\begin{aligned} &\left| \langle \nabla q_i^\varepsilon - \nu \Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} - \int_\Omega \mathbf{T}_k(\mathbf{v})\, \mathrm{d}\mu_i \right| \\ &\le \left| \langle \nabla q_i^\varepsilon - \nu \Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_{\varepsilon,k} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} \right| \\ &+ \left| \langle \nabla q_i^\varepsilon - \nu \Delta \mathbf{w}_i^\varepsilon, \mathbf{T}_k(\mathbf{v}_\varepsilon) \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} - \int_\Omega \mathbf{T}_k(\mathbf{v})\, \mathrm{d}\mu_i \right| \\ &= I_1 + I_2. \end{aligned} \quad (15.62)$$

Since I_2 tends to 0 as $\varepsilon \to 0$ for a fixed k by hypothesis (H4) and Lemma 15.24, and I_1 tends to zero when $k \to \infty$ and $\varepsilon \to 0$ by property (15.58), this concludes the proof. □

Our next step concerns the structural identification of the matrix $\mathbf{M} = \{\mu_{ij}\}$, where

$$\mu_{ij} \text{ is the weak-}* \text{ limit of } \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon \text{ in the the space } \mathcal{M}_b(\Omega). \quad (15.63)$$

Lemma 15.27. *Assume the functions*

$$\mathbf{w}_j^\varepsilon = \left(w_{j,1}^\varepsilon, w_{j,2}^\varepsilon, w_{j,3}^\varepsilon\right)^t \in \mathbf{H}^1(\Omega) \quad (1 \le j \le 3)$$

are such that

$$w_{j,k}^\varepsilon \in H_0^1(\Omega) \quad \text{if} \quad j \ne k. \quad (15.64)$$

Then $\mathbf{M} = \mathrm{diag}(\mu_{11}, \mu_{22}, \mu_{33})$.

Proof. According to hypothesis (H3) and Rellich–Kondrachov's compactness theorem, we conclude that $\mathbf{w}_j^\varepsilon = \left(w_{j,1}^\varepsilon, w_{j,2}^\varepsilon, w_{j,3}^\varepsilon\right)^t$ converges to $\mathbf{e}_j$ strongly in $\mathbf{L}^2(\Omega)$. In addition, due to the initial assumptions (15.64), we also have

$w^\varepsilon_{j,k} \rightharpoonup 0$ in $H^1_0(\Omega)$ and $w^\varepsilon_{j,k} \to 0$ in $L^2(\Omega)$, $\forall\, j \neq k$. Then, by Poincaré inequality on Ω, we can deduce

$$\limsup_{\varepsilon\to 0} \|\nabla w^\varepsilon_{j,k}\|_{\mathbf{L}^2(\Omega)} \le C \limsup_{\varepsilon\to 0} \|w^\varepsilon_{j,k}\|_{L^2(\Omega)} = 0, \quad \forall\, j \neq k. \tag{15.65}$$

Let (i,j) be a pair of indices such that $i \neq j$ and $1 \le i,j \le 3$. Extracting, if necessary, a subsequence, we can assume the existence of a bounded Radon measure μ_{ij} such that $\nabla \mathbf{w}^\varepsilon_i : \nabla \mathbf{w}^\varepsilon_j$ converges to μ_{ij} in the weak-* sense of $\mathcal{M}_b(\Omega)$. It means that

$$\lim_{\varepsilon\to 0} \int_\Omega \left(\nabla \mathbf{w}^\varepsilon_i : \nabla \mathbf{w}^\varepsilon_j\right) \varphi \,\mathrm{d}x = \int_\Omega \varphi \,\mathrm{d}\mu_{ij}, \quad \forall\, \varphi \in C_0(\Omega).$$

Observing that $\nabla \mathbf{w}^\varepsilon_i : \nabla \mathbf{w}^\varepsilon_j = \sum_{k=1}^3 \nabla w^\varepsilon_{i,k} \cdot \nabla w^\varepsilon_{j,k}$, we have the estimate

$$\int_\Omega \left(\nabla \mathbf{w}^\varepsilon_i : \nabla \mathbf{w}^\varepsilon_j\right) \varphi \,\mathrm{d}x \le \|\varphi\|_{C_0(\Omega)} \sum_{k=1}^3 \|\nabla w^\varepsilon_{i,k}\|_{\mathbf{L}^2(\Omega)} \|\nabla w^\varepsilon_{j,k}\|_{\mathbf{L}^2(\Omega)}. \tag{15.66}$$

Since each of the terms $\|\nabla w^\varepsilon_{i,k}\|_{\mathbf{L}^2(\Omega)}$ and $\|\nabla w^\varepsilon_{j,k}\|_{\mathbf{L}^2(\Omega)}$ contains at least one multiplier with noncoinciding indices, we can apply property (15.65). As a result, passing to the limit in (15.66) as $\varepsilon \to 0$, we obtain the required result: $\mu_{ij} = 0$. □

To check hypotheses (H1)–(H5) and obtain a precise description for the measures μ_{ij}, we partition the set $\widetilde{\Omega}$ into squares $\varepsilon\widetilde{Y}$ with edges ε and denote these squares with $\varepsilon\widetilde{Y}_j$. The corresponding cylindrical cells $\varepsilon\widetilde{Y}_j \times (0,\ell)$ are denoted by Z^ε_j. Following the ideas of Cioranescu and Murat [65] and Allaire [5], we introduce the functions $\mathbf{w}^\varepsilon_k \in \mathbf{H}^1(Z^\varepsilon_j)$ and $q^\varepsilon_k \in L^2(Z^\varepsilon_j)$ $(k = 1,2,3)$ with $\int_{Z^\varepsilon_j} q_k \,\mathrm{d}x = 0$ as follows:

(i) For each cell Z^ε_j that meets the boundary Γ_3,

$$\{\mathbf{w}^\varepsilon_k = \mathbf{e}_k, \quad q^\varepsilon_k = 0\} \quad \text{on} \quad Z^\varepsilon_j \cap \Omega. \tag{15.67}$$

(ii) For each cell Z^ε_j entirely included in Ω (precisely, for $Z^\varepsilon_{\mathbf{k}}$ where $\mathbf{k} \in \Theta_\varepsilon$),

$$\begin{aligned}
&\left\{\begin{array}{c} \mathbf{w}^\varepsilon_k = \mathbf{e}_k \\ q^\varepsilon_k = 0 \end{array}\right\} && \text{on} \quad Z^\varepsilon_{\mathbf{k}} \setminus [\varepsilon\,(A+\mathbf{k}) \times (0,\ell)]\,, \\
&\left\{\begin{array}{c} -\nu\Delta \mathbf{w}^\varepsilon_k + \nabla q^\varepsilon_k = 0 \\ \nabla \cdot \mathbf{w}^\varepsilon_k = 0 \end{array}\right\} && \text{on} \quad \varepsilon\,(A \setminus Q^{r_\varepsilon} + \mathbf{k}) \times (0,\ell), \\
&\left\{\begin{array}{c} \mathbf{w}^\varepsilon_k = 0 \\ q^\varepsilon_k = 0 \end{array}\right\} && \text{on} \quad \varepsilon\,(Q^{r_\varepsilon} + \mathbf{k}) \times (0,\ell).
\end{aligned} \tag{15.68}$$

It is now clear that the functions $\mathbf{w}^\varepsilon_k$ are independent of x_3, that is,

$$\mathbf{w}^\varepsilon_k(x_1,x_2,x_3) = \mathbf{w}^\varepsilon_k(x_1,x_2), \quad 1 \le k \le 3. \tag{15.69}$$

Following Allaire [5] closely, a quite similar result can be proved.

Proposition 15.28. *Assume that the size of thin cylinders $T_\varepsilon^{\mathbf{k}}$ satisfies the condition*

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon=\lim_{\varepsilon\to 0}\varepsilon^2\,(\log 1/r_\varepsilon)>0 \tag{15.70}$$

and the functions $(\mathbf{w}_k,q_k^\varepsilon)$ $(k=1,2,3)$ are defined as in (15.67) and (15.68). Then there exists a symmetric positive matrix $\mathbf{M}=(\mu_1,\mu_2,\mu_3)\in L(\mathbb{R}^3;\mathbb{R}^3)$ such that hypotheses (H1)–(H3) are satisfied. Moreover, in this case there exists a linear map R_ε satisfying (H5) such that the extension P_ε of the pressure p_ε turns out to be equal to $P_\varepsilon=p_\varepsilon$ on Ω_ε and

$$P_\varepsilon=\frac{1}{\ell|A^\varepsilon\setminus Q^{\varepsilon r_\varepsilon}|}\int_{(A^\varepsilon\setminus Q^{\varepsilon r_\varepsilon}+\varepsilon\mathbf{k})\times(0,\ell)}p_\varepsilon\,\mathrm{d}x\ \text{ on each cylinder }\ T_\varepsilon^{\mathbf{k}}.$$

Now, it is clear that, in view of Lemma 15.27 and relations (15.67) and (15.68), the matrix $\mathbf{M}=\{\mu_{ij}\}_{1\le i,j\le 3}$, which appears in Proposition 15.28, has a diagonal structure $\mathbf{M}=\operatorname{diag}(\mu_{11},\mu_{22},\mu_{33})$. In fact, the measure $\mu_{ii}\in\mathcal{M}_0^+$ that appeared as the weak limit of $\nabla\mathbf{w}_i^\varepsilon:\nabla\mathbf{w}_i^\varepsilon$ in the space $\mathcal{M}_b(\Omega)$ can be recovered in an explicit form. For this, we give the following result.

Lemma 15.29. *Let Q be a compact subset of $\widetilde{Y}$ with Lipschitz boundary ∂Q and let* int Q *be a strongly connected set,*

$$Q\subset\{x=(x_1,\dots,x_n)\in\mathbb{R}^n\ :\ x_1\ge 0\},$$

and its boundary ∂Q contains the origin. Let $A=B(\mathbf{0},r_0)$ be an open ball centered at the origin with a radius $r_0<1/2$, so that $A\subset\subset\widetilde{Y}$ and $Q\subset\subset A$ (see Fig. 15.1). Then, under condition (15.70) for a sequence $\{\mathbf{w}_i^\varepsilon\in\mathbf{H}^1(\Omega)\}$ $(1\le i\le 3)$ defined by (15.67)–(15.68), we have

$$(\nabla\mathbf{w}_i^\varepsilon:\nabla\mathbf{w}_i^\varepsilon)\rightharpoonup\mu_{ii}^*\quad\text{weakly-}*\text{ in }\ \mathcal{M}_b(\Omega),$$

where

$$\mu_{ii}^*=2\pi\lim_{\varepsilon\to 0}\sigma_\varepsilon^{-1},\quad\forall\,i:1\le i\le 3. \tag{15.71}$$

Proof. The proof follows standard techniques (see [106]) and, in some aspects, it is similar to the one given in [73]. To begin, we use the notation $|\nabla\mathbf{w}_i^\varepsilon|^2=(\nabla\mathbf{w}_i^\varepsilon:\nabla\mathbf{w}_i^\varepsilon)$ and partition the cylindrical domain Ω into cubes εY with edges ε. We denote these cubes by εY_j. It is clear that $|\varepsilon Y_j|=\varepsilon|\varepsilon\widetilde{Y}_j|=\varepsilon^3$.

Let us observe that $|\nabla\mathbf{w}_i^\varepsilon|^2=\sum_{k=1}^3\left|\nabla w_{i,k}^\varepsilon\right|^2$, where $w_{i,i}^\varepsilon\rightharpoonup 1$ in $H^1(\Omega)$ and $w_{i,k}^\varepsilon\rightharpoonup 0$ in $H_0^1(\Omega)$ for all $k\ne i$. Then

$$\begin{aligned}\int_{\varepsilon Y_j}\varphi\,|\nabla\mathbf{w}_i^\varepsilon|^2\,\mathrm{d}x&=\int_{\varepsilon Y_j}\varphi\,|\nabla w_{ii}^\varepsilon|^2\,\mathrm{d}x+\sum_{k\ne i}\int_{\varepsilon Y_j}\varphi\,|\nabla w_{ik}^\varepsilon|^2\,\mathrm{d}x\\&=\int_{\varepsilon Y_j}\varphi\,|\nabla w_{ii}^\varepsilon|^2\,\mathrm{d}x+S_j(\varepsilon)\end{aligned}$$

for any $\varphi \in C_0(\Omega)$. Hence, using (15.69), we have the following obvious relation:

$$\varphi(x_j^\varepsilon)\varepsilon \int_{\varepsilon\widetilde{Y}_j} |\nabla w_{ii}^\varepsilon|^2 \,\mathrm{d}x = \varphi(x_j^\varepsilon) \int_{\varepsilon Y_j} |\nabla w_{ii}^\varepsilon|^2 \,\mathrm{d}x \le \int_{\varepsilon Y_j} \varphi\, |\nabla w_{ii}^\varepsilon|^2 \,\mathrm{d}x$$
$$\le \varphi(y_j^\varepsilon) \int_{\varepsilon Y_j} |\nabla w_{ii}^\varepsilon|^2 \,\mathrm{d}x = \varphi(y_j^\varepsilon)\varepsilon \int_{\varepsilon\widetilde{Y}_j} |\nabla w_{ii}^\varepsilon|^2 \,\mathrm{d}x,$$

where $x_j^\varepsilon, y_j^\varepsilon \in \varepsilon Y_j$. Combining this relation with the previous one, we obtain

$$\varepsilon \int_{\varepsilon\widetilde{Y}_j} |\nabla w_{ii}^\varepsilon|^2 \,\mathrm{d}x + S_j(\varepsilon) \le \int_{\varepsilon Y_j} |\nabla \mathbf{w}_i^\varepsilon|^2 \,\varphi \,\mathrm{d}x$$
$$\le \varphi(y_j^\varepsilon)\varepsilon \int_{\varepsilon\widetilde{Y}_j} |\nabla w_{ii}^\varepsilon|^2 \,\mathrm{d}x + S_j(\varepsilon). \tag{15.72}$$

From the definition of the capacity and its properties, it readily follows that

$$\int_{\varepsilon\widetilde{Y}_j} |\nabla w_{ii}^\varepsilon|^2 \,\mathrm{d}x = \operatorname{cap}\Big(Q^{\varepsilon r(\varepsilon)}, A^\varepsilon\Big)$$
$$= \operatorname{cap}\big(r_\varepsilon Q, A\big) = \operatorname{cap}\big(Q, r_\varepsilon^{-1} A\big). \tag{15.73}$$

Since $0 \in \partial Q$ and using the arguments of [73] (see Lemma 3.3), we can conclude that

$$\operatorname{cap}\big(Q, r_\varepsilon^{-1} A\big) = \frac{2\pi}{\log(1/r_\varepsilon)}\,(1 + c_\varepsilon) = 2\pi\varepsilon^2 \sigma_\varepsilon^{-1}\,(1 + c_\varepsilon)\,, \quad \text{where} \quad \lim_{\varepsilon\to 0} c_\varepsilon = 0. \tag{15.74}$$

Then, summing up inequalities (15.72) for all admissible indices j and taking into account relations (15.73) and (15.74), we obtain

$$2\pi\sigma_\varepsilon^{-1}\,(1 + c_\varepsilon) \sum_j \varepsilon^3 \varphi(x_j^\varepsilon) + \sum_j S_j(\varepsilon)$$
$$\le \sum_j \int_{\varepsilon Y_j} |\nabla \mathbf{w}_i^\varepsilon|^2 \,\varphi \,\mathrm{d}x$$
$$\le 2\pi\sigma_\varepsilon^{-1}\,(1 + c_\varepsilon) \sum_j \varepsilon^3 \varphi(y_j^\varepsilon) + \sum_j S_j(\varepsilon). \tag{15.75}$$

Observing that

$$-\|\varphi\|_{C_0(\Omega)} \lim_{\varepsilon\to 0} \int_\Omega |\nabla w_{ik}^\varepsilon|^2 \,\mathrm{d}x \le \lim_{\varepsilon\to 0} \sum_j \int_{\varepsilon Y_j} \varphi\, |\nabla w_{ik}^\varepsilon|^2 \,\mathrm{d}x$$
$$\le \|\varphi\|_{C_0(\Omega)} \lim_{\varepsilon\to 0} \int_\Omega |\nabla w_{ik}^\varepsilon|^2 \,\mathrm{d}x$$

and taking into account Lemma 15.27 and the construction of the Riemann sum for the integral $\int_\Omega \varphi \, dx$, we can pass to the limit in (15.75) as ε tends to 0. As a result, we obtain

$$2\pi \lim_{\varepsilon\to 0} \sigma_\varepsilon^{-1} \int_\Omega \varphi \, dx \le \lim_{\varepsilon\to 0} \int_\Omega |\nabla \mathbf{w}_i^\varepsilon|^2 \varphi \, dx \le 2\pi \lim_{\varepsilon\to 0} \sigma_\varepsilon^{-1} \int_\Omega \varphi \, dx, \quad \forall \varphi \in C_0(\Omega).$$

Hence,

$$\lim_{\varepsilon\to 0} \int_\Omega |\nabla w_\varepsilon|^2 \varphi \, dx = 2\pi \lim_{\varepsilon\to 0} \sigma_\varepsilon^{-1},$$

and, therefore, Lemma 15.29 is proved. □

As a consequence of this result, we have the following one.

Corollary 15.30. *Let* $\mathbf{w}_k^\varepsilon \in \mathbf{H}^1(Z_j^\varepsilon)$ *and* $q_k^\varepsilon \in L^2(Z_j^\varepsilon)$ $(k = 1, 2, 3)$ *be the functions defined by (15.67)–(15.68). Then*

$$\lim_{\varepsilon\to 0} \langle \nabla q_i^\varepsilon - \nu \Delta \mathbf{w}_i^\varepsilon, \varphi \mathbf{w}_k^\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} = 2\pi \lim_{\varepsilon\to 0} \sigma_\varepsilon^{-1} \delta_{ik} \int_\Omega \varphi \, dx \tag{15.76}$$

for all $\varphi \in C_0^\infty(\Omega)$ *and* $1 \le k, i \le 3$, *where* $\delta_{ik} = 0$ *if* $i \ne k$, *and* $\delta_{ii} = 1$.

Proof. Integrating the expression $\langle \nabla q_i^\varepsilon - \nu \Delta \mathbf{w}_i^\varepsilon, \varphi \mathbf{w}_k^\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)}$ by parts and using property (H2), we can reduce it to the form

$$\begin{aligned} \langle \nabla q_i^\varepsilon - \nu \Delta \mathbf{w}_i^\varepsilon, \varphi \mathbf{w}_k^\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} &= -\int_\Omega q_i^\varepsilon (\mathbf{w}_k^\varepsilon \cdot \nabla \varphi) \, dx \\ &+ \nu \int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_k^\varepsilon \nabla \varphi \, dx \\ &+ \nu \int_\Omega (\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_k^\varepsilon) \varphi \, dx. \end{aligned} \tag{15.77}$$

Then, combining Rellich's theorem, the conditions $\mathbf{w}_k^\varepsilon \rightharpoonup \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$ and $q_k^\varepsilon \rightharpoonup 0$ in $L_0^2(\Omega)$, and Lemmas 15.27–15.29, we can pass to the limit in the above equation as $\varepsilon \to 0$. As a result, we conclude

$$\begin{aligned} &\lim_{\varepsilon\to 0} \langle \nabla q_i^\varepsilon - \nu \Delta \mathbf{w}_i^\varepsilon, \varphi \mathbf{w}_k^\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} \\ &\qquad = \nu \lim_{\varepsilon\to 0} \int_\Omega (\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_k^\varepsilon) \varphi \, dx = 2\pi\nu \lim_{\varepsilon\to 0} \sigma_\varepsilon^{-1} \delta_{ik} \int_\Omega \varphi \, dx. \end{aligned}$$

□

Now, we are able to state and prove the main result of this section concerning the passage to the limit as $\varepsilon \to 0$ in the following integral identities:

$$\begin{aligned} \nu \int_{\Omega_\varepsilon} (\nabla \mathbf{y}_\varepsilon \,:\, \nabla \mathbf{v})\, \mathrm{d}x + \int_{\Omega_\varepsilon} (\mathbf{y}_\varepsilon \cdot \nabla)\mathbf{y}_\varepsilon \cdot \mathbf{v}\, \mathrm{d}x - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} \mathbf{v}\, \mathrm{d}x \\ = \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{v}\, \mathrm{d}x, \qquad \forall\, \mathbf{v} \in \mathbf{H}^1_0(\Omega_\varepsilon), \\ \int_{\Omega_\varepsilon} q \operatorname{div} \mathbf{y}_\varepsilon \, \mathrm{d}x = 0, \quad \forall\, q \in L^2_0(\Omega_\varepsilon), \\ \mathbf{y}_\varepsilon|_{\partial\Omega_\varepsilon} = \mathbf{u}_\varepsilon|_{\partial\Omega_\varepsilon}\,. \end{aligned} \tag{15.78}$$

The scheme of the proof is rather standard and is based on the energy method, introduced by Tartar [236] and adapted later by Allaire [5] for the Navier–Stokes equations.

Theorem 15.31. *Let* $\mathbf{f}_\varepsilon \rightharpoonup \mathbf{f}$ *in* $\mathbf{L}^2(\Omega)$ *and let* $\{(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *by a sequence of admissible triplets for the* $\mathbb{P}_\varepsilon$*-problems such that*

$$(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \xrightarrow{\ w\ } (\mathbf{u}, \mathbf{y}, p). \tag{15.79}$$

Then $\mathbf{u} \in \mathbf{U}$ *and the pair* $(\mathbf{y}, p) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ *is a solution of the variational problem*

$$\mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega), \tag{15.80}$$

$$\begin{aligned} \nu \int_\Omega (\nabla \mathbf{y} \,:\, \nabla \mathbf{v})\, \mathrm{d}x + \frac{2\pi\nu}{C_0} \int_\Omega (\mathbf{y} - \mathbf{u})\mathbf{v}\, \mathrm{d}x + \int_\Omega (\mathbf{y} \cdot \nabla)\mathbf{y}_\varepsilon \cdot \mathbf{v}\, \mathrm{d}x \\ + \int_\Omega \nabla p \cdot \mathbf{v}\, \mathrm{d}x = \int_\Omega f \cdot \mathbf{v}\, \mathrm{d}x, \ \forall\, \mathbf{v} \in \mathbf{H}^1_0(\Omega), \\ \int_\Omega q \operatorname{div} \mathbf{y}\, \mathrm{d}x = 0, \quad \forall\, q \in L^2_0(\Omega). \end{aligned} \tag{15.81}$$

Remark 15.32. The corresponding limit boundary problem to (15.80) and (15.81) can be formally described as

$$-\nu \triangle \mathbf{y} + \frac{2\pi\nu}{C_0}(\mathbf{y} - \mathbf{u}) + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} \quad \text{in} \ \ \Omega, \tag{15.82}$$

$$\operatorname{div} \mathbf{y} = 0 \quad \text{in} \ \ \Omega, \tag{15.83}$$

$$\mathbf{y}|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega}. \tag{15.84}$$

These relations correspond to the so-called Brinkman-type law that was introduced in the late 1940s in [34] as a new set of equations, intermediate between the Darcy and Stokes equations. The Brinkman law is obtained from the Stokes equations by adding to the momentum equation a term proportional to the velocity. In our case, the term $2\pi\nu/C_0(\mathbf{y} - \mathbf{u})$ takes that role and expresses the presence of the cylindrical holes of critical size ($0 < C_0 < +\infty$) and Dirichlet controls supported on their boundaries, which disappeared passing to the limit.

Proof. For a given sequence of admissible solutions, let $\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon) \in \widehat{\Xi}_\varepsilon\}_{\varepsilon>0}$ be a sequence of their prototypes. Here, we suppose that $\{\breve{\mathbf{y}}_\varepsilon\}$ are uniformly bounded in $\mathbf{H}^1(\Omega)$ and each $\breve{p}_\varepsilon$ is defined as $P_\varepsilon(p_\varepsilon) \in L^2_0(\Omega)$ (see Remark 15.11). Due to Theorem 15.19, we may assume that the sequence $\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $\mathbb{X}_\varepsilon$, and hence there exists a triplet

$$(\mathbf{u}, \mathbf{y}, p) \in \big[\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{sol}(\Omega)\big] \times \mathbf{H}^1_{sol}(\Omega) \times L^2(\Omega)$$

satisfying (15.79).

Let $\{(\mathbf{w}^\varepsilon_k, q^\varepsilon_k) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)\}_{\varepsilon>0}$ be a sequence defined by hypotheses (H1)–(H4). Let $\varphi \in C^\infty_0(\Omega)$ be a fixed function. It is clear that $\varphi \mathbf{w}^\varepsilon_k \in H^1_{0,sol}(\Omega)$ and $\varphi q^\varepsilon_k \in L^2_0(\Omega)$ for every $\varepsilon > 0$. Now, we consider the variational problem (15.78) with the test functions $\mathbf{v} = \varphi \mathbf{w}^\varepsilon_k \in H^1_{0,sol}(\Omega_\varepsilon)$ and $q = \varphi q^\varepsilon_k \in L^2_0(\Omega_\varepsilon)$. As a result, we obtain

$$\begin{aligned}
&\nu \int_{\Omega_\varepsilon} \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \,:\, \nabla(\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x + \nu \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon \,:\, \nabla(\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x \\
&\qquad + \int_\Omega \chi_\varepsilon (\mathbf{u}_\varepsilon \cdot \nabla) \breve{\mathbf{y}}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x \\
&\qquad + \int_\Omega \chi_\varepsilon ((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot \nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot (\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x \\
&\qquad + \int_\Omega \chi_\varepsilon (\breve{\mathbf{y}}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x - \int_\Omega \chi_\varepsilon (\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x \\
&\qquad - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div}(\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x = \int_\Omega \chi_\varepsilon \mathbf{f}_\varepsilon \cdot (\varphi \mathbf{w}^{\boldsymbol{\varepsilon}}_{\mathbf{k}})\,\mathrm{d}x,
\end{aligned} \tag{15.85}$$

$$\int_{\Omega_\varepsilon} \varphi\, q^\varepsilon_k \operatorname{div} \mathbf{y}_\varepsilon \,\mathrm{d}x = \int_{\Omega_\varepsilon} \varphi\, q^\varepsilon_k \operatorname{div}(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x = 0. \tag{15.86}$$

Expanding (15.85) and using the fact that $\mathbf{w}^\varepsilon_k$ is divergence-free, we have

$$\begin{aligned}
&\nu \int_{\Omega_\varepsilon} \varphi \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \,:\, \nabla \mathbf{w}^\varepsilon_k\,\mathrm{d}x + \nu \int_{\Omega_\varepsilon} \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \,:\, \mathbf{w}^\varepsilon_k \nabla \varphi\,\mathrm{d}x \\
&+ \nu \int_{\Omega_\varepsilon} \varphi(\nabla \mathbf{u}_\varepsilon \,:\, \nabla \mathbf{w}^\varepsilon_k)\,\mathrm{d}x + \nu \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon \,:\, \mathbf{w}^\varepsilon_k \nabla \varphi\,\mathrm{d}x \\
&+ \int_\Omega \chi_\varepsilon ((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot \nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot (\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x + \int_\Omega \chi_\varepsilon (\mathbf{u}_\varepsilon \cdot \nabla) \breve{\mathbf{y}}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x \\
&+ \int_\Omega \chi_\varepsilon (\breve{\mathbf{y}}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x - \int_\Omega \chi_\varepsilon (\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_k)\,\mathrm{d}x \\
&- \int_{\Omega_\varepsilon} p_\varepsilon \mathbf{w}^\varepsilon_k \cdot \nabla \varphi\,\mathrm{d}x = \int_\Omega \chi_\varepsilon \mathbf{f}_\varepsilon \cdot \boldsymbol{\varphi} \mathbf{w}^{\boldsymbol{\varepsilon}}_{\mathbf{k}}\,\mathrm{d}x.
\end{aligned} \tag{15.87}$$

Since $\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \in \mathbf{H}^1_0(\Omega)$, after integration of (15.86) by parts, we get

$$\langle \nabla q_k^\varepsilon, \varphi\, \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\rangle_{\mathbf{H}^{-1}(\Omega);\mathbf{H}_0^1(\Omega)} + \int_\Omega \chi_\varepsilon q_k^\varepsilon\, \nabla\varphi \cdot (\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, \mathrm{d}x = 0. \quad (15.88)$$

Then adding two last equations and using the fact that

$$\begin{aligned}
\int_{\Omega_\varepsilon} \varphi \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \,:\, \nabla \mathbf{w}_k^\varepsilon \,\mathrm{d}x & \\
&= \int_\Omega \varphi \nabla(\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)) \,:\, \nabla \mathbf{w}_k^\varepsilon \,\mathrm{d}x \\
&= -\langle \Delta \mathbf{w}_k^\varepsilon, \varphi \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\rangle_{\mathbf{H}^{-1}(\Omega);\mathbf{H}_0^1(\Omega)} \\
&\quad - \int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \nabla\varphi : \nabla \mathbf{w}_k^\varepsilon \,\mathrm{d}x, \qquad (15.89)
\end{aligned}$$

we obtain

$$\begin{aligned}
&\left\langle \nabla q_k^\varepsilon - \nu \Delta \mathbf{w}_k^\varepsilon, \varphi \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\right\rangle_{\mathbf{H}^{-1}(\Omega);\mathbf{H}_0^1(\Omega)} + \nu \int_\Omega \chi_\varepsilon \nabla \mathbf{u}_\varepsilon \,:\, \mathbf{w}_k^\varepsilon \nabla\varphi \,\mathrm{d}x \\
&\quad + \int_\Omega \chi_\varepsilon q_k^\varepsilon\, \nabla\varphi \cdot (\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x - \nu \int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\nabla\varphi : \nabla \mathbf{w}_k^\varepsilon \,\mathrm{d}x \\
&\quad + \nu \int_\Omega \chi_\varepsilon \nabla(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \,:\, \mathbf{w}_k^\varepsilon \nabla\varphi \,\mathrm{d}x + \nu \int_\Omega \chi_\varepsilon \varphi(\nabla \mathbf{u}_\varepsilon \,:\, \nabla \mathbf{w}_k^\varepsilon)\,\mathrm{d}x \\
&\quad + \int_\Omega \chi_\varepsilon((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot(\varphi \mathbf{w}_k^\varepsilon)\,\mathrm{d}x \\
&\quad + \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\breve{\mathbf{y}}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\,\mathrm{d}x + \int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\,\mathrm{d}x \\
&\quad - \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\,\mathrm{d}x - \int_\Omega \chi_\varepsilon \breve{p}_\varepsilon \mathbf{w}_k^\varepsilon\cdot\nabla\varphi\,\mathrm{d}x \\
&\quad = \int_\Omega \chi_\varepsilon \mathbf{f}_\varepsilon \cdot \boldsymbol{\varphi}\mathbf{w}_\mathbf{k}^{\boldsymbol{\varepsilon}}\,\mathrm{d}x. \qquad (15.90)
\end{aligned}$$

Now, we can pass to the limit in (15.90) as ε tends to 0. To do so, we recall the following facts: $\mathbf{w}_k^\varepsilon \rightharpoonup \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$; $\nabla \mathbf{w}_j^\varepsilon$ converges pointwise and weakly in $\mathbf{L}^2(\Omega)$ to 0; $\chi_\varepsilon \to 0$ in $L^2(\Omega)$; $q_k^\varepsilon \rightharpoonup 0$ in $L_0^2(\Omega)$; $\breve{p}_\varepsilon = P_\varepsilon(p_\varepsilon) \rightharpoonup p$ in $L_0^2(\Omega)$; the sequence $\{\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\}$ fulfills the conditions of hypothesis (H4); the nonlinear term $((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)$ converges strongly to $((\mathbf{y}-\mathbf{u})\cdot\nabla)(\mathbf{y}-\mathbf{u})$ in $\mathbf{H}^{-1}(\Omega)$; and condition (15.79) implies $\mathbf{u}_\varepsilon \to \mathbf{u}$ in $\mathbf{H}^1(\Omega)$. As a result, we obtain

$$\begin{aligned}
&\left\langle \nabla q_k^\varepsilon - \nu \Delta \mathbf{w}_k^\varepsilon, \varphi \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\right\rangle_{\mathbf{H}^{-1}(\Omega);\mathbf{H}_0^1(\Omega)} \\
&\qquad \to \nu \left\langle \mu_k, \varphi(\mathbf{y}-\mathbf{u})\right\rangle_{\mathbf{H}^{-1}(\Omega);\mathbf{H}_0^1(\Omega)} \\
&\qquad \overset{\text{by Lemma 15.29}}{=} \frac{2\pi\nu}{C_0} \int_\Omega \varphi\, \mathbf{e}_k \cdot (\mathbf{y}-\mathbf{u})\,\mathrm{d}x,
\end{aligned}$$

and hence, using Rellich's theorem, from (15.90) we get

$$\frac{2\pi\nu}{C_0}\int_\Omega \varphi\,\mathbf{e}_k\cdot(\mathbf{y}-\mathbf{u})\,\mathrm{d}x + \nu\int_\Omega \nabla\mathbf{y} : \mathbf{e}_k\nabla\varphi\,\mathrm{d}x + \int_\Omega (\mathbf{y}\cdot\nabla)\mathbf{y}\cdot(\varphi\mathbf{e}_k)\,\mathrm{d}x - \int_\Omega p\,\mathbf{e}_k\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\boldsymbol{\varphi}\mathbf{e}_\mathbf{k}\,\mathrm{d}x, \quad \forall\varphi\in C_0^\infty(\Omega) \text{ and each } k=1,2,3. \tag{15.91}$$

Integrating the term $\int_\Omega p\,\mathbf{e}_k\cdot\nabla\varphi\,\mathrm{d}x$ by parts and regrouping (15.91) and (15.46), we deduce that the limit triplet $(\mathbf{u},\mathbf{y},p)$ must satisfy the following relations:

$$\nu\int_\Omega \nabla\mathbf{y} : \nabla\Phi\,\mathrm{d}x + \frac{2\pi\nu}{C_0}\int_\Omega (\mathbf{y}-\mathbf{u})\cdot\Phi\,\mathrm{d}x + \int_\Omega (\mathbf{y}\cdot\nabla)\mathbf{y}\cdot\Phi\,\mathrm{d}x + \int_\Omega \nabla p\cdot\Phi\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\Phi\,\mathrm{d}x, \quad \forall\Phi\in\mathbf{C}_0^\infty(\Omega), \tag{15.92}$$

$$\mathbf{y}-\mathbf{u}\in\mathbf{H}^1_{0,sol}(\Omega), \quad \mathbf{u}\in\mathbf{U}. \tag{15.93}$$

The proof is complete. □

15.8 Identification of the limit optimal control problem

In this section, we show that for the sequence of constrained minimization problems (15.52), there exists a weak variational limit with respect to the w-convergence, and it can be recovered in an explicit form. We begin with the following result.

Lemma 15.33. *Let* $\{(\overline{\boldsymbol{\alpha}}_\varepsilon,\mathbf{y}_\varepsilon,p_\varepsilon)\in\Xi_\varepsilon\}_{\varepsilon>0}$ *be a bounded sequence of admissible solutions, assumed to be* w*-convergent to a triplet*

$$(\mathbf{u},\mathbf{y},p)\in\left[\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_{sol}(\Omega)\right]\times\mathbf{H}^1_{sol}(\Omega)\times L^2_0(\Omega).$$

Then

$$\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon}|\nabla\mathbf{y}_\varepsilon|^2\,\mathrm{d}x = \int_\Omega|\nabla\mathbf{y}|^2\,\mathrm{d}x + \frac{2\pi}{C_0}\int_\Omega|\mathbf{y}-\mathbf{u}|^2\,\mathrm{d}x. \tag{15.94}$$

Proof. We first observe that

$$\int_{\Omega_\varepsilon}|\nabla\mathbf{y}_\varepsilon|^2\,\mathrm{d}x = \int_\Omega\chi_\varepsilon|\nabla\check{\mathbf{y}}_\varepsilon-\nabla\mathbf{u}_\varepsilon|^2\,\mathrm{d}x + 2\int_\Omega\chi_\varepsilon(\nabla\check{\mathbf{y}}_\varepsilon : \nabla\mathbf{u}_\varepsilon)\,\mathrm{d}x - \int_\Omega\chi_\varepsilon|\nabla\mathbf{u}_\varepsilon|^2\,\mathrm{d}x. \tag{15.95}$$

Then, taking into account the facts that $\nabla\mathbf{u}_\varepsilon\to\nabla\mathbf{u}$ in $\mathbf{L}^2(\Omega)$, $\check{\mathbf{y}}_\varepsilon\rightharpoonup\mathbf{y}$ in $\mathbf{H}^1(\Omega)$, and $\chi_\varepsilon\to1$ in $L^2(\Omega)$, we have

$$\int_\Omega \chi_\varepsilon(\nabla \breve{\mathbf{y}}_\varepsilon : \nabla \mathbf{u}_\varepsilon)\,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \nabla \mathbf{y} : \nabla \mathbf{u}\,\mathrm{d}x, \tag{15.96}$$

$$\int_\Omega \chi_\varepsilon |\nabla \mathbf{u}_\varepsilon|^2\,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega |\nabla \mathbf{u}|^2\,\mathrm{d}x. \tag{15.97}$$

Since $(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon)$ for every $\varepsilon > 0$, it follows that we can take $\mathbf{v} = \mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon$ as a test function in (15.78). Then the following equality is ensured:

$$\begin{aligned}
&\nu \int_{\Omega_\varepsilon} \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \,:\, \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x \\
&= -\nu \int_\Omega \chi_\varepsilon \nabla \mathbf{u}_\varepsilon \,:\, \nabla(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x \\
&\quad - \int_\Omega \chi_\varepsilon((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x \\
&\quad - \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\breve{\mathbf{y}}_\varepsilon\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x \\
&\quad - \int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x + \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x \\
&\quad + \int_\Omega \chi_\varepsilon \breve{p}_\varepsilon \,\mathrm{div}\,(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x + \int_\Omega \chi_\varepsilon \mathbf{f}_\varepsilon\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

By the arguments of the previous theorem, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \nu \int_{\Omega_\varepsilon} \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \,:\, \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x \\
&\quad = -\nu \int_\Omega \nabla \mathbf{u} \,:\, \nabla(\mathbf{y} - \mathbf{u})\,\mathrm{d}x \\
&\qquad - \int_\Omega ((\mathbf{y} - \mathbf{u})\cdot\nabla)(\mathbf{y} - \mathbf{u})\cdot(\mathbf{y} - \mathbf{u})\,\mathrm{d}x - \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{y}\cdot(\mathbf{y} - \mathbf{u})\,\mathrm{d}x \\
&\qquad - \int_\Omega (\mathbf{y}\cdot\nabla)\mathbf{u}\cdot(\mathbf{y} - \mathbf{u})\,\mathrm{d}x + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot(\mathbf{y} - \mathbf{u})\,\mathrm{d}x \\
&\qquad + \int_\Omega p\,\mathrm{div}\,(\mathbf{y} - \mathbf{u})\,\mathrm{d}x + \int_\Omega \mathbf{f}\cdot(\mathbf{y} - \mathbf{u})\,\mathrm{d}x.
\end{aligned} \tag{15.98}$$

We now consider the integral identity (15.91) with the test function $\mathbf{y} - \mathbf{u}$. By a rearrangement, we have

$$\begin{aligned}
&\frac{2\pi\nu}{C_0}\int_\Omega \varphi\,|\mathbf{y} - \mathbf{u}|^2\,\mathrm{d}x + \nu\int_\Omega |\nabla(\mathbf{y} - \mathbf{u})|^2\,\mathrm{d}x + \nu\int_\Omega \nabla\mathbf{u} \,:\, \nabla(\mathbf{y} - \mathbf{u})\,\mathrm{d}x \\
&\quad + \int_\Omega ((\mathbf{y} - \mathbf{u})\cdot\nabla)(\mathbf{y} - \mathbf{u})\cdot(\mathbf{y} - \mathbf{u})\,\mathrm{d}x + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{y}\cdot(\mathbf{y} - \mathbf{u})\,\mathrm{d}x \\
&\quad + \int_\Omega (\mathbf{y}\cdot\nabla)\mathbf{u}\cdot(\mathbf{y} - \mathbf{u})\,\mathrm{d}x - \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot(\mathbf{y} - \mathbf{u})\,\mathrm{d}x \\
&\quad = \int_\Omega p\,\mathrm{div}\,(\mathbf{y} - \mathbf{u})\,\mathrm{d}x + \int_\Omega \mathbf{f}\cdot(\mathbf{y} - \mathbf{u})\,\mathrm{d}x.
\end{aligned} \tag{15.99}$$

The comparison of (15.98) with (15.99) leads to the following equality:

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|\nabla(\mathbf{y}_\varepsilon-\mathbf{u}_\varepsilon)|^2\,\mathrm{d}x=\int_\Omega|\nabla(\mathbf{y}-\mathbf{u})|^2\,\mathrm{d}x+\frac{2\pi}{C_0}\int_\Omega\varphi\,|\mathbf{y}-\mathbf{u}|^2\,\mathrm{d}x,$$

which, together with (15.95)–(15.97), concludes the proof. □

Remark 15.34. Using this approach, we can prove a more general result: Under the supposition of Lemma 15.33, the following relation is valid:

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\nabla\mathbf{y}_\varepsilon:(\nabla\mathbf{y}_\varepsilon)^t\,\mathrm{d}x=\int_\Omega\nabla\mathbf{y}:(\nabla\mathbf{y})^t\,\mathrm{d}x+\frac{2\pi}{C_0}\int_\Omega|\mathbf{y}-\mathbf{u}|^2\,\mathrm{d}x. \quad (15.100)$$

We are now able to establish the identification result of the variational limit for the sequence of constrained minimization problems (15.52).

Theorem 15.35. *For the sequence (15.52), there exists a unique variational w-limit which can be represented in the form (15.53), where the cost functional $\mathcal{J}_0$ and the set of admissible solutions $\Xi_0=\{(\mathbf{u},\mathbf{y},p)\}$ are defined as follows:*

$$\begin{aligned}\mathcal{J}_0(\mathbf{u},\mathbf{y})=&\lambda\int_\Omega|\nabla\mathbf{y}|^2\,\mathrm{d}x\\&+\frac{2\pi\lambda}{C_0}\int_\Omega|\mathbf{y}-\mathbf{u}|^2\,\mathrm{d}x+\beta|\partial Q|_H\int_\Omega|\mathbf{u}|^2\,\mathrm{d}x,\end{aligned} \quad (15.101)$$

$$\Xi_0=\left\{\begin{array}{l}p\in L_0^2(\Omega),\ \mathbf{u}\in\mathbf{H}^2(\Omega),\ \mathbf{y}-\mathbf{u}\in\mathbf{H}^1_{0,sol}(\Omega),\\ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}\le\gamma,\quad \mathbf{u}|_{\Gamma_3}=0,\\ -\nu\triangle\mathbf{y}+\frac{2\pi\nu}{C_0}(\mathbf{y}-\mathbf{u})+(\mathbf{y}\cdot\nabla)\mathbf{y}+\nabla p=\mathbf{f}\quad \textit{in}\ \ \Omega,\\ \operatorname{div}\mathbf{y}=0\quad \textit{in}\ \ \Omega,\\ \mathbf{y}|_{\partial\Omega}=\mathbf{u}|_{\partial\Omega}\end{array}\right\}. \quad (15.102)$$

Proof. The proof of this theorem is divided into two steps, each of them concerns the verification of the corresponding item of Definition 15.22.

Step 1: *Statement (i) of Definition 15.22 is valid.*

Let $\{(\overline{\boldsymbol{\alpha}}_\varepsilon,\mathbf{y}_\varepsilon,p_\varepsilon)\}_{\varepsilon>0}$ be a bounded sequence of admissible triplets which is w-convergent to a triplet $(\mathbf{u},\mathbf{y},p)$. Let $\{\varepsilon_k\}$ be a subsequence of $\{\varepsilon\}$ such that $\varepsilon_k\to 0$ as $k\to\infty$ and $(\overline{\boldsymbol{\alpha}}_k,\mathbf{y}_k,p_k)\in\Xi_{\varepsilon_k}$ for all $k\in\mathbb{N}$. Then, due to Theorem 15.31, we have that the w-limit triplet $(\mathbf{u},\mathbf{y},p)$ satisfies relations (15.92) and (15.93), and moreover,

$$\gamma\ge\liminf_{k\to\infty}\|\mathbf{u}_k\|_{\mathbf{H}^2(\Omega)}\ge\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)},$$

by the lower semicontinuity of $\|\cdot\|_{H^2(\Omega)}$ with respect to the weak convergence in $\mathbf{H}^2(\Omega)$. So, inclusion (15.54) holds true. The fulfillment of inequality (15.54)

immediately follows from Lemma 15.33 and the property of lower semicontinuity of the weak convergence in variable spaces (see Proposition 6.16)

$$\lim_{k\to\infty}\left[\beta|\partial Q|_H\int_\Omega|\mathbf{u}_k|^2\,\mathrm{d}\eta_{\varepsilon_k}^{r(\varepsilon_k)}\right]\geq\beta|\partial Q|_H\int_\Omega|\mathbf{u}|^2\,\mathrm{d}x.$$

Step 2: *Statement (ii) of Definition 15.22 holds true.*

Let $(\mathbf{u},\mathbf{y},p)\in\Xi_0$ be an admissible triplet for the minimization problem (15.53), (15.101), (15.102). In view of Hypothesis (H5), we may suppose that there exist operators $\Lambda_\varepsilon:\mathbf{H}^1_{sol}(\Omega)\mapsto\mathbf{H}^1_{sol}(\Omega_\varepsilon)$ such that

$$\text{if }\ \mathbf{y}\in\mathbf{H}^1_{sol}(\Omega_\varepsilon),\ \ \text{then }\ \Lambda_\varepsilon(\breve{\mathbf{y}})=\mathbf{y}\ \text{ on }\ \Omega_\varepsilon, \tag{15.103}$$

$$\|\Lambda_\varepsilon(\mathbf{u})\|_{\mathbf{H}^1(\Omega_\varepsilon)}\leq C\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)},\ \ \text{where }\ C>0\ \text{is independent of }\ \varepsilon. \tag{15.104}$$

Hence, for a given $\mathbf{u}\in\mathbf{U}$, we have

$$\nabla\cdot\Lambda_\varepsilon(\mathbf{u})=0\ \ \text{and}\ \ \int_{\partial\Omega_\varepsilon}\Lambda_\varepsilon(\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^2=0.$$

As follows from the structure of the set of admissible controls $\mathbf{U}$,

$$\int_{\partial\Omega_\varepsilon}\Lambda_\varepsilon(\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^2=0\quad\Leftrightarrow\quad\int_{\partial T_\varepsilon^{\mathbf{k}_j}}\Lambda_\varepsilon(\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^2=0, \tag{15.105}$$

$\forall\, j=1,\dots,J_\varepsilon$. Let us define the sequence

$$\left\{\overline{\boldsymbol{\alpha}}_\varepsilon^*=\left(\alpha^*_{\mathbf{k}_1},\alpha^*_{\mathbf{k}_2},\dots,\alpha^*_{\mathbf{k}_{J_\varepsilon}}\right)=\Lambda_\varepsilon(\mathbf{u})|_{\partial T_\varepsilon}\right\}_{\varepsilon>0}. \tag{15.106}$$

Here, we suppose that the sequence $\{\mathbf{w}_\varepsilon=\Lambda_\varepsilon(\mathbf{u})\}$ can be extended to the whole domain Ω with uniformly bounded norms in $\mathbf{H}^1(\Omega)$ (see Remark 15.10). It is easy to see that each of the functions $\{\overline{\boldsymbol{\alpha}}_\varepsilon^*\}$ satisfy conditions (15.50) and $\Lambda_\varepsilon(\mathbf{u})\in\mathbf{L}^2(\Omega,\mathrm{d}\eta_\varepsilon^{r_\varepsilon})$ (see Remark 15.7). Moreover, in view of the estimates $\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}\leq\gamma$ and (15.104), the sequence $\{\Lambda_\varepsilon(\mathbf{u})\}$ is uniformly bounded in the variable space $\mathbf{L}^2(\Omega,\mathrm{d}\eta_\varepsilon^{r_\varepsilon})$. Hence, we may suppose the existence of a function $\mathbf{u}^*\in L^2(\Omega)$ such that

$$\{\Lambda_\varepsilon(\mathbf{u})\}\to\mathbf{u}^*\ \ \text{weakly in}\ \ \mathbf{L}^2(\Omega,\mathrm{d}\eta_\varepsilon^{r_\varepsilon}). \tag{15.107}$$

On the other hand, thanks to Lemmas 15.15–15.17, we have

$$\mathbf{u}\in\mathbf{L}^2(\Omega,\mathrm{d}\eta_\varepsilon^{r_\varepsilon}),\quad\forall\,\varepsilon>0, \tag{15.108}$$

$$\mathbf{u}\rightharpoonup\mathbf{u}\ \ \text{in}\ \ \mathbf{L}^2(\Omega,\mathrm{d}\eta_\varepsilon^{r_\varepsilon})\ \text{as}\ \ \varepsilon\to0, \tag{15.109}$$

$$\lim_{\varepsilon\to0}\int_\Omega|\mathbf{u}|^2\,\mathrm{d}\eta_\varepsilon^{r_\varepsilon}=\int_\Omega|\mathbf{u}|^2\,\mathrm{d}x, \tag{15.110}$$

and hence, $\mathbf{u}|_{\partial T_\varepsilon}\xrightarrow{w_\alpha}\mathbf{u}$. Let us show that $\mathbf{u}=\mathbf{u}^*$.

To do so, we note that $\mathbf{u} \in \mathbf{C}(\Omega)$, by the imbedding results for Sobolev spaces. Hence, for every $\delta > 0$, there exists $\varepsilon_0 > 0$ such that

$$\left| \int_{\partial T_\varepsilon^{\mathbf{k}_j}} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^2 \right| < \delta, \quad j = 1, \ldots, J_\varepsilon \text{ and } \forall \varepsilon < \varepsilon_0. \tag{15.111}$$

Let us partition the set Ω into cubes εY with edges ε and denote these cubes with εY_i. Then, combining (15.105) with (15.111), we obtain that the following estimate holds true $\forall \varepsilon < \varepsilon_0$:

$$\left| \int_{\partial T_\varepsilon^{\mathbf{k}_j} \cap \varepsilon Y_i} (\mathbf{u} - \Lambda_\varepsilon(\mathbf{u})) \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^2 \right| < \delta, \quad \forall j = 1, \ldots, J_\varepsilon, \ \forall i \ : \ \partial T_\varepsilon^{\mathbf{k}_j} \cap \varepsilon Y_i \neq \emptyset.$$

Hence, for every fixed indices (i, j), there exist points $x^i \in \partial T_\varepsilon^{\mathbf{k}_j} \cap \varepsilon Y_i$ such that

$$\begin{aligned} &\left| \mathbf{u}(x^i) - \frac{1}{\varepsilon^3} \int_{\varepsilon Y_i} \Lambda_\varepsilon(\mathbf{u}) \, \mathrm{d}\eta_\varepsilon^{r_\varepsilon} \right| \\ &\qquad = \left| \mathbf{u}(x^i) - \frac{1}{\varepsilon^3} \int_{\partial T_\varepsilon^{\mathbf{k}_j} \cap \varepsilon Y_i} \Lambda_\varepsilon(\mathbf{u}) \, \mathrm{d}\mathcal{H}^2 \right| < \delta. \end{aligned} \tag{15.112}$$

We are able to estimate the difference

$$\left| \int_\Omega \varphi \, \mathbf{u} \, \mathrm{d}x - \int_\Omega \varphi \, \mathbf{u}^* \, \mathrm{d}x \right|,$$

where $\varphi \in C_0(\Omega)$. As a result, we have

$$\begin{aligned} \left| \int_\Omega \varphi \, \mathbf{u} \, \mathrm{d}x - \int_\Omega \varphi \, \mathbf{u}^* \, \mathrm{d}x \right| &\leq \left| \int_\Omega \varphi \, \mathbf{u} \, \mathrm{d}x - \int_\Omega \varphi \, \mathbf{u} \, \mathrm{d}\eta_\varepsilon^{r_\varepsilon} \right| \\ + \left| \int_\Omega \varphi \, \mathbf{u} \, \mathrm{d}\eta_\varepsilon^{r_\varepsilon} - \int_\Omega \varphi \, \Lambda_\varepsilon(\mathbf{u}) \, \mathrm{d}\eta_\varepsilon^{r_\varepsilon} \right| &+ \left| \int_\Omega \varphi \, \Lambda_\varepsilon(\mathbf{u}) \, \mathrm{d}\eta_\varepsilon^{r_\varepsilon} - \int_\Omega \varphi \, \mathbf{u}^* \, \mathrm{d}x \right| \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Taking into account the weak convergence of the sequences $\left\{ \mathbf{u} \in \mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^{r_\varepsilon}) \right\}$ and $\left\{ \Lambda_\varepsilon(\mathbf{u}) \in \mathbf{L}^2(\Omega, \mathrm{d}\eta_\varepsilon^{r_\varepsilon}) \right\}$ (see (15.107), (15.109), and (15.110)), we observe that $I_1 \to 0$ and $I_3 \to 0$ as ε tends to 0. Now, we show that $I_2 \to 0$. Since $\mathbf{u} \in \mathbf{C}(\Omega)$, it follows that there exists a constant C^* such that

$$\begin{aligned} I_2 &\leq \sum_i \left| \int_{\varepsilon Y_i} \varphi \, \mathbf{u} \, \mathrm{d}\eta_\varepsilon^{r_\varepsilon} - \int_{\varepsilon Y_i} \varphi \, \Lambda_\varepsilon(\mathbf{u}) \, \mathrm{d}\eta_\varepsilon^{r_\varepsilon} \right| \\ &\leq C^* \varepsilon^3 \sum_i \left| \varphi(x^i) \, \mathbf{u}(x^i) - \frac{1}{\varepsilon^3} \int_{\varepsilon Y_i} \varphi \, \Lambda_\varepsilon(\mathbf{u}) \, \mathrm{d}\eta_\varepsilon^{r_\varepsilon} \right| \\ &\leq C^* \varepsilon^3 \|\varphi\|_{C(\Omega)} \sum_i \left| \mathbf{u}(x^i) - \frac{1}{\varepsilon^3} \int_{\varepsilon Y_i} \frac{\varphi(x)}{\varphi(x^i)} \, \Lambda_\varepsilon(\mathbf{u}) \, \mathrm{d}\eta_\varepsilon^{r_\varepsilon} \right|. \end{aligned}$$

Let us suppose the converse, that is,

$$\lim_{\varepsilon\to 0}\varepsilon^3\sum_i\left|\mathbf{u}(x^i)-\frac{1}{\varepsilon^3}\int_{\varepsilon Y_i}\frac{\varphi(x)}{\varphi(x^i)}\,\Lambda_\varepsilon(\mathbf{u})\,\mathrm{d}\eta_\varepsilon^{r_\varepsilon}\right|>0,$$

then there exist a constant D^* and a value $\varepsilon^*>0$ such that

$$\left|\mathbf{u}(x^i)-\frac{1}{\varepsilon^3}\int_{\varepsilon Y_i}\frac{\varphi(x)}{\varphi(x^i)}\,\Lambda_\varepsilon(\mathbf{u})\,\mathrm{d}\eta_\varepsilon^{r_\varepsilon}\right|\geq D^*,\quad \varepsilon<\varepsilon^*.$$

Assuming that $\varphi(x)/\varphi(x^i)=1$, we just come into conflict with (15.112). So, our supposition was wrong, and we get $\lim_{\varepsilon\to 0} I_2=0$. Thus, $\mathbf{u}=\mathbf{u}^*$. As a result, we have (see Lemma 15.15)

$$\overline{\boldsymbol{\alpha}}_\varepsilon^*=\left(\alpha^*_{\mathbf{k}_1},\alpha^*_{\mathbf{k}_2},\dots,\alpha^*_{\mathbf{k}_{\mathrm{J}_\varepsilon}}\right)=\Lambda_\varepsilon(\mathbf{u})|_{\partial T_\varepsilon}\overset{w_a}{\longrightarrow}\mathbf{u}.\tag{15.113}$$

Using the fact that the weak limit of $\Lambda_\varepsilon(\mathbf{u})$ in $\mathbf{H}^1(\Omega)$ coincides with its weak limit in the variable space $\mathbf{L}^2(\Omega,\mathrm{d}\eta_\varepsilon^{r(\varepsilon)})$ (see [86]) and closely following the previous arguments, it can be proved that

$$\lim_{\varepsilon\to 0}\int_\Omega|\Lambda_\varepsilon(\mathbf{u})|^2\,\mathrm{d}\eta_\varepsilon^{r_\varepsilon}=\int_\Omega|\mathbf{u}^*|^2\,\mathrm{d}x.\tag{15.114}$$

Let us consider the sequence of triplets $\{(\overline{\boldsymbol{\alpha}}_\varepsilon^*,\widehat{\mathbf{y}}_\varepsilon,\widehat{p})\in\Xi_\varepsilon\}_{\varepsilon>0}$, where $(\widehat{\mathbf{y}}_\varepsilon,\widehat{p})=(\widehat{\mathbf{y}}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon^*),\widehat{p}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon^*))$ are the corresponding solutions of the boundary value problem (15.9)–(15.12). Due to Theorem 15.19, this sequence is relatively compact with respect to the w-convergence. Hence, taking into account Lemma 15.33 and property (15.114), we deduce that $(\overline{\boldsymbol{\alpha}}_\varepsilon^*,\widehat{\mathbf{y}}_\varepsilon,\widehat{p})\overset{\varepsilon\to 0}{\longrightarrow}(\mathbf{u},\widehat{\mathbf{y}},\widehat{p})$, where $(\mathbf{u},\widehat{\mathbf{y}},\widehat{p})$ and $(\mathbf{u},\mathbf{y},p)$ belong to the same class of equivalence (here we suppose that the problem (15.53) may have more than one solution). Thus, to end the proof, it remains only to apply Lemma 15.33 and property (15.114). □

Remark 15.36. It is now clear that the structure of the limit problem (15.53) essentially depends on the parameter C_0. If $C_0=+\infty$ (which corresponds to the case when the cylinders have smaller cross-size), then the limit minimization problem (15.53), (15.101), (15.102) takes the form of the calculus variation problem (15.18)–(15.19). If $0<C_0<+\infty$ (which corresponds to the case when the cylinders are of critical cross-size), then the limit minimization problem (15.53), (15.101), (15.102) can be recovered in the form of the OCP (15.20)–(15.24).

Remark 15.37. Following the above approach, we can obtain the explicit mathematical description for the variational w-limit problem (15.53) in the case when the cost functional to the $\mathbb{P}_\varepsilon$-problem takes the form

$$\mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon,\mathbf{y}_\varepsilon)=\lambda\int_{\Omega_\varepsilon}|\mathrm{curl}\,\mathbf{y}_\varepsilon|^2\,\mathrm{d}x+\frac{\beta\varepsilon}{r_\varepsilon}\sum_{j=1}^{J_\varepsilon}\int_{\partial T_\varepsilon^{\mathbf{k}_j}}\left|\alpha_{\mathbf{k}_j}\right|^2\mathrm{d}\mathcal{H}^2.\tag{15.115}$$

Indeed, to do this, we have to use relation (15.17) and to apply identity (15.100). As a result, we obtain the following structure for the limit cost functional:

$$\mathcal{J}_0(\mathbf{u},\mathbf{y}) = \lambda \int_\Omega |\operatorname{curl}\mathbf{y}|^2 \,\mathrm{d}x + \beta|\partial Q|_H \int_\Omega |\mathbf{u}|^2 \,\mathrm{d}x,$$

in which the relaxed term

$$+\frac{2\pi\lambda}{C_0}\int_\Omega |\mathbf{y}-\mathbf{u}|^2 \,\mathrm{d}x$$

does not appear.

To conclude this section, we combine the results of Theorem 15.23 and Theorem 15.35 and come to the following conclusion concerning the variational properties of the limit problem (15.53), (15.101), (15.102).

Theorem 15.38. *Let* $\left\{(\overline{\boldsymbol{\alpha}}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ *be the optimal solutions of the problems* $(\mathbb{P}_\varepsilon)$. *Then each w-cluster triplet* $(\mathbf{u}^0,\mathbf{y}^0,p^0)$ *of this sequence satisfies the conditions*

$$(\mathbf{u}^0,\mathbf{y}^0,p^0) \in \Xi_0, \qquad \inf_{(\mathbf{u},\mathbf{y},p)\in\,\Xi_0} \mathcal{J}_0(\mathbf{u},\mathbf{y}) = \mathcal{J}_0\left(\mathbf{u}^0,\mathbf{y}^0\right) = \lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}^0_\varepsilon,\mathbf{y}^0_\varepsilon).$$

15.9 Suboptimal controls and their approximation properties

In this section, we deal with the construction of suboptimal controls to the original optimal boundary control problem (15.8)–(15.12). The main result can be formulated as follows.

Theorem 15.39. *Let* $\Lambda_\varepsilon : \mathbf{H}^1_{sol}(\Omega) \mapsto \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ *be the linear continuous map such that properties (15.103) and (15.104) hold true, and let* $(\mathbf{u}^0,\mathbf{y}^0,p^0) \in \Xi_0$ *be an optimal solution to the limit problem (15.53). Then the function*

$$\overline{\boldsymbol{\alpha}}^{sub}_\varepsilon = \left(\alpha^{sub}_{\mathbf{k}_1}, \alpha^{sub}_{\mathbf{k}_2}, \ldots, \alpha^{sub}_{\mathbf{k}_{\mathrm{J}_\varepsilon}}\right) = \Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon} \tag{15.116}$$

is an asymptotically suboptimal control for the original problem $(\mathbb{P}_\varepsilon)$ *in the sense of Definition 15.21.*

Proof. To begin, we note that, following the proof of Theorem 15.35, we can immediately establish the following approximation property: If $(\mathbf{u}^0,\mathbf{y}^0,p^0)$ is an optimal solution to the limit minimization problem (15.53), then

$$\Lambda_\varepsilon(\mathbf{u}^0) \in \mathbf{L}^2(\Omega, \mathrm{d}\eta^{r(\varepsilon)}_\varepsilon), \quad \forall\,\varepsilon>0, \tag{15.117}$$

$$\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon} \xrightarrow{w_a} \mathbf{u}^0 \text{ as } \varepsilon\to 0, \tag{15.118}$$

$$\lim_{\varepsilon\to 0}\int_\Omega \left|\Lambda_\varepsilon(\mathbf{u}^0)\right|^2 \mathrm{d}\eta^{r(\varepsilon)}_\varepsilon = \int_\Omega \left|\mathbf{u}^0\right|^2 \mathrm{d}x. \tag{15.119}$$

Let us consider now the sequence of triplets

$$\left\{ (\,\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}\,, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon\,) \in \Xi_\varepsilon \right\}_{\varepsilon>0},$$

where

$$(\widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon\,) = \Big(\widehat{\mathbf{y}}_\varepsilon(\,\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}), \widehat{p}_\varepsilon(\,\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon})\Big)$$

are the corresponding solutions of the boundary value problem (15.9)–(15.12). Then following the motivation given in the proof of Theorem 15.35, we conclude that this sequence is relatively compact with respect to the w-convergence, and extracting, if necessary, a subsequence, it satisfies condition

$$(\widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon\,) = \Big(\widehat{\mathbf{y}}_\varepsilon(\,\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}), \widehat{p}_\varepsilon(\,\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon})\Big) \overset{\varepsilon\to 0}{\longrightarrow} (\mathbf{u}^0, \widehat{\mathbf{y}}^{\,0}, \widehat{p}^{\,0}),$$

where $(\mathbf{u}^0, \widehat{\mathbf{y}}^{\,0}, \widehat{p}^{\,0})$ and $(\mathbf{u}^0, \mathbf{y}^0, p^0)$ belong to the same equivalence class. Consequently,

$$\mathcal{J}_0(\mathbf{u}^0, \mathbf{y}^0) = \mathcal{J}_0(\mathbf{u}^0, \widehat{\mathbf{y}}^{\,0}).$$

Let $\left\{(\overline{\boldsymbol{\alpha}}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be the optimal solutions to the $\mathbb{P}_\varepsilon$-problems. We observe that

$$\begin{aligned}
&\Big|\inf_{(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)\in \Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}_\varepsilon, \mathbf{y}_\varepsilon) - \mathcal{J}_\varepsilon(\,\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}\,, \widehat{\mathbf{y}}_\varepsilon)\Big| \\
&= \Big|\mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}^0_\varepsilon, \mathbf{y}^0_\varepsilon) - \mathcal{J}_\varepsilon(\,\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}\,, \widehat{\mathbf{y}}_\varepsilon)\Big| \le \big|\mathcal{J}_\varepsilon(\overline{\boldsymbol{\alpha}}^0_\varepsilon, \mathbf{y}^0_\varepsilon) - \mathcal{J}_0(\mathbf{u}^0, \mathbf{y}^0)\big| \\
&+ \lambda \left| \int_\Omega |\nabla \widehat{\mathbf{y}}^{\,0}|^2 \,\mathrm{d}x + \frac{2\pi}{C_0} \int_\Omega \big|\widehat{\mathbf{y}}^{\,0} - \mathbf{u}^0\big|^2 \,\mathrm{d}x - \int_{\Omega_\varepsilon} |\nabla\, \widehat{\mathbf{y}}_\varepsilon|^2 \,\mathrm{d}x \right| \\
&+ \beta \left| |\partial Q|_H \int_\Omega \big|\mathbf{u}^0\big|^2 \,\mathrm{d}x - \frac{\varepsilon}{r_\varepsilon} \sum_{j=1}^{J_\varepsilon} \int_{\partial T_\varepsilon^{\mathbf{k}_j}} \Big|\,\Lambda_\varepsilon(\mathbf{u}^0)\Big|_{\partial T_\varepsilon^{\mathbf{k}_j}}\Big|^2 \,\mathrm{d}\mathcal{H}^2 \right| \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

To conclude the proof, we note that for a given $\delta > 0$, one can always find the following: (i) $\varepsilon_1 > 0$ such that $J_1 < \delta/3$ for all $\varepsilon < \varepsilon_1$ by Theorem 15.23; (ii) $\varepsilon_2 > 0$ such that $J_2 < \delta/3$ for all $\varepsilon < \varepsilon_2$ by the Lemma 15.33; and (iii) $\varepsilon_3 > 0$ such that $J_3 < \delta/3$ for all $\varepsilon < \varepsilon_3$ by property (15.119). Thus, as expected, the estimate of suboptimality (15.51) is valid for all $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

16

Optimal Control Problems in Coefficients: Sensitivity Analysis and Approximation

In this chapter, we are mainly interested in optimal $L^\infty(\Omega)$-control of the coefficients of an elliptic Dirichlet problem. We prove an existence result for this problem using the direct method of Calculus of Variations, and then we provide a sensitivity analysis of this problem on a reticulated structured with respect to the domain perturbation and discuss possible ways for the approximation of its solutions.

Let Ω be a fixed open subset of D which we associate with a thick lattice structure or a network contained in D. As a particular example of a nonlinear optimal control problem (OCP) we consider in this chapter is minimizing the discrepancy between a given distribution $y_d \in L^2(D)$ and the solution of a Dirichlet problem by choosing an appropriate matrix of coefficients $A \in L^\infty(D;\mathbb{R}^{n\times n})$. More precisely, we consider the following minimization problem:

$$\text{Minimize } \Big\{ I_\Omega(A,y) = \int_\Omega |y(x) - y_d(x)|^2 \,\mathrm{d}x + \int_\Omega |\nabla y(x)|^2_{\mathbb{R}^n} \,\mathrm{d}x + J(A) \Big\} \tag{16.1}$$

subject to the constraints

$$A \in \mathcal{M}, \quad y \in H_0^1(\Omega), \tag{16.2}$$

$$A \geq \alpha I, \quad \alpha > 0, \tag{16.3}$$

$$-\text{div } (A(x)\nabla y) + a_0(x)y = f \quad \text{in } \ \Omega, \tag{16.4}$$

where $\mathcal{M}$ is a weakly-$*$ closed subset of $L^\infty(D;\mathbb{R}^{n\times n})$, and $J : \mathcal{M} \to \mathbb{R}$ is a lower semicontinuous functional with respect to the weak-$*$ topology of $L^\infty(D;\mathbb{R}^{n\times n})$.

In (16.3), I is the identity matrix in $\mathbb{R}^{n\times n}$, and the above inequality is in the sense of the quadratic forms defined by $(A\xi,\xi)_{\mathbb{R}^n}$ for $\xi \in \mathbb{R}^n$.

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_16,
© Springer Science+Business Media, LLC 2011

Thus, we seek a matrix of coefficients $A \in L^\infty(D;\mathbb{R}^{n\times n})$ such that the corresponding weak solution $y = y_{\Omega,A,f}$ of (16.2)–(16.4) is as close as possible to a desired state y_d.

Note that the range of OCPs in coefficients is very wide, including, for example, optimal shape design problems, topology optimization and material optimization, and many others. This topic has been widely studied by many authors. We mainly mention Allaire [7], Buttazzo and Dal Maso [45, 46], Calvo-Jurado and Casado-Díaz [50, 51, 52], Haslinger and Neittaanmäki [119], Lions [169], Litvinov [176], Lurie [181], Murat [193], Murat and Tartar [195], Pironneau [214], Raytum [219], Serovaĭskiĭ [226], Sokolowski and Zolesio [233], and Tiba [240].

However, the principal feature of OCPs in coefficients is the fact that an optimal solution for (16.1)–(16.4) does not exist, in general (see, e.g., [45, 50, 193, 219]). We have once again a typical situation of optimal control theory, namely the original control object is described by the well-posed boundary value problem, but the associated OCP is ill-posed and requires relaxation.

Taking this fact into account, we will relax the problem (16.1)–(16.4) by introducing the solenoidal controls $A \in U_{sol}$. Notice that this class of admissible controls does not belong to the Sobolev space $W^{1,\infty}(\Omega)$ but is still a uniformly bounded subset of $L^\infty(\Omega)$. We give a precise definition of such controls and prove that, in this case, the original OCP admits at least one solution. Note that we do not invoke in this process the homogenization method and the relaxation procedure.

16.1 Notation and preliminaries

Throughout this chapter, Ω and D are bounded open subsets of $\mathbb{R}^n$, $n \geq 1$, such that $\Omega \subset\subset D$. We will, as always, associate Ω with some lattice structure. Let χ_Ω be the characteristic function of the set Ω. For two real numbers $1 < p < +\infty$ and $1 < q < +\infty$ such that $1/p + 1/q = 1$, the space $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$ and $W^{-1,q}(\Omega)$ is the space of all distributions of the form $f = f_0 + \sum_j D_j f_j$, with $f_0, f_1, \ldots, f_n \in L^q(\Omega)$ (i.e., $W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$).

We recall that for any vector field $\mathbf{v} \in \mathbf{L}^p(\Omega) = [L^p(\Omega)]^n$, the divergence is an element of the space $W^{-1,q}(\Omega)$ defined by the formula

$$\langle \operatorname{div} \mathbf{v}, \varphi \rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = -\int_\Omega (\mathbf{v}, D\varphi)_{\mathbb{R}^n}\, \mathrm{d}x, \quad \forall\, \varphi \in W_0^{1,p}(\Omega), \tag{16.5}$$

where $\langle \cdot,\cdot \rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)}$ denotes the duality pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the scalar product of two vectors in $\mathbb{R}^n$.

Using the terminology of Sect. 2.3 (see also Sect. 6.4.2 for the periodic case), we say that a vector field $\mathbf{v}$ is solenoidal if $\operatorname{div} \mathbf{v} = 0$. For any vector

field $\mathbf{v} \in \mathbf{L}^p(\Omega)$, the relations

$$\langle \operatorname{curl} \mathbf{v}, \varphi \rangle^{i\,j}_{W^{-1,\,q}(\Omega);W_0^{1,\,p}(\Omega)} = -\int_\Omega (v_i D_j \varphi - v_j D_i \varphi)\ \mathrm{d}x,$$
$$\forall\, \varphi \in W_0^{1,\,p}(\Omega), \quad i,j = 1,\dots,n,$$

define a skew-symmetric matrix $\operatorname{curl} \mathbf{v}$, whose elements belong to the space $W^{-1,\,q}(\Omega)$. We recall that a vector field $\mathbf{v}$ is said to be vortex-free if $\operatorname{curl} \mathbf{v} = 0$. We say that a vector field $\mathbf{v} \in \mathbf{L}^p(\Omega)$ is potential if $\mathbf{v}$ can be represented in the form $\mathbf{v} = Du$, where $u \in W^{1,\,p}(\Omega)$. Obviously, any potential vector is vortex-free.

In order to identify a weakly-$*$ closed subset $\mathcal{M}$ of $L^\infty(D;\mathbb{R}^{n\times n})$ (see (16.2) and (16.3)), we fix two constants α and β such that

$$0 < \alpha \le \beta < +\infty.$$

We define $\mathcal{M}$ as the set $M_\alpha^\beta(D)$ of all symmetric matrices $A = [a_{i\,j}]$ in $L^\infty(D;\mathbb{R}^{n\times n})$ such that

$$\left.\begin{aligned} \|A(x)\|_{\mathbb{R}^{n\times n}} &\le \beta \quad \text{for a.e. } x \in D, \\ \big(A(x)[\zeta^{p-2}]\zeta,\zeta\big)_{R^n} &\ge \alpha|\zeta|_p^p \quad \text{a.e. on } \Omega,\ \forall\,\zeta \in \mathbb{R}^n, \\ \big(A(x)([\zeta^{p-2}]\zeta - [\eta^{p-2}]\eta),\zeta-\eta\big)_{\mathbb{R}^n} &\ge 0 \quad \text{a.e. on } \Omega,\ \forall\,\zeta \ne \eta \in \mathbb{R}^n, \end{aligned}\right\} \tag{16.6}$$

where $|\eta|_p = \big(\sum_{k=1}^n |\eta|_k^p\big)^{1/p}$ and

$$[\eta^{p-2}] = \operatorname{diag}\{|\eta_1|^{p-2}, |\eta_2|^{p-2}, \dots, |\eta_n|^{p-2}\}, \quad \forall \eta \in \mathbb{R}^n, \tag{16.7}$$

$$\big(A(x)[\zeta^{p-2}]\zeta,\zeta\big)_{\mathbb{R}^n} = \sum_{i,j=1}^n a_{ij}(x)|\eta_j|^{p-2}\,\eta_j\,\eta_i. \tag{16.8}$$

Remark 16.1. It is clear that $M_\alpha^\beta(D)$ is a nonempty subset of $L^\infty(D;\mathbb{R}^{n\times n})$ and its typical representatives are the diagonal matrixes

$$A(x) = \operatorname{diag}\{\delta_1(x), \delta_2(x), \dots, \delta_n(x)\}$$

such that $\alpha \le \delta_i(x) \le \beta$ almost everywhere on Ω, $\forall\, i \in \{1,\dots,n\}$. Indeed, let ζ and η be any two fixed vectors of $\mathbb{R}^n$. Then the estimates $(16.6)_1$ and $(16.6)_2$ are obvious. As for the verification of the monotonicity property $(16.6)_3$, we make use of the following chain of transformations:

$$
\begin{aligned}
&\big(A(x)([\zeta^{p-2}]\zeta - [\eta^{p-2}]\eta), \zeta - \eta\big)_{\mathbb{R}^n} \\
&\qquad = \big([\zeta^{p-2}]A(x)\zeta - [\eta^{p-2}]A(x)\eta, \zeta - \eta\big)_{\mathbb{R}^n} \\
&\qquad = \big([\zeta^{p-2}]A(x)\zeta, \zeta\big)_{\mathbb{R}^n} - \big([\zeta^{p-2}]A(x)\zeta, \eta\big)_{\mathbb{R}^n} \\
&\qquad - \big([\eta^{p-2}]A(x)\eta, \zeta\big)_{\mathbb{R}^n} + \big([\eta^{p-2}]A(x)\eta, \eta\big)_{\mathbb{R}^n} \\
&\qquad = \sum_{i=1}^{n} \delta_i(x)|\zeta_i|^{p-2}\zeta_i^2 - \sum_{i=1}^{n} \delta_i(x)|\zeta_i|^{p-2}\zeta_i\eta_i \\
&\qquad - \sum_{i=1}^{n} \delta_i(x)|\eta_i|^{p-2}\zeta_i\eta_i + \sum_{i=1}^{n} \delta_i(x)|\eta_i|^{p-2}\eta_i^2 \\
&\qquad = \sum_{i=1}^{n} \delta_i(x)|\zeta_i|^{p-2}\zeta_i(\zeta_i - \eta_i) - \sum_{i=1}^{n} \delta_i(x)|\eta_i|^{p-2}\eta_i(\zeta_i - \eta_i) \\
&\qquad = \sum_{i=1}^{n} \delta_i(x)\left(|\zeta_i|^{p-2}\zeta_i - |\eta_i|^{p-2}\eta_i\right)(\zeta_i - \eta_i)\,. \qquad (16.9)
\end{aligned}
$$

It remains to use the following estimates (see, for instance, [39] and [55]):

$$
(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq 2^{2-p}|a-b|^p,\ p \geq 2, \qquad (16.10)
$$

$$
(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq (|a| + |b|)^{p-2}\,|a-b|^2,\ 1 < p \leq 2, \qquad (16.11)
$$

for every $a, b \in \mathbb{R}$. As a result, the monotonicity property $(16.6)_3$ is a direct consequence of the coercivity assumption $(16.6)_2$. However, in a general case, these properties should be considered as independent.

Remark 16.2. If $p = 2$ and a matrix $A(x)$ possesses the coerciveness property $(16.6)_2$, then $A(x)$ satisfies the strong monotonicity property

$$
(A(x)(\zeta - \eta), \zeta - \eta)_{\mathbb{R}^n} \geq \alpha|\zeta - \eta|^2_{\mathbb{R}^n} \quad \text{a.e. on } \ \Omega,\ \forall\, \zeta, \eta \in \mathbb{R}^n.
$$

Hence, in this case, the strong monotonicity $(16.6)_3$ is a direct consequence of the coerciveness property $(16.6)_2$.

For a fixed matrix $A \in M_\alpha^\beta(D)$ and a real number $1 < p < +\infty$, we define the mapping $a : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ as follows: $a(x, \zeta) = A(x)[\zeta^{p-2}]\zeta$. Then, as follows from the definition of the set $M_\alpha^\beta(D)$ (see, (16.6)) for a.e. $x \in \mathbb{R}^n$ and for every $\zeta_1 \neq \zeta_2 \in \mathbb{R}^n$, we have:

$$
(a(x, \zeta_1) - a(x, \zeta_2), \zeta_1 - \zeta_2)_{\mathbb{R}^n} \geq 0, \qquad (16.12)
$$

$$
(a(x, \zeta), \zeta)_{\mathbb{R}^n} \geq \alpha|\zeta|_p^p, \qquad |a(x, \zeta)|_{\mathbb{R}^n} \leq \beta|\zeta|_p^{p-1}. \qquad (16.13)
$$

For a given function $a_0 \in L^\infty(D)$ such that $a_0(x) \geq 0$ for a.e. $x \in D$, we consider the operator

$$
\mathcal{A}y = -\text{div}\ (a(x, Dy)) + a_0(x)|y|^{p-2}y
$$

defined via the paring

$$\langle \mathcal{A}y, v\rangle_{W^{-1,\,q}(\Omega),W_0^{1,\,p}(\Omega)} = \int_\Omega \left(A(x)[Dy^{p-2}]Dy, Dv\right)_{\mathbb{R}^n}\,\mathrm{d}x + \int_\Omega a_0(x)|y|^{p-2}yv\,\mathrm{d}x \quad \forall\, y, v \in W_0^{1,\,p}(\Omega).$$

As a result, since $A \in M_\alpha^\beta(D)$, the following estimate takes place

$$\begin{aligned}\int_\Omega a_0(x)\left(|y|^{p-2}y(y-v) - |v|^{p-2}v(y-v)\right)\mathrm{d}x &\geq \int_\Omega a_0(x)\left(|y|^{p-2}(y^2-|y||v|) - |v|^{p-2}(|y||v|-v^2)\right)\mathrm{d}x\\ &= \int_\Omega a_0(x)\left(|y|^{p-1}-|v|^{p-1}\right)(|y|-|v|)\,\mathrm{d}x \geq 0, \end{aligned} \tag{16.14}$$

the operator $\mathcal{A}$ turns out to be coercive and monotone from $W_0^{1,\,p}(\Omega)$ into its dual $W^{-1,\,q}(\Omega)$, and demicontinuous in the following sense: $y_k \to y_0$ strongly in $W_0^{1,\,p}(\Omega)$ implies that $Ay_k \to Ay_0$ weakly in $W^{-1,\,q}(\Omega)$ (see [113, 168]). Moreover, if the element $a_0(\cdot) \in L^\infty(\Omega)$ is strictly separated from 0 in Ω, then inequality (16.14) and estimates (16.10) and (16.11) imply that the monotonicity property for the operator $\mathcal{A} : W_0^{1,\,p}(\Omega) \to W^{-1,\,q}(\Omega)$ can be specified as follows:

Proposition 16.3. *If there exists a positive value β' such that $a_0(x) \geq \beta' > 0$ a.e. on Ω, then the operator $\mathcal{A} : W_0^{1,\,p}(\Omega) \to W^{-1,\,q}(\Omega)$ is strictly monotone, that is,*

$$\langle \mathcal{A}(y) - \mathcal{A}(v), y - v\rangle_{W^{-1,\,q}(\Omega);W_0^{1,\,p}(\Omega)} > 0,$$

$\forall\, y, v \in W_0^{1,\,p}(\Omega)$ such that $y \neq v$ in $L^p(\Omega)$.

Then by the well-known existence results for nonlinear elliptic equations with strictly monotone demicontinuous coercive operators (see [113, 126]), one can easily see that for every open set $\Omega \subset D$ and every $f \in L^q(D)$, the nonlinear Dirichlet boundary value problem

$$\mathcal{A}y = f \quad \text{in} \quad \Omega, \qquad y \in W_0^{1,\,p}(\Omega), \tag{16.15}$$

has a weak solution in $W_0^{1,\,p}(\Omega)$. Let us recall that a function y is the weak solution of (16.15) if

$$y \in W_0^{1,\,p}(\Omega), \tag{16.16}$$

$$\int_\Omega \left(A(x)[Dy^{p-2}]Dy, Dv\right)_{\mathbb{R}^n}\,\mathrm{d}x + \int_\Omega a_0(x)|y|^{p-2}yv\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x, \quad \forall\, v \in W_0^{1,\,p}(\Omega). \tag{16.17}$$

Following [39] and [82], we conclude this section by giving a couple results concerning the estimates for the solution of the Dirichlet problem (16.15).

Proposition 16.4. *For every $A \in M_\alpha^\beta(D)$, $a_0 \in L^\infty(D)$ $(a_0(x) \geq 0$ a.e. $x \in D)$, and $f \in W^{-1,q}(\Omega)$, a weak solution $y_{\Omega,A,f}$ to the problem (16.15) satisfies the estimate*

$$\int_\Omega |Dy_{\Omega,A,f}|_{\mathbb{R}^n}^p \, \mathrm{d}x \leq C\|f\|_{W^{-1,q}(\Omega)}^q, \tag{16.18}$$

where C is a constant depending only on p, α, and β.

Proposition 16.5. *For every open set $\Omega \subseteq D$ let $y_{\Omega,A,f}$ be a weak solution of (16.15), and let $\widetilde{y}_{\Omega,A,f}$ denote the zero-extension of $y_{\Omega,A,f}$ to D. Then*

$$\|\widetilde{y}_{\Omega,A,f}\|_{W_0^{1,p}(D)} \leq C,$$

where C depends on f, a_0, p, α, and β.

16.2 *H*-Convergence and a counterexample of Murat

Murat showed in 1970 (see [193, 194]) that the problem (16.1)–(16.4) has no solution, in general. It turns out that this feature is typical for the majority of OCPs with respect to coefficients. In addition, this fact is not just a mathematical problem, but it is also very restrictive from the point of view of numerical applications.

The initial problem that Murat considered was to minimize the cost functional

$$I(a,y) = \int_0^1 \left|y(x) - 1 - x^2\right|^2 \, \mathrm{d}x, \tag{16.19}$$

where y is the solution of the boundary value problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}y}{\mathrm{d}x}\right) + ay = 0 \quad \text{in} \quad (0,1), \tag{16.20}$$

$$y(0) = 1, \quad y(1) = 2, \quad y \in H^1(0,1) \tag{16.21}$$

and the control a belongs to the following admissible control set

$$\mathcal{A} = \{a \mid a \in L^\infty(0,1),\ \alpha \leq a \leq \beta \text{ a.e. on } (0,1)\}. \tag{16.22}$$

Trying to apply the direct method of the Calculus of Variations, he noticed that for a sequence $\{a_n\}_{n=1}^\infty \subset \mathcal{A}$ such that

$$a_n \rightharpoonup a_+ \quad \text{and} \quad \frac{1}{a_n} \rightharpoonup \frac{1}{a_-} \quad \text{weakly-}* \text{ in } \ L^\infty(0,1), \tag{16.23}$$

the corresponding sequence of solutions $y_n = y(a_n)$ of (16.20) and (16.21) converges weakly in $H^1(0,1)$ to the solution y_∞ of the problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(a_-\frac{\mathrm{d}y_\infty}{\mathrm{d}x}\right)+a_+y_\infty=0 \quad \text{in} \quad (0,1), \tag{16.24}$$

$$y_\infty(0)=1, \quad y_\infty(1)=2, \quad y_\infty\in H^1(0,1), \tag{16.25}$$

with the properties

$$a_n\frac{\mathrm{d}y_n}{\mathrm{d}x}\longrightarrow a_-\frac{\mathrm{d}y_\infty}{\mathrm{d}x} \quad \text{strongly in} \quad L^2(0,1), \tag{16.26}$$

$$I(a_n,y_n)\to\widetilde{I}(a_-,a_+,y_\infty)=\int_0^1|y_\infty(x)-1-x^2|^2\,\mathrm{d}x. \tag{16.27}$$

Indeed, y_n is bounded in $H^1(0,1)$ and $v_n=a_n\mathrm{d}y_n/\mathrm{d}x$ is bounded in $L^2(0,1)$, but as its derivative is a_ny_n which is bounded in $L^2(0,1)$, v_n is actually bounded in $H^1(0,1)$. Therefore, one can extract a subsequence such that y_m weakly converges in $H^1(0,1)$ and strongly in $L^2(0,1)$ to y_∞, and v_m strongly converges in $L^2(0,1)$ to v_∞. Then a_my_m and $\mathrm{d}y_m/\mathrm{d}x=(1/a_m)v_m$ weakly converge in $L^2(0,1)$, respectively to a_+y_∞ and to $\mathrm{d}y_\infty/\mathrm{d}x=(1/a_-)v_\infty$, showing (16.26), and therefore (16.24) and (16.25); the fact that y_∞ is uniquely determined by (16.24) and (16.25) shows that the extraction of a subsequence is not necessary.

Moreover, in this case, it can be shown that all possible pairs (a_-,a_+) which could appear in (16.23) satisfy the inequality

$$\alpha\le a_-(x)\le a_+(x)\le\frac{a_-(x)(\alpha+\beta)-\alpha\beta}{a_-(x)}\le\beta \quad \text{for a.e.} \quad x\in(0,1),$$

or equivalently,

$$\frac{1}{a_+(x)}\le\frac{1}{a_-(x)}\le\frac{\alpha+\beta-a_+(x)}{\alpha\beta} \quad \text{for a.e.} \quad x\in(0,1).$$

In particular, if $\alpha=\left(\sqrt{2}-1\right)/\sqrt{2}$, $\beta=\left(\sqrt{2}+1\right)/\sqrt{2}$, and

$$a_n(x)=\begin{cases}1-\sqrt{\dfrac{1}{2}-\dfrac{x^2}{6}} & \text{if} \quad x\in\left(\dfrac{2k}{2n},\dfrac{2k+1}{2n}\right),\ k=0,\dots,n-1,\\[2ex] 1+\sqrt{\dfrac{1}{2}-\dfrac{x^2}{6}} & \text{if} \quad x\in\left(\dfrac{2k+1}{2n},\dfrac{2k+2}{2n}\right),\ k=0,\dots,n-1,\end{cases}$$

then Murat has shown in [193] that

$$a_-=\frac{1}{2}+\frac{x^2}{6};\ a_+=1;\ y_\infty=1+x^2 \quad \text{on} \quad (0,1), \tag{16.28}$$

$$\lim_{n\to\infty}I(a_n,y_n)=\inf_{(a,y)\in\Xi}I(a,y)=0, \tag{16.29}$$

where Ξ denotes the set of all admissible pairs for the problem (16.19)–(16.22). However, it is not possible to have $y=1+x^2$ in (16.20) and (16.21) for some

$a \in \mathcal{A}$. It means that a pair $(a, 1 + x^2)$ does not belong to the set Ξ for any $a \in \mathcal{A}$.

In order to clarify this situation, we recall the notion of H-convergence introduced by Tartar in 1977 (see [236]) and later developed by Murat and Tartar (see [196]). Let $\{A_\varepsilon\}_{\varepsilon>0}$ be a sequence of matrices in $M_\alpha^\beta(\Omega)$ which are not necessary symmetric.

Definition 16.6. *A sequence $\{A_\varepsilon\}_{\varepsilon>0}$ in $M_\alpha^\beta(\Omega)$ with $p = 2$ is said to be H-convergent to $A \in M_\alpha^\beta(\Omega)$ (in symbols, $A_\varepsilon \overset{H}{\to} A$) as $\varepsilon \to 0$ if the following condition holds. Whenever vector fields $\mathbf{d}_\varepsilon$, $\mathbf{d}$, $\mathbf{v}_\varepsilon$, $\mathbf{v} \in \mathbf{L}^2(\Omega)$ satisfy*

$$\mathbf{d}_\varepsilon = A_\varepsilon \mathbf{v}_\varepsilon, \quad \forall\, \varepsilon > 0, \tag{16.30}$$

$$\mathbf{d}_\varepsilon \to \mathbf{d} \text{ weakly in } \mathbf{L}^2(\Omega), \tag{16.31}$$

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly in } \mathbf{L}^2(\Omega), \tag{16.32}$$

$$\{\operatorname{div} \mathbf{d}_\varepsilon\}_{\varepsilon>0} \text{ is relatively compact in } W^{-1,2}(\Omega), \tag{16.33}$$

$$\{\operatorname{div} \mathbf{v}_\varepsilon\}_{\varepsilon>0} \text{ is relatively compact in } \left[W^{-1,2}(\Omega)\right], \tag{16.34}$$

then

$$\mathbf{d} = A\mathbf{v}. \tag{16.35}$$

It can be proved that the above definition of H-convergence is equivalent to the slightly different definition given in [196] concerning the limit behavior of the elliptic equations governed by matrix A_ε (see Definition 16.7).

Definition 16.7. *A sequence $\{A_\varepsilon\}_{\varepsilon>0}$ of matrices in $M_\alpha^\beta(\Omega)$ with $p = 2$ H-converges to a matrix $A \in M_\alpha^\beta(\Omega)$ if for every $f \in W^{-1,2}(\Omega)$, the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of the solutions to the Dirichlet problems*

$$\begin{cases} u_\varepsilon \in W_0^{1,2}(\Omega), \\ -\operatorname{div}(A_\varepsilon\, Du_\varepsilon) = f \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{16.36}$$

satisfies

$$u_\varepsilon \to u \quad \text{weakly in } W_0^{1,2}(\Omega),$$
$$A_\varepsilon\, Du_\varepsilon \to A\, Du \quad \text{weakly in } L^2(\Omega, \mathbb{R}^n),$$

where u is the solution to the problem

$$\begin{cases} u \in W_0^{1,2}(\Omega), \\ -\operatorname{div}(A\, Du) = f \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{16.37}$$

We list some of the basic properties of H-convergence:

(i) The H-limit of an H-converging sequence is unique.
(ii) The set $M_\alpha^\beta(\Omega)$ with $p = 2$ is sequentially compact with respect to the H-convergence.

(iii) H-convergence is stable under transposition of the matrices.

(iv) If $\{A_\varepsilon\}_{\varepsilon>0}$, $\{B_\varepsilon\}_{\varepsilon>0} \subset M_\alpha^\beta(\Omega)$, $A_\varepsilon \overset{H}{\to} A$, $B_\varepsilon \overset{H}{\to} B$, and for some open set $U \subset \Omega$, one has $A_\varepsilon = B_\varepsilon$ in U for every $\varepsilon > 0$, then $A = B$ in U.

Remark 16.8. As follows from the counterexample given above, in general, the H-limit of a bounded sequence $\{A_\varepsilon\}_{\varepsilon>0}$ and its weak-$*$ limit in $L^\infty(\Omega, \mathbb{R}^{n\times n})$ may be drastically different. Moreover, this example indicates that the set of admissible solutions Ξ to the problem (16.19)–(16.22) is not sequentially closed with respect to the product of H-convergence in $L^\infty(\Omega, \mathbb{R}^{n\times n})$ and the weak convergence in $W_0^{1,2}(\Omega)$ and also closedness of Ξ is lacking with respect to the product of weak-$*$ convergence in $L^\infty(\Omega, \mathbb{R}^{n\times n})$ and the weak convergence in $W_0^{1,2}(\Omega)$. Indeed, in this case, we have $(a_-, y_\infty) \notin \Xi$ and $(a_+, y_\infty) \notin \Xi$.

At the same time, Proposition 16.5, the definition of H-convergence, and its properties immediately imply that the set of admissible solutions $\Xi \subset L^\infty(\Omega, \mathbb{R}^{n\times n}) \times W_0^{1,2}(\Omega)$ to the OCP

$$\text{Minimize} \quad \left\{ I(A,y) = \int_\Omega |y(x) - y_d(x)|^2 \, \mathrm{d}x + \int_\Omega |Dy(x)|^2_{\mathbb{R}^n} \, \mathrm{d}x \right\} \tag{16.38}$$

subject to the constraints

$$A \in M_\alpha^\beta(\Omega), \quad p = 2, \quad y \in H_0^1(\Omega), \tag{16.39}$$

$$-\mathrm{div}\,(A(x)Dy) + a_0(x)y = f \quad \text{in} \quad \Omega \tag{16.40}$$

is sequentially compact with respect to the product of H-convergence in $L^\infty(\Omega, \mathbb{R}^{n\times n})$ and the weak convergence in $W_0^{1,2}(\Omega)$. Moreover, in this case, the cost functional (16.38) is lower semicontinuous in the following sense:

$$\left.\begin{array}{l} A_k \overset{H}{\to} A_* \ \text{ in } L^\infty(\Omega; \mathbb{R}^{n\times n}) \\ y_k \rightharpoonup y_* \ \text{ in } W_0^{1,2}(\Omega) \end{array}\right\} \Longrightarrow \liminf_{k\to\infty} I(A_k, y_k) \geq I(A_*, y_*). \tag{16.41}$$

As a result, we come to the conclusion.

Theorem 16.9. *The problem (16.38)–(16.40) possesses at least one solution.*

However, as for the OCP (16.1)–(16.4), the set $\mathcal{M} = M_\alpha^\beta(\Omega)$ is not H-closed in general, and the property of lower semicontinuity for the cost functional $I_\Omega(A, y)$ like (16.41) cannot be guaranteed. Thus, the existence of optimal solutions for the problem (16.1)–(16.4) is, in general, an open question.

16.3 Setting of the optimal control problem and existence theorem

Let us associate with (16.15) the following minimization problem:

$$\text{Minimize} \quad \Big\{ I_{\Omega}(A,y) = \int_{\Omega} |y(x) - y_d(x)|^p \, \mathrm{d}x + \int_{\Omega} |Dy(x)|^p_{\mathbb{R}^n} \, \mathrm{d}x + J(A) \Big\} \tag{16.42}$$

subject to the constraints

$$A \in M^{\beta}_{\alpha}(D), \quad y \in W^{1,\,p}_0(\Omega), \tag{16.43}$$

$$\int_{\Omega} \big(A(x)[Dy^{p-2}]Dy, Dv\big)_{\mathbb{R}^n} \, \mathrm{d}x + \int_{\Omega} a_0(x)|y|^{p-2}yv \, \mathrm{d}x = \int_{\Omega} fv \, \mathrm{d}x, \quad \forall\, v \in W^{1,\,p}_0(\Omega), \tag{16.44}$$

where $f \in L^q(D)$, $y_d \in L^p(D)$, $a_0 \in L^{\infty}(D)$, $a_0(x) \geq 0$ a. e. on D, and $J : L^{\infty}(D;\mathbb{R}^{n\times n}) \to \mathbb{R}$ is a lower semicontinuous functional with respect to the weak-$*$ topology of $L^{\infty}(D;\mathbb{R}^{n\times n})$ in the following sense:

$$A_k \overset{*}{\rightharpoonup} A_* \text{ in } L^{\infty}(D;\mathbb{R}^{n\times n}) \Longrightarrow \liminf_{k\to\infty} J(A_k) \geq J(A_*).$$

It is well known that for both physical and mathematical reasons, the size of the controls should be constrained. Physically, one cannot realize controls of arbitrary size. On the other hand, limits on the size of the control are needed in order to obtain a mathematically meaningful problem (e.g., to guarantee the existence of an optimal solution in the prescribed functional space).

Let $\{Q_1, \ldots, Q_n\}$ be a collection of nonempty compact subsets of the space $W^{-1,\,q}(D)$. To define the class of admissible controls, we introduce two sets (see [131])

$$U_b = \big\{ A = [a_{i\,j}] \in M^{\beta}_{\alpha}(D) \big\}, \tag{16.45}$$

$$U_{sol} = \big\{ A = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in M^{\beta}_{\alpha}(D) \big| \operatorname{div} \mathbf{a}_i \in Q_i, \ \forall\, i = 1, \ldots, n \big\}, \tag{16.46}$$

assuming that the intersection $U_b \cap U_{sol} \subset L^{\infty}(D;\mathbb{R}^{n\times n})$ is a nonempty set.

Definition 16.10. *We say that a matrix $A = [a_{i\,j}]$ is an admissible control to the nonlinear Dirichlet problem (16.15) if $A \in U_{ad} := U_b \cap U_{sol}$.*

Hereinafter we suppose that the set of admissible controls U_{ad} is sufficiently rich in some sense; otherwise, in view of Proposition 16.4, the OCP

$$I_{\Omega}(A,y) = \int_{\Omega} |y(x) - y_d(x)|^p \, \mathrm{d}x + \int_{\Omega} |Dy(x)|^p_{\mathbb{R}^n} \, \mathrm{d}x + J(A) \ \longrightarrow \inf, \tag{16.47}$$

$$-\operatorname{div}\big(A(x)[Dy^{p-2}]Dy\big) + a_0|y|^{p-2}y = f \quad \text{in} \quad \Omega, \tag{16.48}$$

$$y \in W^{1,\,p}_0(\Omega), \quad A \in U_{ad} \tag{16.49}$$

becomes trivial. The existence of admissible controls is important both from a theoretical and an application point of view. Usually, this type of controls arises in the optimization of materials (represented by the matrix A). This question is largely open, except for some special cases, and an affirmative answer is usually just put as a hypothesis (see [7, 51, 181]).

Let us denote by $\Xi \subset L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,\,p}(\Omega)$ the set of all admissible pairs to the OCP (16.47)–(16.49). The first question to be answered for the problem (16.47)–(16.49) is about solvability: Does there exists an optimal pair (A^0, y^0) in $L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,\,p}(\Omega)$ satisfying (16.48) and (16.49)? To begin with, we prove the following result.

Proposition 16.11. *The set U_{ad} is sequentially compact with respect to the weak-$*$ topology of $L^\infty(D;\mathbb{R}^{n\times n})$.*

Proof. Let $\{A_k = [\mathbf{a}_{1\,k}, \ldots, \mathbf{a}_{n\,k}]\}_{k\in\mathbb{N}} \subset U_{ad}$ be an arbitrary sequence of admissible controls. Since $U_{ad} \subset U_b$ and U_b is the sequentially weakly-$*$ compact subset of $L^\infty(D;\mathbb{R}^{n\times n})$, we may suppose that there exist a matrix $A_0 = [\mathbf{a}_{1\,0}, \ldots, \mathbf{a}_{n\,0}] \in U_b$ and elements $f_i \in Q_i$, $i = 1, \ldots, n$, such that

$$\int_D (\mathbf{a}_{i\,k}, \phi)_{\mathbb{R}^n}\,\mathrm{d}x \overset{k\to\infty}{\longrightarrow} \int_D (\mathbf{a}_{i\,0}, \phi)_{\mathbb{R}^n}\,\mathrm{d}x, \quad \forall \phi \in \mathbf{L}^1(D) = [L^1(D)]^n,\ \forall\, i = 1, 2, \ldots, n, \tag{16.50}$$

$$\operatorname{div} \mathbf{a}_{i\,k} \overset{k\to\infty}{\longrightarrow} f_i \ \text{ in } \ W^{-1,\,q}(D),\ \forall\, i = 1, \ldots, n. \tag{16.51}$$

It remains to prove that $\operatorname{div} \mathbf{a}_{i\,0} = f_i$ for all $i = 1, \ldots, n$. To do this, we choose ϕ in (16.50) as a potential vector, that is, $\phi = Dv$, where $v \in W_0^{1,\,p}(D)$. Then, in view of (16.5), relation (16.51) implies

$$\begin{aligned}\int_D (\mathbf{a}_{i\,k}, Dv)_{\mathbb{R}^n}\,\mathrm{d}x &= -\,\langle \operatorname{div} \mathbf{a}_{i\,k}, v\rangle_{W^{-1,\,q}(D),W_0^{1,\,p}(D)} \\ &\overset{k\to\infty}{\longrightarrow} -\,\langle f_i, v\rangle_{W^{-1,\,q}(D);W_0^{1,\,p}(D)}\,,\ \forall i = 1, \ldots, n.\end{aligned}$$

Using this and relation (16.50), we finally get

$$\begin{aligned}\lim_{k\to\infty} \int_D (\mathbf{a}_{i\,k}, Dv)_{\mathbb{R}^n}\,\mathrm{d}x &= \int_D (\mathbf{a}_{i\,0}, Dv)_{\mathbb{R}^n}\,\mathrm{d}x \\ &= -\,\langle \operatorname{div} \mathbf{a}_{i\,0}, v\rangle_{W^{-1,\,q}(D);W_0^{1,\,p}(D)} \\ &= -\,\langle f_i, v\rangle_{W^{-1,\,q}(D);W_0^{1,\,p}(D)}\,, \quad \forall\, i = 1, \ldots, n.\end{aligned}$$

As a result, we have $A_0 = [\mathbf{a}_{1\,0}, \ldots, \mathbf{a}_{n\,0}] \in U_{sol}$. This concludes the proof. □

In order to discuss the existence of solutions for the problem (16.48)–(16.49), we make use of the following result (for comparison, we refer to the Lemma on Compensated Compactness in [261]).

Lemma 16.12. *Let* $\{\mathbf{f}_k\}_{k\in\mathbb{N}} \subset \mathbf{L}^q(D)$, $\{\mathbf{g}_k\}_{k\in\mathbb{N}} \subset \mathbf{L}^p(D)$ *be a bounded sequences of vector functions such that* $\mathbf{f}_k \rightharpoonup \mathbf{f}_0$ *in* $\mathbf{L}^q(D)$ *and* $\mathbf{g}_k \rightharpoonup \mathbf{g}_0$ *in* $\mathbf{L}^p(D)$. *If*

$$\{\operatorname{div}\mathbf{f}_k\}_{k\in\mathbb{N}} \quad \begin{array}{l}\textit{is compact with respect}\\ \textit{to the strong topology of } W^{-1,\,q}(D),\end{array} \tag{16.52}$$

$$\textit{and}\quad \operatorname{curl}\mathbf{g}_k = 0, \quad \forall\, k\in\mathbb{N}, \tag{16.53}$$

then

$$\lim_{k\to\infty}\int_D \phi\ (\mathbf{f}_k,\mathbf{g}_k)_{\mathbb{R}^n}\ \mathrm{d}x = \int_D \phi\ (\mathbf{f}_0,\mathbf{g}_0)_{\mathbb{R}^n}\ \mathrm{d}x, \quad \forall\,\phi\in C_0^\infty(D). \tag{16.54}$$

Remark 16.13. Equality (16.54) can be interpreted as the weak-$*$ convergence in $L^1(D)$ of the sequence $\{(\mathbf{f}_k,\mathbf{g}_k)_{\mathbb{R}^n}\}_{k\in\mathbb{N}}$ to the element $(\mathbf{f}_0,\mathbf{g}_0)_{\mathbb{R}^n}$.

Proof. We have

$$(\mathbf{f}_k,\mathbf{g}_k)_{\mathbb{R}^n} = (\mathbf{f}_k-\mathbf{f}_0,\mathbf{g}_k-\mathbf{g}_0)_{\mathbb{R}^n} - (\mathbf{f}_0,\mathbf{g}_0)_{\mathbb{R}^n} + (\mathbf{f}_k,\mathbf{g}_0)_{\mathbb{R}^n} + (\mathbf{f}_0,\mathbf{g}_k)_{\mathbb{R}^n}\,.$$

The sum of the last three terms obviously has a weak-$*$ limit in $L^1(D)$ which is equal to $(\mathbf{f}_0,\mathbf{g}_0)_{\mathbb{R}^n}$. Therefore, it is sufficient to consider only the case $\mathbf{f}_0 = \mathbf{g}_0 = 0$.

Notice that the weak-$*$ convergence in $L^1(D)$ has a local property and, thus, for the simplicity, the domain D may be assumed to be a ball. Then, as follows from (16.53), $\mathbf{g}_k$ is a potential vector field for every $k\in\mathbb{N}$ (i.e., $\mathbf{g}_k = Dv_k$) and one can assume that the potential function v_k satisfies the condition $\int_D v_k\,\mathrm{d}x = 0$, $\forall\, k\in\mathbb{N}$. By the initial assumptions, we have $\mathbf{g}_k = Dv_k \rightharpoonup 0$ in $\mathbf{L}^p(D)$. Hence, by the Poincaré inequality

$$\int_D v_k^p\,\mathrm{d}x \le C\left\{\left(\int_D v_k\,\mathrm{d}x\right)^p + \int_D |Dv_k|_{\mathbb{R}^n}^p\,\mathrm{d}x\right\} = C\int_D |Dv_k|_{\mathbb{R}^n}^p\,\mathrm{d}x$$

and the Sobolev embedding theorem, it follows that

$$v_k \rightharpoonup 0 \ \text{ in } \ W^{1,\,p}(D), \quad v_k \to 0 \ \text{ in } \ L^p(D).$$

Notice also that due to condition (16.52), we may suppose that (passing to a subsequence if necessary) $\operatorname{div}\mathbf{f}_k \to 0$ strongly in $W^{-1,\,q}(D)$.

Now, for every fixed function $\phi\in C_0^\infty(D)$, we have

$$\begin{aligned}\int_D (\mathbf{f}_k,\mathbf{g}_k)_{\mathbb{R}^n}\ \phi\,\mathrm{d}x &= \int_D (\mathbf{f}_k, D(\phi\,v_k))_{\mathbb{R}^n}\ \mathrm{d}x - \int_D v_k\,(\mathbf{f}_k, D\phi)_{\mathbb{R}^n}\ \mathrm{d}x\\ &= \langle \operatorname{div}\mathbf{f}_k, \phi\,v_k\rangle_{W^{-1,\,q}(D);W_0^{1,\,p}(D)}\\ &\quad - \int_D v_k\,(\mathbf{f}_k, D\phi)_{\mathbb{R}^n}\ \mathrm{d}x.\end{aligned} \tag{16.55}$$

Since $\operatorname{div} \mathbf{f}_k \to 0$ strongly in $W^{-1,\,q}(D)$ and the sequence $\{\phi\, v_k\}_{k\in\mathbb{N}}$ is bounded in $W_0^{1,\,p}(D)$, it follows that

$$\langle \operatorname{div} \mathbf{f}_k, \phi\, v_k\rangle_{W^{-1,\,q}(D);W_0^{1,\,p}(D)} \overset{k\to\infty}{\longrightarrow} 0.$$

The last integral converges to 0, since $v_k \to 0$ strongly in $L^p(D)$, and the sequence $\{(\mathbf{f}_k, D\phi)_{\mathbb{R}^n}\}_{k\in\mathbb{N}}$ is bounded in $L^q(D)$. □

We are now in a position to study the topological properties of the set

$$\Xi \subset L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,\,p}(\Omega)$$

of all admissible pairs to the OCP (16.47)–(16.49). Let τ be the topology on the set $L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,\,p}(\Omega)$ which we define as the product of the weak-$*$ topology of $L^\infty(D;\mathbb{R}^{n\times n})$ and the weak topology of $W_0^{1,\,p}(\Omega)$.

Theorem 16.14 ([139]). *For every $f \in W^{-1,\,q}(\Omega)$, the set Ξ is sequentially τ-closed.*

Proof. Let $\{(A_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ be any τ-convergent sequence of admissible pairs to the problem (16.47)–(16.49). Let (A_0, y_0) be its τ-limit. Our aim is to prove that $(A_0, y_0) \in \Xi$. Let us set

$$\mathcal{A}(A,y) = -\operatorname{div}\left(A(x)\,[D\,y^{p-2}]\,D\,y\right) + a_0(x)\,|y|^{p-2}\,y = \mathcal{A}_1(A,y) + \mathcal{A}_2(y),$$
$$\mathcal{A}_1(A,y) = -\operatorname{div}\left(A(x)\,[D\,y^{p-2}]\,D\,y\right) = -\operatorname{div} a(A(x), D\,y).$$

By Proposition 16.11 and the initial assumptions, we have $A_0 \in U_{ad}$; therefore,

$$A_k \rightharpoonup A_0 = [\mathbf{a}_{1\,0}, \ldots, \mathbf{a}_{n\,0}] \quad \text{weakly-}* \ \text{ in } \ L^\infty(D, R^{n\times n}), \tag{16.56}$$
$$\operatorname{div} \mathbf{a}_{i\,k} \to \operatorname{div} \mathbf{a}_{i\,0} \ \text{ strongly in } \ W^{-1,\,q}(D), \ \forall\, i = 1, \ldots, n, \tag{16.57}$$
$$y_k \rightharpoonup y_0 \ \text{ in } W_0^{1,\,p}(\Omega). \tag{16.58}$$

Hence,

$$\left\{[Dy_k^{p-2}]y_k\right\}_{k\in\mathbb{N}} \ \text{ is bounded in } \ \mathbf{L}^q(\Omega), \ q = p/(p-1),$$
$$\left\{|y_k|^{p-2}y_k\right\}_{k\in\mathbb{N}} \ \text{ is bounded in } \ L^q(\Omega), \tag{16.59}$$
$$y_k \to y_0 \ \text{ in } \ L^p(\Omega), \quad y_k(x) \to y_0(x) \ \text{ for a.e. } x \in \Omega. \tag{16.60}$$

Then, by (16.60) and the monotonicity of the function $g(\zeta) = |\zeta|^{p-2}\zeta$, we have

$$|y_k|^{p-2}y_k \to |y_0|^{p-2}y_0 \ \text{ almost everywhere on } \Omega.$$

Using this and (16.59), we conclude (see [168])

$$|y_k|^{p-2}y_k \rightharpoonup |y_0|^{p-2}y_0 \quad \text{in} \quad L^q(\Omega).$$

Consider the sequence

$$\left\{f_k := f - a_0|y_k|^{p-2}y_k\right\}_{k\in\mathbb{N}}.$$

It is clear that $f_k \in W^{-1,\,q}(\Omega)$ for every $k \in \mathbb{N}$ and, since the embedding $L^q(\Omega) \hookrightarrow W^{-1,\,q}(\Omega)$ is compact, it follows that

$$f_k \to f_0 = f - a_0|y_0|^{p-2}y_0 \quad \text{strongly in} \quad W^{-1,\,q}(\Omega). \tag{16.61}$$

In view of this and the fact that

$$-\operatorname{div} a(A_k(x), D\,y_k) = f_k \ \text{ in } \ \Omega,\ \forall\, k \in \mathbb{N},$$

we come to the following conclusion (see (16.13)): $\{a(A_k(x), D\,y_k)\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbf{L}^q(\Omega)$. So, passing to a subsequence, we may assume that there exists a vector function $\boldsymbol{\xi} \in \mathbf{L}^q(\Omega)$ such that

$$a(A_k(x), D\,y_k) = A_k\,[D\,y_k^{p-2}]\,D\,y_k =: \boldsymbol{\xi}_k \rightharpoonup \boldsymbol{\xi} \quad \text{on} \ \ \mathbf{L}^q(\Omega). \tag{16.62}$$

Passing to the limit in the relation as $k \to \infty$ (in the sense of the distributions)

$$-\operatorname{div}\boldsymbol{\xi}_k = f - a_0|y_k|^{p-2}|y_k| \quad \text{in} \quad \mathcal{D}'(\Omega)$$

yields

$$-\operatorname{div}\boldsymbol{\xi} = f - a_0|y_0|^{p-2}|y_0| \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{16.63}$$

It remains to show that $\boldsymbol{\xi} = A_0\,|D\,y_0|^{p-2}_{\mathbb{R}^n}\,D\,y_0$. To do so, we consider the scalar function

$$v(x) = (z, x)_{\mathbb{R}^n}, \tag{16.64}$$

where z is a fixed element of $\mathbb{R}^n$. Since the operator $\mathcal{A}_1$ is monotone, it follows that for every $z \in \mathbb{R}^n$ and every positive function $\phi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \phi(x)\,(a(A_k, Dy_k) - a(A_k, Dv), Dy_k - Dv)_{\mathbb{R}^n}\ \mathrm{d}x \geq 0,$$

or, taking into account (16.64), this inequality can be rewritten as

$$\int_\Omega \phi(x)\,(a(A_k, Dy_k) - a(A_k, z), Dy_k - z)_{\mathbb{R}^n}\ \mathrm{d}x \geq 0. \tag{16.65}$$

Our next intention is to pass to the limit in (16.65) as $k \to \infty$ using Lemma 16.12. Since

$$\left.\begin{aligned} -\operatorname{div} a(A_k, Dy_k) \to f - a_0|y_0|^{p-2}|y_0| \ \text{ strongly in } W^{-1,\,q}(\Omega),\\ \operatorname{curv}(D\,y_k - z) = \operatorname{curv} D\,y_k = 0, \quad \forall\, k \in \mathbb{N}, \end{aligned}\right\} \tag{16.66}$$

it remains to show that the sequence $\{\operatorname{div} a(A_k, z)\}_{k\in\mathbb{N}}$ is compact with respect to the strong topology of $W^{-1,\,q}(\Omega)$. Indeed, for every $\phi \in C_0^\infty(\Omega)$, we have

$$
\begin{aligned}
&\big\langle -\operatorname{div} a(A_k, z), \phi\big\rangle_{W^{-1,\,q}(\Omega);W_0^{1,\,p}(\Omega)} \\
&\quad = \int_\Omega (a(A_k, z), D\phi)_{\mathbb{R}^n}\,\mathrm{d}x = \int_\Omega \big(A_k(x)[z^{p-2}]z, D\phi\big)\,\mathrm{d}x \\
&\quad = \int_\Omega \left(\begin{bmatrix} (\mathbf{a}_{1\,k}(x), [z^{p-2}]z)_{\mathbb{R}^n} \\ \dots \\ (\mathbf{a}_{n\,k}(x), [z^{p-2}]z)_{\mathbb{R}^n} \end{bmatrix}, D\phi \right)_{\mathbb{R}^n} \mathrm{d}x \\
&\quad = \int_\Omega \sum_{i=1}^n \big(\mathbf{a}_{i\,k}(x), [z^{p-2}]z\big)_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i}\,\mathrm{d}x \\
&\quad = \int_\Omega \sum_{i=1}^n \sum_{j=1}^n a_{i\,j}^k(x) \frac{\partial \phi}{\partial x_i} |z_j|^{p-2} z_j\,\mathrm{d}x \\
&\quad = \sum_{j=1}^n |z_j|^{p-2} z_j \int_\Omega (\mathbf{a}_{j\,k}(x), D\phi)_{\mathbb{R}^n}\,\mathrm{d}x \\
&\quad = \sum_{j=1}^n |z_j|^{p-2} z_j \int_D (\mathbf{a}_{j\,k}(x), D\widetilde{\phi})_{\mathbb{R}^n}\,\mathrm{d}x \\
&\quad = -\sum_{j=1}^n |z_j|^{p-2} z_j \left\langle \operatorname{div} \mathbf{a}_{j\,k}, \widetilde{\phi} \right\rangle_{W^{-1,\,q}(D);W_0^{1,\,p}(D)} = J_k. \qquad (16.67)
\end{aligned}
$$

Then, using (16.57), we get

$$
\begin{aligned}
\lim_{k\to\infty} J_k &= \sum_{j=1}^n |z_j|^{p-2} z_j \lim_{k\to\infty} \left\langle -\operatorname{div} \mathbf{a}_{j\,k}, \widetilde{\phi} \right\rangle_{W^{-1,\,q}(D);W_0^{1,\,p}(D)} \\
&= \sum_{j=1}^n |z_j|^{p-2} z_j \left\langle -\operatorname{div} \mathbf{a}_{j\,0}, \widetilde{\phi} \right\rangle_{W^{-1,\,q}(D);W_0^{1,\,p}(D)}. \qquad (16.68)
\end{aligned}
$$

Making the reverse transformations with (16.68) as we did it in (16.67), we come to the relation

$$
\begin{aligned}
&\lim_{k\to\infty} \big\langle -\operatorname{div} a(A_k, z), \phi\big\rangle_{W^{-1,\,q}(\Omega);W_0^{1,\,p}(\Omega)} \\
&\qquad = \big\langle -\operatorname{div} a(A_0, z), \phi\big\rangle_{W^{-1,\,q}(\Omega);W_0^{1,\,p}(\Omega)}. \qquad (16.69)
\end{aligned}
$$

Since, for every $i = 1, \dots, n$, the sequences $\{\operatorname{div} \mathbf{a}_{i\,k}\}_{k\in\mathbb{N}}$ are strongly convergent in $W^{-1,\,q}(D)$, it follows from (16.67)–(16.69) that

$$
\begin{aligned}
&\lim_{k\to\infty} \big\langle -\operatorname{div} a(A_k, z), \phi_k\big\rangle_{W^{-1,\,q}(\Omega);W_0^{1,\,p}(\Omega)} \\
&\qquad = \big\langle -\operatorname{div} a(A_0, z), \phi\big\rangle_{W^{-1,\,q}(\Omega);W_0^{1,\,p}(\Omega)} \qquad (16.70)
\end{aligned}
$$

for each sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that $\phi_k \rightharpoonup \phi$ in $W_0^{1,\,p}(\Omega)$. Thus, summing up the above results, we obtain

$$\left.\begin{aligned} \operatorname{div} a(A_k, z) &\to \operatorname{div} a(A_0, z) \quad \text{strongly in} \quad W^{-1,\,q}(\Omega), \\ a(A_k, z) = A_k[z^{p-2}]z &\rightharpoonup A_0[z^{p-2}]z \quad \text{weakly-}* \quad \text{in} \quad \mathbf{L}^\infty(\Omega). \end{aligned}\right\} \tag{16.71}$$

As a result, combining properties (16.66) and (16.71), it has been shown that all suppositions of Lemma 16.12 are fulfilled. So, taking into account (16.58), (16.66), and (16.71) and passing to the limit in inequality (16.65) as $k \to \infty$, we get

$$\int\limits_{\Omega} \phi(x)\,(\xi - a(A_0, z), D\,y_0 - z)_{\mathbb{R}^n}\,\mathrm{d}x \geq 0, \quad \forall z \in \mathbb{R}^n,$$

for all positive $\phi \in C_0^\infty(\Omega)$. After localization, we have

$$(\xi - a(A_0, z), Dy_0 - z)_{\mathbb{R}^n} \geq 0, \quad \forall z \in \mathbb{R}^n.$$

Since the operator $\mathcal{A}_1$ is monotone, it follows that

$$\xi = a(A_0, Dy_0) = A_0(x)[Dy_0^{p-2}]Dy_0. \tag{16.72}$$

As a result, (16.63) takes the form

$$-\operatorname{div}\left(A_0(x)[Dy_0^{p-2}]Dy_0\right) + a_0|y_0|^{p-2}|y_0| = f \quad \text{in} \quad \mathcal{D}'(\Omega).$$

Hence, the τ-limit pair (A_0, y_0) is an admissible to the problem (16.47)–(16.49), and this concludes the proof. □

Now, we can turn to the existence of optimal pairs.

Theorem 16.15. *Under the control admissibility hypothesis*

$$U_{ad} = U_b \cap U_{sol} \neq \emptyset,$$

the optimal control problem (16.47)–(16.49) admits at least one solution

$$(A^{opt}, y^{opt}) \in \Xi \subset L^\infty(D; \mathbb{R}^{n\times n}) \times W_0^{1,\,p}(\Omega)$$

for every open set $\Omega \subseteq D$.

Proof. Due to Proposition 16.4, the control admissibility condition ensures the existence of a minimizing sequence $\{(A_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$, that is,

$$\lim_{k\to\infty} I_\Omega(A_k, y_k) = \inf_{(A,y)\in\,\Xi} I_\Omega(A, y) < +\infty. \tag{16.73}$$

Since the controls $\{A_k \in U_{ad}\}_{k\in\mathbb{N}}$ are uniformly bounded in $L^\infty(D; \mathbb{R}^{n\times n})$, by the a priori estimate (16.18) it follows that the minimizing sequence is

bounded in $L^\infty(D;\mathbb{R}^{n\times n})\times W_0^{1,p}(\Omega)$ and, by taking a subsequence, we may assume that there exists a pair (A^*, y^*) such that

$$A_k \rightharpoonup A^* \ \text{ weakly-}* \ \text{ in } \ L^\infty(D, R^{n\times n}), \quad y_k \rightharpoonup y^* \ \text{ in } \ W_0^{1,p}(\Omega).$$

By Theorem 16.14, the pair (A^*, y^*) is admissible to the problem (16.47)–(16.49). Moreover, since the cost functional I is lower τ-semicontinuous, we get

$$I_\Omega(A^*, y^*) \le \liminf_{k\to\infty} I_\Omega(A_k, y_k) = \inf_{(A,\,y)\in\,\Xi} I_\Omega(A, y).$$

Hence, (A^*, y^*) is an optimal pair. □

Remark 16.16. The argument used in the proof of Theorem 16.15 is related to the so-called "direct method" in the Calculus of Variations which, roughly speaking, aims to construct a minimizing sequence $\{(A_k, y_k)\in\Xi\}_{k\in\mathbb{N}}$. However, from the practical point of view, it makes sense to use an "indirect method" in the study of control problems, based on the first order necessary optimality conditions (see [169, 219, 226]).

Since there is no good topology a priory given on the set of all open subsets of $\mathbb{R}^n$, we study the stability properties of the original control problem imposing some constraints on the domain perturbation. Namely, we consider three types of domain perturbations: the so-called topologically admissible perturbations (following Dancer [83]), perturbations in the Hausdorff complementary topology (following Bucur and Zolesio [40]), and the so-called parametrical perturbations (following Liu, Neittaanmäki, and Tiba [178]). In particular, we show that in this case the sequence of the sets of admissible pairs $\{\Xi_\varepsilon\}_{\varepsilon>0}$ is compact with respect to the Mosco-set convergence in $L^\infty(D;\mathbb{R}^{n\times n})\times H_0^1(D)$. Further, we study the stability properties of the optimal control problem (16.1)–(16.4) under the domain perturbation. Our treatment of this question is based on a new stability concept for optimal control problems. We show that Mosco-stable optimal control problems possess "good" variational properties which allow using the optimal solutions to the perturbed problems in more "simple" domains as a basis for the construction of approximate controls for the original control problem.

16.4 H^c- and t-admissible domain perturbations for optimal control problems in coefficients

The aim of this section is to study how the solutions $(A_\varepsilon^{opt}, y_\varepsilon^{opt})$ of the OCP

$$I_{\Omega_\varepsilon}(A_\varepsilon, y_\varepsilon) = \int_{\Omega_\varepsilon} |y_\varepsilon(x) - y_d(x)|^p \,\mathrm{d}x + \int_{\Omega_\varepsilon} |Dy_\varepsilon(x)|^p_{\mathbb{R}^n} \,\mathrm{d}x + J(A_\varepsilon) \ \longrightarrow \inf, \tag{16.74}$$

$$-\text{div}\,\big(A_\varepsilon(x)[Dy_\varepsilon^{p-2}]Dy_\varepsilon\big) + a_0|y_\varepsilon|^{p-2}y_\varepsilon = f \quad \text{in} \quad \Omega_\varepsilon, \tag{16.75}$$

$$y_\varepsilon \in W_0^{1,\,p}(\Omega_\varepsilon), \quad A_\varepsilon \in U_{ad}, \tag{16.76}$$

depend on a perturbation Ω_ε of a fixed domain $\Omega \subseteq D$. As before, we suppose that $f \in L^q(D)$ and $y_d \in L^p(D)$ are given functions, $J : L^\infty(D;\mathbb{R}^{n\times n}) \to \mathbb{R}$ is a lower semicontinuous functional with respect to the weak-$*$ topology of $L^\infty(D;\mathbb{R}^{n\times n})$ and this functional is continuous with respect to the strong topology of $L^\infty(D;\mathbb{R}^{n\times n})$, and the set of admissible controls U_{ad} and, hence, the corresponding set of admissible solutions $\Xi_\varepsilon \subset L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,\,p}(\Omega_\varepsilon)$ are nonempty for every $\varepsilon > 0$.

Before we give the precise definition of the shape stability of the above problem and admissible perturbations for the open set Ω, we remark that neither the set convergence $\Omega_\varepsilon \overset{H^c}{\longrightarrow} \Omega$ in the Hausdorff complementary topology (see Definition 7.36) nor the topological set convergence $\Omega_\varepsilon \overset{\text{top}}{\longrightarrow} \Omega$ (see Definition 7.43) is a sufficient condition to prove the shape stability of control problem (16.47)–(16.49). In general, a limit pair for the sequence $\{(A_\varepsilon^{opt}, y_\varepsilon^{opt})\}_{\varepsilon>0}$, under H^c-perturbations of Ω, can be a nonadmissible pair to the original problem (16.47)–(16.49). We refer to [82] for simple counterexamples. So, we have to impose some constraints on the moving domain. In view of this, we begin with the following concepts.

Definition 16.17. *Let Ω and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be open subsets of D. We say that the sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ form an H^c-admissible perturbation of Ω if the following hold:*

(i) $\Omega_\varepsilon \overset{H^c}{\longrightarrow} \Omega$ as $\varepsilon \to 0$;
(ii) $\Omega_\varepsilon \in \mathcal{W}_w(D)$ for every $\varepsilon > 0$, where the class of subsets $\mathcal{W}_w(D)$ is defined in (7.36).

Definition 16.18. *Let Ω and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be open subsets of D. We say that the sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ form a topologically admissible perturbation of Ω (in short, t-admissible), if $\Omega_\varepsilon \overset{\text{top}}{\longrightarrow} \Omega$ in the sense of Definition 7.43.*

Remark 16.19. As Theorem 7.41 indicates, a subset $\Omega \subset D$ admits the existence of H^c-admissible perturbations if and only if Ω belongs to the family $\mathcal{W}_w(D)$. However, this condition is not very restrictive. Indeed, it turns out that the assertion

$$\text{“}y \in W^{1,\,p}(\mathbb{R}^n),\ \Omega \in \mathcal{W}_w(D), \quad \text{and} \quad \operatorname{supp} y \subset \overline{\Omega} \quad \text{imply} \quad y \in W_0^{1,\,p}(\Omega)\text{”}$$

is not true, in general. In particular, it fails in the case when the open domain Ω has a crack. So, $\mathcal{W}_w(D)$ is a rather general class of open subsets of the domain D.

Remark 16.20. The above remark motivates us to say that $\Omega \subset D$ is p-stable if for any $y \in W^{1,\,p}(\mathbb{R}^n)$ such that $y = 0$ almost everywhere on $\operatorname{int}\Omega^c$, we get $y|_{\,\Omega} \in W_0^{1,\,p}(\Omega)$. Note that this property holds for all reasonably regular domains such as Lipschitz domains. A more precise discussion of this property may be found in [83].

We begin with the following result.

Proposition 16.21. *Let $\Omega \in \mathcal{W}_w(D)$ be a fixed subdomain of D and let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be an H^c-admissible perturbation of Ω. Let $\{(A_\varepsilon, y_{\Omega_\varepsilon, A_\varepsilon}) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ be a sequence of admissible pairs for the problems (16.74)–(16.76). Then*

$$\{(A_\varepsilon, \widetilde{y}_{\Omega_\varepsilon, A_\varepsilon})\}_{\varepsilon>0} \quad \text{is uniformly bounded in} \quad L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,p}(D),$$

and for every τ-cluster pair $(A^*, y^*) \in L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,p}(D)$ of this sequence, we have

$$A^* \in U_{ad}, \tag{16.77}$$

$$\begin{aligned}\int_D \left(A^*[D(y^*)^{p-2}]Dy^*, D\widetilde{\varphi}\right)_{\mathbb{R}^n} \mathrm{d}x + \int_D a_0|y^*|^{p-2}y^*\widetilde{\varphi}\,\mathrm{d}x \\ = \int_D f\widetilde{\varphi}\,\mathrm{d}x, \quad \forall\,\varphi \in C_0^\infty(\Omega).\end{aligned} \tag{16.78}$$

Proof. For simplicity, we write $y_\varepsilon = y_{\Omega_\varepsilon, A_\varepsilon}$. As usual, the zero-extension of y_ε to $\mathbb{R}^n$ is denoted by $\widetilde{y}_\varepsilon$. Since each of the pairs $(A_\varepsilon, y_\varepsilon)$ is admissible to the corresponding problem (16.74)–(16.76), the uniform boundedness of the sequence $\{(A_\varepsilon, \widetilde{y}_\varepsilon)\}_{\varepsilon>0}$ with respect to the norm of $L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,p}(D)$ is a direct consequence of Proposition 16.5 and Definition 16.10. Therefore, we may assume that there exists a pair (A^*, y^*) such that (up to passing to a subsequence still indexed by ε) $(A_\varepsilon, \widetilde{y}_\varepsilon) \xrightarrow{\tau} (A^*, y^*)$ in $L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,p}(D)$. Then, in view of Proposition 16.11, we have $A^* \in U_{ad}$.

Let us take as a test function $\varphi \in C_0^\infty(\Omega)$. Since $\Omega_\varepsilon \xrightarrow{H^c} \Omega$, then by Theorem 7.41, the Sobolev spaces $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ converge in the sense of Mosco to $W_0^{1,p}(\Omega)$. Hence, for the function $\varphi \in W_0^{1,p}(\Omega)$ fixed earlier, there exists a sequence $\left\{\varphi_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ such that $\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi}$ strongly in $W^{1,p}(D)$ (see property (M_1) of Definition 7.39). Since $(A_\varepsilon, y_\varepsilon)$ is an admissible pair for the corresponding problem in Ω_ε, we can write

$$\int_{\Omega_\varepsilon} \left(A_\varepsilon \left[D\,(y_\varepsilon)^{p-2}\right] Dy_\varepsilon, D\varphi_\varepsilon\right)_{\mathbb{R}^n} \mathrm{d}x + \int_{\Omega_\varepsilon} a_0|y_\varepsilon|^{p-2}y_\varepsilon\,\varphi_\varepsilon\,\mathrm{d}x = \int_{\Omega_\varepsilon} f\,\varphi_\varepsilon\,\mathrm{d}x,$$

and hence,

$$\int_D \left(A_\varepsilon \left[D\,(\widetilde{y}_\varepsilon)^{p-2}\right] D\widetilde{y}_\varepsilon, D\widetilde{\varphi}_\varepsilon\right)_{\mathbb{R}^n} \mathrm{d}x + \int_D a_0|\widetilde{y}_\varepsilon|^{p-2}\widetilde{y}_\varepsilon\,\widetilde{\varphi}_\varepsilon\,\mathrm{d}x \tag{16.79}$$

$$= \int_D f\,\widetilde{\varphi}_\varepsilon\,\mathrm{d}x, \ \forall\,\varepsilon > 0. \tag{16.80}$$

To prove (16.78), we pass to the limit in the integral (16.80) identity as $\varepsilon \to 0$. Using the arguments of the proof of Theorem 16.14, we have

$$\operatorname{div} \mathbf{a}_{i\,\varepsilon} \to \operatorname{div} \mathbf{a}_i^* \text{ strongly in } W^{-1,\,q}(D),\ \forall\, i = 1, \dots, n,$$
$$\left\{[D(\widetilde{y}_\varepsilon)^{p-2}]D\widetilde{y}_\varepsilon\right\}_{\varepsilon>0} \text{ is bounded in } \mathbf{L}^q(D),\ q = p/(p-1),$$
$$\left\{|\widetilde{y}_\varepsilon|^{p-2}\widetilde{y}_\varepsilon\right\}_{\varepsilon>0} \text{ is bounded in } L^q(D),$$
$$\widetilde{y}_\varepsilon \to y^* \text{ in } L^p(D), \quad \widetilde{y}_\varepsilon(x) \to y^*(x) \text{ for a.e. } x \in D,$$
$$|\widetilde{y}_\varepsilon|^{p-2}\widetilde{y}_\varepsilon \rightharpoonup |y^*|^{p-2}y^* \quad \text{in} \quad L^q(D),$$

where $A_\varepsilon = [\mathbf{a}_{1\,\varepsilon}, \dots, \mathbf{a}_{n\,\varepsilon}]$ and $A^* = [\mathbf{a}_1^*, \dots, \mathbf{a}_n^*]$.

Consider the sequence $\left\{f_\varepsilon := f - a_0|\widetilde{y}_\varepsilon|^{p-2}\widetilde{y}_\varepsilon\right\}_{\varepsilon>0}$. It is clear that

$$f_\varepsilon \to f_0 = f - a_0|y^*|^{p-2}y^* \quad \text{strongly in} \quad W^{-1,\,q}(D).$$

In view of this and the a priori estimate (16.18), the sequence $\{a(A_\varepsilon(x), D\,\widetilde{y}_\varepsilon)\}_{\varepsilon>0}$ is bounded in $\mathbf{L}^q(D)$. So, up to a subsequence, we may suppose that there exists a vector function $\boldsymbol{\xi} \in \mathbf{L}^q(D)$ such that

$$\boldsymbol{\xi}_\varepsilon := a(A_\varepsilon(x), D\,\widetilde{y}_\varepsilon) = A_\varepsilon\,[D\,(\widetilde{y}_\varepsilon)^{p-2}]\,D\,\widetilde{y}_\varepsilon \rightharpoonup \boldsymbol{\xi} \quad \text{in} \quad \mathbf{L}^q(D).$$

Using the strong convergence $\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi}$ in $W^{1,\,p}(D)$ and passing to the limit as $\varepsilon \to 0$ in relation (16.80), we obtain

$$\int\limits_D (\boldsymbol{\xi}, D\widetilde{\varphi})_{\mathbb{R}^n}\,\mathrm{d}x = \int\limits_D \left(f - a_0|y^*|^{p-2}y^*\right)\widetilde{\varphi}\,\mathrm{d}x. \tag{16.81}$$

It remains to show that

$$\boldsymbol{\xi} = A^*\left[D\,(y^*)^{p-2}\right]Dy^*. \tag{16.82}$$

This, however, can be done as in (16.72). In particular, we should repeat all the arguments of that proof, replacing Ω by D, A_k by A_ε, y_k by $\widetilde{y}_\varepsilon$, A_0 by A^*, y_0 by y^*, and ϕ by $\widetilde{\phi}$.

As a result, we show that representation (16.82) holds true, and hence the integral identity (16.81) takes the form of the desired equality (16.78). The proof is completed. □

Our next intention is to prove that every τ-cluster pair $(A^*, y^*) \in L^\infty(D; \mathbb{R}^{n\times n}) \times W_0^{1,\,p}(D)$ for the sequence $\{(A_\varepsilon, y_{\,\Omega_\varepsilon,\,A_\varepsilon}) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ (see Proposition 16.21) is admissible to the original OCP (16.47)–(16.49). As follows from (16.77) and (16.78), it remains to show that $y^*|_\Omega \in W_0^{1,\,p}(\Omega)$ and, hence, $(A^*, y^*) \in \Xi$. To do so, we recall the following result which is the direct consequence of Theorem 1.1 from [39].

Lemma 16.22. *Let $\Omega, \{\Omega_\varepsilon\}_{\varepsilon>0} \in \mathcal{W}_w(D)$ and $\Omega_\varepsilon \xrightarrow{H^c} \Omega$ as $\varepsilon \to 0$. Let A_0 be any fixed matrix of the set $M_\alpha^\beta(D)$. Then*

$$\widetilde{v}_{\,\Omega_\varepsilon,\,h} \to \widetilde{v}_{\,\Omega,\,h} \text{ strongly in } W_0^{1,\,p}(D), \quad \forall\, h \in W_0^{1,\,p}(D), \tag{16.83}$$

where $v_{\Omega_\varepsilon,h}$ *and* $v_{\Omega,h}$ *are the unique weak solutions to the boundary value problems*

$$\left.\begin{aligned} -\operatorname{div}\left(A_0[Dv^{p-2}]Dv\right)+a_0|v|^{p-2}v=0 \quad &\text{in} \quad \Omega_\varepsilon,\\ v-h\in W_0^{1,p}(\Omega_\varepsilon),& \end{aligned}\right\} \tag{16.84}$$

and

$$\left.\begin{aligned} -\operatorname{div}\left(A_0[Dv^{p-2}]Dv\right)+a_0|v|^{p-2}v=0 \quad &\text{in} \quad \Omega,\\ v-h\in W_0^{1,p}(\Omega),& \end{aligned}\right\} \tag{16.85}$$

respectively.

Remark 16.23. Notice that this assertion is not true in the case when

$$\Omega_\varepsilon \xrightarrow{\text{top}} \Omega.$$

We are now in a position to prove the desired property.

Proposition 16.24. *Let* $\{(A_\varepsilon, y_{\Omega_\varepsilon,A_\varepsilon})\in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ *be a sequence of admissible pairs for the family of the problems (16.74)–(16.76), where* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *is some* H^c*-admissible perturbation of the set* $\Omega\in\mathcal{W}_w(D)$*. If for a subsequence of the given sequence* $\{(A_\varepsilon, y_{\Omega_\varepsilon,A_\varepsilon})\in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ *(still denoted by indices* ε*), we have*

$$(A_\varepsilon, \widetilde{y}_{\Omega_\varepsilon,A_\varepsilon}) \xrightarrow{\tau} (A^*, y^*),$$

then

$$y^*=\widetilde{y}_{\Omega,A^*}, \quad \text{and hence,} \quad (A^*, y^*|_\Omega)\in\Xi,$$

where by y_{Ω,A^*} *we denote the weak solution of the boundary value problem (16.48)–(16.49) with* $A=A^*$*.*

Proof. For simplicity, we use the following notation: $y_\varepsilon = y_{\Omega_\varepsilon,A_\varepsilon}$, $y=y_{\Omega,A^*}$. By Propositions 16.4, 16.5, and 16.21, we have (up to passing to a subsequence)

$$A_\varepsilon \rightharpoonup A^*=[\mathbf{a}_1^*,\dots,\mathbf{a}_n^*]\in U_{ad} \quad \text{weakly-}* \quad \text{in} \quad L^\infty(D,R^{n\times n}), \tag{16.86}$$

$$\widetilde{y}_\varepsilon \rightharpoonup y^* \quad \text{in} \quad W_0^{1,p}(D), \tag{16.87}$$

$$y\in W_0^{1,p}(\Omega), \quad \widetilde{y}\in W_0^{1,p}(D).$$

We prove that $y^*=\widetilde{y}$. Following Bucur and Trebeschi [39], for every $\varepsilon>0$ we consider the new boundary value problem

$$\left.\begin{aligned} -\operatorname{div}\left(A^*[D\varphi_\varepsilon^{p-2}]D\varphi_\varepsilon\right)+a_0|\varphi_\varepsilon|^{p-2}\varphi_\varepsilon=0 \quad &\text{in} \quad \Omega_\varepsilon,\\ \varphi_\varepsilon=-y^* \quad \text{on} \quad D\setminus\Omega_\varepsilon.& \end{aligned}\right\} \tag{16.88}$$

In the weak sense, it means that

$$\int\limits_D \left(A^*\left[D\,\widetilde{\varphi}_\varepsilon^{p-2}\right] D\widetilde{\varphi}_\varepsilon, D\widetilde{\psi}_\varepsilon\right)_{\mathbb{R}^n} \mathrm{d}x + \int\limits_D a_0|\widetilde{\varphi}_\varepsilon|^{p-2}\widetilde{\varphi}_\varepsilon\,\widetilde{\psi}_\varepsilon\,\mathrm{d}x = 0,$$
$$\forall\,\psi \in C_0^\infty(\Omega_\varepsilon),\ \forall\,\varepsilon > 0. \quad (16.89)$$

Taking (16.89) as the test function $\widetilde{\psi}_\varepsilon = \widetilde{\varphi}_\varepsilon + y^* - \widetilde{y}_\varepsilon$, we have

$$\int\limits_D \left(A^*\left[D\,\widetilde{\varphi}_\varepsilon^{p-2}\right] D\widetilde{\varphi}_\varepsilon, D\left(\widetilde{\varphi}_\varepsilon + y^* - \widetilde{y}_\varepsilon\right)\right)_{\mathbb{R}^n} \mathrm{d}x$$
$$+ \int\limits_D a_0|\widetilde{\varphi}_\varepsilon|^{p-2}\widetilde{\varphi}_\varepsilon\,\left(\widetilde{\varphi}_\varepsilon + y^* - \widetilde{y}_\varepsilon\right)\mathrm{d}x = 0,\ \forall\,\varepsilon > 0. \quad (16.90)$$

Let $\varphi \in W^{1,\,p}(\Omega)$ be a weak solution of the problem

$$\left.\begin{aligned} -\mathrm{div}\,\left(A^*[D\varphi^{p-2}]D\varphi\right) + a_0|\varphi|^{p-2}\varphi = 0 \quad &\text{in} \quad \Omega, \\ \varphi = -y^* \ \text{ on } \ D\setminus\Omega. \quad & \end{aligned}\right\}$$

Then, by Lemma 16.22, we have $\widetilde{v}_{\Omega_\varepsilon,h} \to \widetilde{v}_{\Omega,h}$ strongly in $W_0^{1,\,p}(D)$. Hence,

$$D\,\widetilde{\varphi}_\varepsilon \to D\,\widetilde{\varphi} \ \text{ strongly in } \ \mathbf{L}^p(D),$$
$$\|\left[D\,\widetilde{\varphi}_\varepsilon^{p-2}\right] D\widetilde{\varphi}_\varepsilon\|^q_{\mathbf{L}^q(D)} = \|D\,\widetilde{\varphi}_\varepsilon\|^p_{\mathbf{L}^p(D)} \to \|D\,\widetilde{\varphi}\|^p_{\mathbf{L}^p(D)} = \|\left[D\,\widetilde{\varphi}^{p-2}\right] D\widetilde{\varphi}\|^q_{\mathbf{L}^q(D)},$$
$$D\,\widetilde{\varphi}_\varepsilon(x) \to D\,\widetilde{\varphi}(x) \ \text{ for a.e. } \ x \in D,$$

and

$$\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi} \ \text{ strongly in } \ L^p(D),$$
$$\|\,|\widetilde{\varphi}_\varepsilon|^{p-2}\,\widetilde{\varphi}_\varepsilon\|^q_{L^q(D)} = \|\widetilde{\varphi}_\varepsilon\|^p_{L^p(D)} \to \|\widetilde{\varphi}\|^p_{L^p(D)} = \|\,|\widetilde{\varphi}|^{p-2}\,\widetilde{\varphi}\|^q_{L^q(D)},$$
$$\widetilde{\varphi}_\varepsilon(x) \to \widetilde{\varphi}(x) \ \text{ for a.e. } \ x \in D.$$

Since the norm convergence together with pointwise convergence implies the strong convergence, it follows that

$$\left[D\,\widetilde{\varphi}_\varepsilon^{p-2}\right] D\widetilde{\varphi}_\varepsilon \to |D\,\widetilde{\varphi}|^{p-2}_{\mathbb{R}^n}\,D\widetilde{\varphi} \ \text{ strongly in } \ \mathbf{L}^q(D),$$
$$|\widetilde{\varphi}_\varepsilon|^{p-2}\,\widetilde{\varphi}_\varepsilon \to |\widetilde{\varphi}|^{p-2}\,\widetilde{\varphi} \ \text{ strongly in } \ L^q(D),$$
$$D\left(\widetilde{\varphi}_\varepsilon + y^* - \widetilde{y}_\varepsilon\right) \rightharpoonup D\widetilde{\varphi} \ \text{ weakly in } \ \mathbf{L}^p(D) \ \text{(see (16.87))},$$
$$\left(\widetilde{\varphi}_\varepsilon + y^* - \widetilde{y}_\varepsilon\right) \to \widetilde{\varphi} \quad \text{strongly in } \ L^p(D),$$

Hence, the integral identity (16.90) contains only the products of weakly and strongly convergent sequences. So passing to the limit in (16.90) as ε tends to 0, we get

$$\int\limits_D \left(A^* \left[D\,\widetilde{\varphi}^{p-2}\right] D\widetilde{\varphi}, D\widetilde{\varphi}\right)_{\mathbb{R}^n} \mathrm{d}x + \int\limits_D a_0 |\widetilde{\varphi}|^p \,\mathrm{d}x = 0.$$

Taking into account the properties of A^* and a_0 prescribed above, this implies $\widetilde{\varphi} = 0$ a.e. on D. However, by definition, $\widetilde{\varphi} = -y^*$ on $D \setminus \Omega$. So $y^* = 0$ on $D \setminus \Omega$, and we obtain the required property $y_{A^*,\,\Omega} = y^*|_{\Omega} \in W_0^{1,\,p}(\Omega)$. This completes the proof. □

The results given above motivate us to study the asymptotic behavior of sequences of admissible pairs $\{(A_\varepsilon, y_{\Omega_\varepsilon,\,A_\varepsilon}) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ for the case of t-admissible perturbations of the set Ω.

Proposition 16.25. *Let Ω be a p-stable open subset of D. Let*

$$\{(A_\varepsilon, y_{\Omega_\varepsilon,\,A_\varepsilon}) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$$

be a sequence of admissible pairs for the family of the OCPs (16.74)–(16.76), where $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset D$ form a t-admissible perturbation of Ω. Then

$$\{(A_\varepsilon, \widetilde{y}_{\Omega_\varepsilon,\,A_\varepsilon})\}_{\varepsilon>0} \quad \textit{is uniformly bounded in} \quad L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,\,p}(D),$$

and for every τ-cluster pair $(A^*, y^*) \in L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,\,p}(D)$ of this sequence, we have the following:

(j) The pair (A^*, y^*) satisfies relations (16.77)–(16.78).

(jj) The pair $(A^*,\, y^*|_{\Omega})$ is an admissible to the problem (16.47)–(16.49), that is, $y^* = \widetilde{y}_{\Omega,\,A^*}$, where by $y_{\Omega,\,A^*}$ we denote the weak solution of the boundary value problem (16.48)–(16.49) with $A = A^*$.

Proof. Since $\Omega_\varepsilon \xrightarrow{\text{top}} \Omega$ in the sense of Definition 7.43, it follows that for any $\varphi \in C_0^\infty(\Omega \setminus K_0)$, we have $\operatorname{supp}\varphi \subset \Omega_\varepsilon$ for all $\varepsilon > 0$ small enough. Moreover, as the set K_0 has zero p-capacity, it follows that $C_0^\infty(\Omega \setminus K_0)$ is dense in $W_0^{1,\,p}(\Omega)$. Therefore, the verification of item (j) can be done in a way analogous to the proof of Proposition 16.21, replacing therein the sequence $\{\varphi_\varepsilon \in W_0^{1,\,p}(\Omega_\varepsilon)\}_{\varepsilon>0}$ by the function φ. As for the remaining assertions, we have to repeat all of the arguments of that proof.

To prove assertion (jj), we have to show that $y^*|_{\Omega} \in W_0^{1,\,p}(\Omega)$. To do so, let B_0 be any closed ball not intersecting $\overline{\Omega} \cup K_1$. Then it follows from (16.75) and (16.76) that for the pair $(A_\varepsilon, y_{\Omega_\varepsilon,\,A_\varepsilon}) \in \Xi_{\Omega_\varepsilon}$ we have $\widetilde{y}_{\Omega_\varepsilon,\,A_\varepsilon} = 0$ almost everywhere in B_0, whenever the parameter ε is small enough. Since by (j) and the Sobolev embedding theorem, $\widetilde{y}_{\Omega_\varepsilon,\,A_\varepsilon}$ converges to y^* strongly in $L^p(D)$, it follows that $\widetilde{y}^* = 0$ almost everywhere in B_0. As the ball B_0 was chosen arbitrary and K_1 is of Lebesgue measure zero, it follows that $\operatorname{supp} y^* \subset \Omega$. Then, by Fubini's theorem, we have $\operatorname{supp} y^* \subset \overline{\Omega}$. Hence, using the properties of p-stable domains (see Remark 16.20), we just come to the desired conclusion: $y^*|_{\Omega} \in W_0^{1,\,p}(\Omega)$. This completes the proof. □

Corollary 16.26. *Let $\{A_\varepsilon \equiv A^*\}_{\varepsilon>0}$ be a constant sequence, where $A^* \in U_{ad}$ is any admissible control. Let $\{y_{\Omega_\varepsilon, A^*} \in W_0^{1,p}(\Omega_\varepsilon)\}_{\varepsilon>0}$ be the corresponding solutions of (16.75) and (16.76). Then, under the assumptions of Proposition 16.24 or Proposition 16.25, we have*

$$\widetilde{y}_{\Omega_\varepsilon, A^*} \to \widetilde{y}_{\Omega, A^*} \quad \text{strongly in} \quad W_0^{1,p}(D).$$

Proof. As follows from Propositions 16.24 and 16.25, for the sequence of admissible pairs $\{(A^*, y_{\Omega_\varepsilon, A^*}) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$, there exists a τ-limit pair (A^*, y^*) such that $y^*|_\Omega = y_{A^*,\Omega}$. Denote $y_\varepsilon = y_{\Omega_\varepsilon, A_\varepsilon}$ and $y = y_{\Omega, A^*}$. We prove strong convergence of $\widetilde{y}_\varepsilon$ to $\widetilde{y}$ in $W_0^{1,p}(D)$ proving the convergence of the norms, that is, we show that

$$\|\widetilde{y}_\varepsilon\|_{W^{1,p}(D)} \to \|\widetilde{y}\|_{W^{1,p}(D)} \quad \text{as} \quad \varepsilon \to 0. \tag{16.91}$$

Since $A^* \in M_\alpha^\beta(D)$ and $L^\infty(D) \ni a_0 \geq 0$ a.e. in D, it follows that we can take as an equivalent norm in $W_0^{1,p}(D)$ the following:

$$\|y\|_{W_0^{1,p}(D)}^{A^*, a_0} = \left(\int_D \left(A^*[Dy^{p-2}]Dy, Dy \right)_{\mathbb{R}^n} \, dx + \int_D a_0 |y|^p \, dx \right)^{1/p}.$$

As a result, the space $\left\langle W_0^{1,p}(D), \| \cdot \|_{W_0^{1,p}(D)}^{A^*, a_0} \right\rangle$ endowed with this norm is uniformly convex (see [2]). Hence, instead of (16.91), we establish that

$$\|\widetilde{y}_\varepsilon\|_{W^{1,p}(D)}^{A^*, a_0} \to \|\widetilde{y}\|_{W^{1,p}(D)}^{A^*, a_0} \quad \text{as} \quad \varepsilon \to 0. \tag{16.92}$$

Using (16.48) and (16.75), we take as the test functions $\widetilde{y}$ and $\widetilde{y}_\varepsilon$, respectively. Then passing to the limit in (16.75), we get

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \left(\int_D \left(A^*[D\widetilde{y}_\varepsilon^{p-2}]D\widetilde{y}_\varepsilon, D\widetilde{y}_\varepsilon \right)_{\mathbb{R}^n} \, dx + \int_D a_0 |\widetilde{y}_\varepsilon|^p \, dx \right) \\
&\quad = \lim_{\varepsilon \to 0} \left(\|\widetilde{y}_\varepsilon\|_{W^{1,p}(D)}^{A^*, a_0} \right)^p = \lim_{\varepsilon \to 0} \int_D f \widetilde{y}_\varepsilon \, dx = \int_D f \widetilde{y} \, dx \\
&\quad = \int_D \left(A^*[D\widetilde{y}^{p-2}]D\widetilde{y}, D\widetilde{y} \right)_{\mathbb{R}^n} \, dx + \int_D a_0 |\widetilde{y}|^p \, dx \\
&\quad = \left(\|\widetilde{y}\|_{W^{1,p}(D)}^{A^*, a_0} \right)^p.
\end{aligned}$$

Since (16.92) together with the weak convergence in $W_0^{1,p}(D)$ implies the strong convergence, we obtain the required conclusion. □

We a now in a position to prove the main result of this section.

Theorem 16.27. *Let Ω, $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be open subsets of D. Let*

$$\Xi_{\Omega_\varepsilon} \subset L^\infty(D; \mathbb{R}^{n\times n}) \times W_0^{1,p}(\Omega_\varepsilon) \quad \text{and} \quad \Xi_\Omega \subset L^\infty(D; \mathbb{R}^{n\times n}) \times W_0^{1,p}(\Omega)$$

be the sets of admissible solutions to the OCPs (16.74)–(16.76) and (16.47)–(16.49), respectively. Assume that at least one of the following suppositions holds true:

1. $\Omega \in \mathcal{W}_w(D)$ *and* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *is an* H^c*-admissible perturbation of* Ω*.*
2. Ω *is a* p*-stable domain and* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *is an* t*-admissible perturbation of* Ω*.*

Then the sequence $\{\Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ *converges to* Ξ_Ω *in the sense of Mosco, that is, the following conditions hold true:*

(ΞM_1) *For every pair* $(A, y) \in \Xi_\Omega$*, there exists a sequence*

$$\{(A_\varepsilon, y_\varepsilon) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$$

such that $A_\varepsilon \to A$ *strongly in* $L^\infty(D;\mathbb{R}^{n\times n})$ *and* $\widetilde{y}_\varepsilon \to \widetilde{y}$ *strongly in* $W_0^{1,p}(D)$*.*

(ΞM_2) *If* $\{\varepsilon_k\}_{k\in\mathbb{N}}$ *is a numerical sequence converging to* 0*, and the sequence* $\{(A_k, y_k)\}_{k\in\mathbb{N}}$ *is such that*

$$(A_k, y_k) \in \Xi_{\Omega_{\varepsilon_k}} \quad \forall\, k \in \mathbb{N}, \quad \textit{and}$$
$$(A_k, \widetilde{y}_k) \xrightarrow{\tau} (A, \psi) \quad \textit{in} \quad L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,p}(D),$$

then there exists a function $y \in W_0^{1,p}(\Omega)$ *such that* $y = \psi|_\Omega$ *and* $(A, y) \in \Xi_\Omega$*.*

Proof. To begin, we note that property (ΞM_2) is a direct consequence of Propositions 16.24 and 16.25. So it remains to check property (ΞM_1).

In view of our initial assumptions, the set of admissible pairs Ξ_Ω to the problem (16.47)–(16.49) is nonempty. Let $(A, y) \in \Xi_\Omega$ be any pair. To construct a sequence $\{(A_\varepsilon, y_\varepsilon) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ satisfying property (ΞM_1), we proceed as follows: $A_\varepsilon = A\ \forall\, \varepsilon > 0$ and $y_\varepsilon = y_{\Omega_\varepsilon, A}$ is the corresponding solution of the boundary value problem (16.75)–(16.76). Note that such a choice is possible since the matrix A is an admissible control to the problem (16.74)–(16.76) for every $\varepsilon > 0$. Then, by Corollary 16.26, we have

$$\widetilde{y}_{\Omega_\varepsilon, A} \to \widetilde{y}_{\Omega, A} \quad \text{strongly in} \quad W_0^{1,p}(D).$$

Since y_{Ω, A^*} is a unique solution to (16.48) and (16.49) and $(A, y) \in \Xi_\Omega$, it follows that $y = y_{\Omega, A}$, and we come to the required conclusion:

$$(A, \widetilde{y}_\varepsilon) \longrightarrow (A, \widetilde{y}) \quad \text{strongly in} \quad L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,p}(D).$$

This concludes the proof. □

16.5 p-Perturbation of elliptic optimal control problems in coefficients

In this section, we proceed the study of perturbed elliptic OCPs in coefficients. However, for the sake of simplicity, we restrict our analysis to the linear case,

namely the main question is: How the admissible solutions $(A_\varepsilon, y_\varepsilon) \in \Xi_{\Omega_\varepsilon}$ to the OCP

$$I_{\Omega_\varepsilon}(A_\varepsilon, y_\varepsilon) = \int_{\Omega_\varepsilon} |y_\varepsilon(x) - y_d(x)|^2 \, dx + \int_{\Omega_\varepsilon} |Dy_\varepsilon(x)|^2_{\mathbb{R}^n} \, dx + J(A_\varepsilon) \longrightarrow \inf, \tag{16.93}$$

$$-\text{div}\,(A_\varepsilon(x) Dy_\varepsilon) + a_0 y_\varepsilon = f \quad \text{in} \quad \Omega_\varepsilon, \tag{16.94}$$

$$y_\varepsilon \in H_0^1(\Omega_\varepsilon), \quad A_\varepsilon \in U_{ad}, \quad p = 2, \tag{16.95}$$

depend on a perturbation Ω_ε of a fixed domain $\Omega \subseteq D$ if $\Omega = p - \lim_{\varepsilon \to 0} \Omega_\varepsilon$ in the sense of Definition 7.47. Here, the set U_{ad} is defined in Definition 16.10 with the modification that $\{Q_1, \ldots, Q_n\}$ is a collection of nonempty compact subsets of $H^{-1}(D)$. Thus, in contrast to the previous section, we suppose that a perturbation $\{\Omega_\varepsilon\}_{\varepsilon > 0}$ of a fixed domain $\Omega \subset D$ forms a parametrically convergent sequence. In view of this, we begin with the following concepts.

Definition 16.28. *Let Ω and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be open subsets of D. We say that the sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ form a p-admissible perturbation of Ω if the following hold:*

(i) $\Omega = p - \lim_{\varepsilon \to 0} \Omega_\varepsilon$.

(ii) $\Omega_\varepsilon \in \mathcal{O}$ *for every* $\varepsilon > 0$, *where the class* $\mathcal{O}$ *is defined in Definition 7.55.*

(iii) The family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *satisfies the uniform Hölder condition, that is,* $\partial\Omega_\varepsilon \in C^{1,\alpha}$ $\forall\, \varepsilon > 0$ *(see Definition 1.7.1 in [214]).*

Remark 16.29. Condition (iii) of Definition 16.28 implies that the p-limit domain Ω possesses a similar Hölder condition of the boundary (see [178]).

To begin, we prove the following result.

Proposition 16.30. *Let $\Omega \in \mathcal{O}$ be a fixed subdomain of D and let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a p-admissible perturbation of Ω. Let $\{(A_\varepsilon, y_{\Omega_\varepsilon, A_\varepsilon}) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ be a sequence of admissible pairs for the problems (16.74) and (16.76). Then*

$$\{(A_\varepsilon, \widetilde{y}_{\Omega_\varepsilon, A_\varepsilon})\}_{\varepsilon>0} \quad \textit{is uniformly bounded in} \quad L^\infty(D; \mathbb{R}^{n\times n}) \times H_0^1(D)$$

and there is a pair $(A^*, y^*) \in \Xi_\Omega \subset L^\infty(D; \mathbb{R}^{n\times n}) \times H_0^1(\Omega)$ such that (up to a subsequence) the following hold:

(i) $A_\varepsilon \rightharpoonup A^*$ *weakly-* in* $L^\infty(D, R^{n\times n})$.

(ii) $y^* = L\text{–}\lim y_{\Omega_\varepsilon, A_\varepsilon}$ *as* $\varepsilon \to 0$ *in the sense of Definition 7.52.*

(iii) $y^* = y_{\Omega, A^*}$ *is a solution of the boundary value problem (16.48)–(16.49) with* $A = A^*$.

Proof. As usual, we set $y_\varepsilon = y_{\Omega_\varepsilon, A_\varepsilon}$. Due to Proposition 16.5 and Definition 16.10, we see that the sequence $\{(A_\varepsilon, \widetilde{y}_\varepsilon)\}_{\varepsilon>0}$ of admissible pairs to the problems (16.74) and (16.76) is uniformly bounded in $L^\infty(D;\mathbb{R}^{n\times n}) \times H^1_0(D)$. So, in view of Theorem 7.53, we may assume that there exists a pair (A^*, y^*) such that (up to passing to a subsequence still indexed by ε) $A_\varepsilon \rightharpoonup A^*$ weakly-$*$ in $L^\infty(D;\mathbb{R}^{n\times n})$ and $y_\varepsilon \in H^1_0(\Omega_\varepsilon)$ locally converge to $y^* \in H^1(\Omega)$ in the sense of Definition 7.52. Then, by Proposition 16.11, we have $A^* \in U_{ad}$. Let us show that $y^* \in H^1_0(\Omega)$ and $y^* = y_{\Omega, A^*}$.

We now make use of the Γ-property of parametrical convergence (see Proposition 7.53). Then for any open set $G \subset\subset \Omega$, there is a value $\varepsilon_G > 0$ such that $G \subset \Omega_\varepsilon$ for all $\varepsilon \le \varepsilon_G$. Hence, using the weak formulation of the boundary value problem (16.94)–(16.95), we have that the integral identity

$$\int_G (A_\varepsilon Dy_\varepsilon, D\varphi)_{\mathbb{R}^n}\,\mathrm{d}x + \int_G a_0 y_\varepsilon\,\varphi\,\mathrm{d}x - \int_D \chi_{\Omega_\varepsilon} f\widetilde{\varphi}\,\mathrm{d}x = \int_{\Omega_\varepsilon\setminus G} \left((A_\varepsilon Dy_\varepsilon, D\varphi)_{\mathbb{R}^n} + a_0 y_\varepsilon\,\varphi\right),\mathrm{d}x, \quad \forall\,\varphi \in C^1_0(G) \tag{16.96}$$

holds true for all $\varepsilon \in (0, \varepsilon_G]$. We can estimate the right-hand side of (16.96):

$$\begin{aligned}\Big|\int_{\Omega_\varepsilon\setminus G} \big((A_\varepsilon Dy_\varepsilon, D\varphi)_{\mathbb{R}^n} + a_0 y_\varepsilon\,\varphi\big)\,\mathrm{d}x\Big| &\le \|\varphi\|_{C^1(\overline{D})}\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}\sqrt{\mathcal{L}_n(\Omega_\varepsilon\setminus G)}\\ &\times \max\left\{\|A_\varepsilon\|_{L^\infty(D;\mathbb{R}^{n\times n})}, \|a_0\|_{L^\infty(D)}\right\}. \end{aligned}\tag{16.97}$$

Let $y^* \in H^1(\Omega)$ be the L-limit of the sequence $\left\{y_\varepsilon \in H^1_0(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ given by Theorem 7.53. Then, due to Theorem 16.14 and the strong convergence of the characteristic functions $\chi_{\Omega_\varepsilon}$ in $L^2(D)$ (see Proposition 7.51), we can pass to the limit as $\varepsilon \to 0$ in (16.97). As a result, (16.96) implies the estimate

$$\left|\int_G (A^* Dy^*, D\varphi)_{\mathbb{R}^n}\,\mathrm{d}x + \int_G a_0 y^*\,\varphi\,\mathrm{d}x - \int_\Omega f\widetilde{\varphi}\,\mathrm{d}x\right| \le M\|\varphi\|_{C^1(\overline{D})}\sqrt{\mathcal{L}_n(\Omega\setminus G)}. \tag{16.98}$$

We now can take an increasing sequence of open sets $\{G_j \subset\subset \Omega\}_{j=1}^\infty$ such that $\cup_j G_j = \Omega$. Since the set Ω has the segment property (see Definition 7.55 and Theorem 7.57), it follows that for any fixed function $\varphi \in H^1_0(\Omega)$, there exists a sequence $\left\{\varphi_j \in C^1_0(G_j)\right\}_{\varepsilon>0}$ such that $\widetilde{\varphi}_j \to \widetilde{\varphi}$ strongly in $H^1_0(D)$ (see [178]). Then (16.98) yields

$$\int_\Omega (A^* Dy^*, D\varphi)_{\mathbb{R}^n}\,\mathrm{d}x + \int_\Omega a_0 y^*\,\varphi\,\mathrm{d}x = \int_\Omega f\widetilde{\varphi}\,\mathrm{d}x, \quad \forall\,\varphi \in H^1_0(\Omega).$$

Thus, $y^* = y_{\Omega, A^*}$, and this concludes the proof. □

We a now in a position to prove the main result of this section.

Theorem 16.31. *Let Ω, $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be open subsets of D. Let*

$$\Xi_{\Omega_\varepsilon} \subset L^\infty(D;\mathbb{R}^{n\times n}) \times H_0^1(\Omega_\varepsilon) \quad \text{and} \quad \Xi_\Omega \subset L^\infty(D;\mathbb{R}^{n\times n}) \times H_0^1(\Omega)$$

be the sets of admissible solutions to the OCPs (16.93)–(16.95) and

$$I_\Omega(A,y) = \int_\Omega |y(x) - y_d(x)|^2 \,\mathrm{d}x + \int_\Omega |Dy(x)|^2_{\mathbb{R}^n} \,\mathrm{d}x + J(A) \;\longrightarrow \inf, \tag{16.99}$$

$$-\mathrm{div}\,(A(x)Dy) + a_0 y = f \quad \text{in} \quad \Omega, \tag{16.100}$$

$$y \in H_0^1(\Omega), \quad A \in U_{ad}, \tag{16.101}$$

respectively. Assume that $\Omega \in \mathcal{O}$ and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is a p-admissible perturbation of Ω. Then, the sequence $\{\Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ converges to Ξ_Ω in the following sense:

(P_1) *For every pair $(A,y) \in \Xi_\Omega$, there exists a sequence $\{(A_\varepsilon, y_\varepsilon) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ such that $A_\varepsilon \to A$ strongly in $L^\infty(D;\mathbb{R}^{n\times n})$ and $y = L\text{–}\lim y_\varepsilon$.*

(P_2) *If $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is a numerical sequence converging to 0 and $\{(A_k,y_k)\}_{k\in\mathbb{N}}$ is a sequence satisfying*

$$(A_k,y_k) \in \Xi_{\Omega_{\varepsilon_k}} \quad \forall\, k \in \mathbb{N}, \quad \text{and}$$
$$A_k \rightharpoonup A \text{ weakly-}* \text{ in } L^\infty(D;\mathbb{R}^{n\times n}), \quad L\text{–}\lim y_k = y,$$

then $(A,y) \in \Xi_\Omega$.

Proof. Since property (P_2) is a direct consequence of Propositions 16.30, it remains to check property (P_1).

In view of our initial assumptions, the set of admissible pairs Ξ_Ω to the problem (16.99)–(16.101) is nonempty. Let $(A,y) \in \Xi_\Omega$ be any pair. To construct a sequence $\{(A_\varepsilon,y_\varepsilon) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ satisfying property (P_1), we make it as follows: $A_\varepsilon = A$, $\forall\, \varepsilon > 0$, and $y_\varepsilon = y_{\Omega_\varepsilon,A}$ is the corresponding solution of the boundary value problem (16.94)–(16.95). Note that such a choice is possible since the matrix A is an admissible control to the problem (16.93)–(16.95) for every $\varepsilon > 0$. Then, by Propositions 16.30, we have $y^* = L\text{–}\lim y_\varepsilon$ as $\varepsilon \to 0$ in the sense of Definition 7.52. Since y^* is a unique solution to (16.100)–(16.101) and $(A,y) \in \Xi_\Omega$, it follows that $y = y^*$, and we come to the required conclusion. □

16.6 Mosco-stability of optimal control problems

Typically, the mathematical description of an OCP consists of independent mathematical objects such as a state equation, control and state constraints,

and a cost functional. As a rule, each of these ingredients depends on a domain Ω where the control process is studied. Hence, when the domain Ω changes, we obtain, roughly speaking, an absolutely different OCP, probably with another constraints, another cost functional, and so on. Following the classical approach (see, for instance, [38, 39, 80, 83, 84]), if a sequence $\{\Omega_\varepsilon\}_{\varepsilon>0}$ converges to Ω in some sense, then the shape stability of the OCP (16.47)–(16.49) should be achieved by proving that any sequence of optimal pairs $\{(A_\varepsilon^{opt}, y_\varepsilon^{opt})\}_{\varepsilon>0}$ to the perturbed problems (16.74)–(16.76) is relatively compact with respect to an appropriate topology and each of its limits pairs is an optimal one to the original nonlinear problem (16.47)–(16.49) on the domain Ω. However, the optimal pair may not fully characterize the OCP. As a rule, the full identification of an OCP (including a cost functional, a state equation, and existing restrictions on the controls and state function) is impossible by means of the optimal solution only. Moreover, such an approach to the study of the shape stability is unacceptable when the existence of optimal solutions may fail [38] or when these solutions have nonclassical character (solutions satisfying only the requirements of Tonelli's partial regularity theorem and they do not have a "variational character") [17].

Therefore, in order to study the stability of the OCP (16.47)–(16.49) with respect to the variation of the open set $\Omega \subset D$, we should be able to compare two control problems defined on two different domains. In general, this question is open now. However, having taken the "asymptotic behavior" of OCPs, when $\Omega_\varepsilon \to \Omega$ in some sense, as a basis to study the stability properties, we come to the following concept (see [139]).

Definition 16.32. *We say that the OCP (16.47)–(16.49) on Ω is Mosco-stable in $L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,p}(D)$ along the sequence $\{\Omega_\varepsilon\}_{\varepsilon>0}$ which forms some perturbation of Ω if the following conditions are satisfied:*

(MS_1) *The set of admissible pairs Ξ_Ω for (16.47)–(16.49) is the limit in the sense of Mosco of the sequence $\{\Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ of admissible sets for the perturbed problems (16.74)–(16.76).*

(MS_2) *If $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is a numerical sequence converging to 0 and $\{(A_k, y_k)\}_{k\in\mathbb{N}}$ is a sequence such that*

$$(A_k, y_k) \in \Xi_{\Omega_{\varepsilon_k}} \quad \forall\, k \in \mathbb{N}, \quad \text{and}$$
$$(A_k, \widetilde{y}_k) \xrightarrow{\tau} (A, y) \quad \text{in} \quad L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,p}(D),$$
$$\text{where} \quad (A, y|_\Omega) \in \Xi_\Omega,$$

then

$$\liminf_{k\to\infty} I_{\Omega_{\varepsilon_k}}(A_k, y_k) \geq I_\Omega(A, y|_\Omega).$$

(MS_3) *For every pair $(A, y) \in \Xi_\Omega$, there exists a sequence*

$$\{(A_\varepsilon, y_\varepsilon) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$$

such that $A_\varepsilon \to A$ *strongly in* $L^\infty(D;\mathbb{R}^{n\times n})$, $\widetilde{y}_\varepsilon \to \widetilde{y}$ *strongly in* $W_0^{1,\,p}(D)$, *and*

$$\limsup_{\varepsilon\to 0} I_{\,\Omega_\varepsilon}(A_\varepsilon, y_\varepsilon) \le I_{\,\Omega}(A, y).$$

Remark 16.33. Notice that Definition 16.32 can be interpreted as a natural extension of the well-known notion of Γ-convergence of functionals (see Definition 8.9). We do not want here to enter into details of Γ-convergence theory but only emphasize that the Mosco-stable optimal control problem in the sense of Definition 16.32 possesses the fine variational properties that are similar to those of a Γ-limit.

We begin with the following result concerning the variational properties of Mosco-stable OCPs.

Theorem 16.34 ([139]). *Assume that for a given perturbation* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *of the domain* Ω, *the OCP (16.47)–(16.49) on* Ω *is Mosco-stable in*

$$L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,\,p}(D).$$

Let $\left\{(A_\varepsilon^0, y_\varepsilon^0) \in \Xi_{\Omega_\varepsilon}\right\}_{\varepsilon>0}$ *be a sequence of optimal solutions for the corresponding perturbed problems (16.74)–(16.76). Then this sequence is relatively* τ*-compact in* $L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,\,p}(D)$ *and every* τ*-cluster pair is an optimal solution to the original problem (16.47)–(16.49). Moreover, if*

$$(A_\varepsilon^0, \widetilde{y}_\varepsilon^0) \xrightarrow{\ \tau\ } (A^0, y^0), \tag{16.102}$$

then $(A^0,\, y^0\big|_{\,\Omega}) \in \Xi_{\,\Omega}$ *and*

$$\inf_{(A,\,y)\in\,\Xi_{\,\Omega}} I_{\,\Omega}(A, y) = I_{\,\Omega}(A^0,\, y^0\big|_{\,\Omega}) = \lim_{\varepsilon\to 0} \inf_{(A_\varepsilon, y_\varepsilon)\in\,\Xi_{\,\Omega_\varepsilon}} I_{\,\Omega_\varepsilon}(A_\varepsilon, y_\varepsilon). \tag{16.103}$$

Proof. Taking the a priori estimate (16.18) into account, we immediately conclude that any sequence of optimal pairs $\left\{(A_\varepsilon^0, y_\varepsilon^0) \in \Xi_{\Omega_\varepsilon}\right\}_{\varepsilon>0}$ to the perturbed problems (16.74)–(16.76) is relatively τ-compact in $L^\infty(D;\mathbb{R}^{n\times n}) \times W_0^{1,\,p}(D)$. So we may suppose that there exist a subsequence $\left\{(A_{\varepsilon_k}^0, y_{\varepsilon_k}^0)\right\}_{k\in\mathbb{N}}$ and a pair (A^*, y^*) such that $(A_{\varepsilon_k}^0, \widetilde{y}_{\varepsilon_k}^0) \xrightarrow{\ \tau\ } (A^*, y^*)$ as $k \to \infty$. Then by Theorem 16.27 (see property (ΞM_2)), we have $(A^*,\, y^*|_{\,\Omega}) \in \Xi_{\,\Omega}$. Hence, in view of property (MS_2) of Definition 16.32, we get

$$\begin{aligned}\liminf_{k\to\infty} \min_{(A,\,y)\in\,\Xi_{\Omega_{\varepsilon_k}}} I_{\Omega_{\varepsilon_k}}(A, y) &= \liminf_{k\to\infty} I_{\Omega_{\varepsilon_k}}(A_{\varepsilon_k}^0, y_{\varepsilon_k}^0)\\ &\ge I_{\Omega}(A^*,\, y^*|_{\,\Omega}) \ge \min_{(A,\,y)\in\,\Xi_{\,\Omega}} I_{\,\Omega}(A, y)\\ &= I_{\,\Omega}(A^{opt}, y^{opt}).\end{aligned} \tag{16.104}$$

However, condition (MS_3) implies the existence of a sequence

$$\left\{(\widehat{A}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi_{\Omega_\varepsilon}\right\}_{\varepsilon>0}$$

such that

$$(\widehat{A}_\varepsilon, \widetilde{\widehat{y}}_\varepsilon) \xrightarrow{\tau} (A^{opt}, \widetilde{y}^{opt}) \text{ and } I_\Omega(A^{opt}, y^{opt}) \geq \limsup_{\varepsilon\to 0} I_{\Omega_\varepsilon}(\widehat{A}_\varepsilon, \widehat{y}_\varepsilon).$$

Using this fact, we have

$$\begin{aligned}\min_{(A,\,y)\in \Xi_\Omega} I_\Omega(A,y) = I_\Omega(A^{opt}, y^{opt}) &\geq \limsup_{\varepsilon\to 0} I_{\Omega_\varepsilon}(\widehat{A}_\varepsilon, \widehat{y}_\varepsilon)\\ &\geq \limsup_{\varepsilon\to 0} \min_{(A,\,y)\in \Xi_{\Omega_\varepsilon}} I_{\Omega_\varepsilon}(A,y)\\ &\geq \limsup_{k\to\infty} \min_{(A,\,y)\in \Xi_{\Omega_{\varepsilon_k}}} I_{\Omega_{\varepsilon_k}}(A,y)\\ &= \limsup_{k\to\infty} I_{\Omega_{\varepsilon_k}}(A^0_{\varepsilon_k}, y^0_{\varepsilon_k}). \end{aligned} \tag{16.105}$$

From this and (16.104), we deduce

$$\liminf_{k\to\infty} I_{\Omega_{\varepsilon_k}}(A^0_{\varepsilon_k}, y^0_{\varepsilon_k}) \geq \limsup_{k\to\infty} I_{\Omega_{\varepsilon_k}}(A^0_{\varepsilon_k}, y^0_{\varepsilon_k}).$$

Thus, combining the relations (16.104) and (16.105) and rewriting them in the form of equalities, we obtain

$$I_\Omega(A^*, y^*|_\Omega) = I_\Omega(A^{opt}, y^{opt}) = \min_{(A,\,y)\in \Xi_\Omega} I_\Omega(A,y), \tag{16.106}$$

$$I_\Omega(A^{opt}, y^{opt}) = \lim_{k\to\infty} \min_{(A,\,y)\in \Xi_{\Omega_{\varepsilon_k}}} I_{\Omega_{\varepsilon_k}}(A,y). \tag{16.107}$$

Since (16.106) and (16.107) hold true for every τ-convergent subsequence of the original sequence of optimal solutions $\left\{(A^0_\varepsilon, y^0_\varepsilon) \in \Xi_{\Omega_\varepsilon}\right\}_{\varepsilon>0}$, it follows that the limits in (16.106) and (16.107) coincide and, therefore, the value $I_\Omega(A^{opt}, y^{opt})$ is the limit of the whole sequence of minimal values

$$\left\{I_{\Omega_\varepsilon}(A^0_\varepsilon, y^0_\varepsilon) = \inf_{(A,y)\in \Xi_{\Omega_\varepsilon}} I_{\Omega_\varepsilon}(A,y)\right\}_{\varepsilon>0}.$$

This concludes the proof. □

Our next intention is to derive the sufficient conditions for the Mosco-stability of the OCP (16.47)–(16.49). To do so, we provide the following result.

Lemma 16.35. *Let Ω be an open subset of D. Assume that a sequence $\{\Omega_\varepsilon\}_{\varepsilon>0}$ forms an admissible perturbation of Ω (in the sense of either Definition 16.17 or 16.18). Let $\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ be the sequence of the associated characteristic functions. Let χ^* be a weak-$*$ limit point of this sequence in $L^\infty(D;[0,1])$. Then*

$$\chi_\Omega\,(1-\chi^*) = 0 \quad a.e.\ on \quad D. \tag{16.108}$$

Proof. It is clear that for a fixed perturbation $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of the set Ω, up to a subsequence, there exists a function χ^* such that $\chi_{\Omega_\varepsilon}$ converges weakly-$*$ to χ^* in $L^\infty(D;[0,1])$.

Let $\{y_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)\}_{\varepsilon>0}$ be any sequence such that $\widetilde{y}_\varepsilon \rightharpoonup y^*$ in $W_0^{1,p}(D)$ and $y^*|_\Omega \in W_0^{1,p}(\Omega)$. Due to Propositions 16.24 and 16.25, such a choice is always possible. Then

$$\lim_{\varepsilon\to 0}\int_D \widetilde{y}_\varepsilon\varphi\,\mathrm{d}x = \int_D y^*\varphi\,\mathrm{d}x = \int_D \chi_\Omega y^*\varphi\,\mathrm{d}x, \quad \forall\,\varphi\in L^q(D).$$

On the other hand, using the fact that $\widetilde{y}_\varepsilon \to y^*$ strongly in $L^p(D)$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_D \widetilde{y}_\varepsilon\varphi\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_D \chi_{\Omega_\varepsilon}\widetilde{y}_\varepsilon\varphi\,\mathrm{d}x\\ &= \int_D \chi^* y^*\varphi\,\mathrm{d}x = \int_D \chi^*\chi_\Omega y^*\varphi\,\mathrm{d}x, \quad \forall\,\varphi\in L^q(D)\end{aligned}$$

as a limit of the product of strongly and weakly-$*$ convergent sequences. This concludes the proof. □

We are now in a position to prove the result concerning sufficient conditions for Mosco-stability for the class of the OCPs (16.47)–(16.49).

Theorem 16.36. *Let Ω be an open subset of D. Assume that the distribution $y_d \in L^p(D)$ in the cost functional (16.47) is such that*

$$y_d(x) = y_d(x)\chi_\Omega(x) \quad \text{for a.e. } x\in D. \tag{16.109}$$

Assume also that at least one of the following suppositions holds true:

1. $\Omega \in \mathcal{W}_w(D)$ and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is an H^c-admissible perturbation of Ω.
2. Ω is a p-stable domain and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is an t-admissible perturbation of Ω.

Then the OCP (16.47)–(16.49) is Mosco-stable in $L^\infty(D;\mathbb{R}^{n\times n})\times W_0^{1,p}(D)$.

Proof. We prove (MS_1)–(MS_3) of Definition 16.32. (MS_1) has been proved in Theorem 16.27. Let $\{(A_k,y_k)\}_{k\in\mathbb{N}}$ be a sequence with the properties prescribed in (MS_2) and let (A,y) be its τ-limit. Then $|\widetilde{y}_{\varepsilon_k}-y_d|^p \to |y-y_d|^p$ strongly in $L^1(D)$ and

$$\liminf_{k\to\infty}\|D\widetilde{y}_{\varepsilon_k}\|^p_{\mathbf{L}^p(D)} \geq \|Dy\|^p_{\mathbf{L}^p(D)},$$

by the lower semicontinuity of the norm with respect to the weak convergence. Hence,

$$
\begin{aligned}
&\liminf_{k\to\infty} I_{\Omega_{\varepsilon_k}}(A_k, y_k) \\
&\quad = \liminf_{k\to\infty} \left(\int_D \chi_{\Omega_{\varepsilon_k}} |\widetilde{y}_{\varepsilon_k} - y_d|^p \, \mathrm{d}x + \int_D |D\widetilde{y}_{\varepsilon_k}|^p_{\mathbb{R}^n} \, \mathrm{d}x + J(A_k) \right) \\
&\quad \geq \int_D \chi^* |y - y_d|^p \, \mathrm{d}x + \int_D |Dy|^p_{\mathbb{R}^n} \, \mathrm{d}x + J(A) = \ \{\text{by (16.109)}\} \\
&\quad = \int_D \chi_{\,\Omega} \chi^* |y - y_d|^p \, \mathrm{d}x + \int_\Omega |Dy|^p_{\mathbb{R}^n} \, \mathrm{d}x + J(A) = \ \{\text{by (16.108)}\} \\
&\quad = \int_\Omega |y - y_d|^p \, \mathrm{d}x + \int_\Omega |Dy|^p_{\mathbb{R}^n} \, \mathrm{d}x + J(A) = I_{\,\Omega}(A, y|_{\,\Omega}).
\end{aligned}
$$

This completes the verification of (MS_2). It remains only to verify the last item of Definition 16.32. However, this fact seems to be rather obvious because of the strong convergence $(A_\varepsilon, y_\varepsilon) \to (A, y)$ in $L^\infty(D; \mathbb{R}^{n\times n}) \times W_0^{1,\,p}(D)$ and properties (16.108) and (16.109). Indeed, in this case we have

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} I_{\Omega_\varepsilon}(A_\varepsilon, y_\varepsilon) \\
&\qquad = \lim_{\varepsilon\to 0} \left(\int_D \chi_{\Omega_\varepsilon} |\widetilde{y}_\varepsilon - y_d|^p \, \mathrm{d}x + \int_D |D\widetilde{y}_\varepsilon|^p_{\mathbb{R}^n} \, \mathrm{d}x + J(A_\varepsilon) \right) \\
&\qquad = \int_D \chi^* |y - y_d|^p \, \mathrm{d}x + \int_D |Dy|^p_{\mathbb{R}^n} \, \mathrm{d}x + J(A) \\
&\qquad = \int_D \chi_{\,\Omega} \chi^* |y - y_d|^p \, dx + \int_\Omega |Dy|^p_{\mathbb{R}^n} \, \mathrm{d}x + J(A) \\
&\qquad = \int_\Omega |y - y_d|^p \, \mathrm{d}x + \int_\Omega |Dy|^p_{\mathbb{R}^n} \, \mathrm{d}x + J(A) = I_{\,\Omega}(A, y|_{\,\Omega}).
\end{aligned}
$$

This concludes the proof. □

To end of this section, we provide more precise information concerning the variational properties of the OCP (16.47)–(16.49) under its Mosco-stable perturbation.

Proposition 16.37. *Assume that all suppositions of Theorem 16.34 hold true. Let (A^0, y^0) be an optimal pair to the control problem (16.47)–(16.49) and let $\{(A^0_\varepsilon, y^0_\varepsilon) \in \Xi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ be a sequence of optimal pairs for the perturbed problems (16.74)–(16.76) such that*

$$
(A^0_\varepsilon, \widetilde{y}^0_\varepsilon) \xrightarrow{\ \tau\ } (A^0, \widetilde{y}^0) \quad \text{in} \quad L^\infty(D; \mathbb{R}^{n\times n}) \times W_0^{1,\,p}(D). \tag{16.110}
$$

Then condition (16.109) implies that

$$
\widetilde{y}^0_\varepsilon \to \widetilde{y}^0 \ \text{strongly in} \ W_0^{1,\,p}(D), \tag{16.111}
$$

$$
\lim_{\varepsilon\to 0} J(A^0_\varepsilon) = J(A^0), \tag{16.112}
$$

and

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\left(A_\varepsilon^0[D(y_\varepsilon^0)^{p-2}|Dy_\varepsilon^0, Dy_\varepsilon^0\right)_{\mathbb{R}^n}\,\mathrm{d}x$$
$$=\int_\Omega\left(A^0[D(y^0)^{p-2}]Dy^0, Dy^0\right)_{\mathbb{R}^n}\,\mathrm{d}x. \quad (16.113)$$

Proof. As follows from Theorem 16.36, for a given perturbation $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of the domain Ω the OCP (16.47)–(16.49) on Ω is Mosco-stable in $L^\infty(D;\mathbb{R}^{n\times n})\times W_0^{1,\,p}(D)$. Moreover, due to Theorem 16.34, any sequence of optimal pairs for the perturbed problems (16.74)–(16.76) is relatively τ-compact in

$$L^\infty(D;\mathbb{R}^{n\times n})\times W_0^{1,\,p}(D)$$

and its every τ-cluster pair is an optimal solution to the original problem (16.47)–(16.49). So the supposition (16.110) is not restrictive.

To prove (16.111), we make use relation (16.103). Then

$$\lim_{\varepsilon\to 0}\left(\int_D\chi_{\Omega_\varepsilon}|\widetilde{y}_\varepsilon^0-y_d|^p\,\mathrm{d}x+\int_D|D\widetilde{y}_\varepsilon^0|_{\mathbb{R}^n}^p\,\mathrm{d}x+J(A_\varepsilon^0)\right)$$
$$=\int_D\chi_\Omega|\widetilde{y}^0-y_d|^p\,\mathrm{d}x+\int_D|D\widetilde{y}^0|_{\mathbb{R}^n}^p\,\mathrm{d}x+J(A^0). \quad (16.114)$$

By the Sobolev embedding theorem, we have $\widetilde{y}_\varepsilon^0\to\widetilde{y}^0$ strongly in $L_0^p(D)$. As a result, using properties (16.108) and (16.109), we obtain

$$\lim_{\varepsilon\to 0}\int_D\chi_{\Omega_\varepsilon}|\widetilde{y}_\varepsilon^0-y_d|^p\,\mathrm{d}x=\int_D\chi^*\chi_\Omega|\widetilde{y}^0-y_d|^p\,\mathrm{d}x$$
$$=\int_D\chi_\Omega|\widetilde{y}^0-y_d|^p\,\mathrm{d}x. \quad (16.115)$$

Combining this with (16.114), we come to the relation

$$\lim_{\varepsilon\to 0}\left(\int_D|D\widetilde{y}_\varepsilon^0|_{\mathbb{R}^n}^p\,\mathrm{d}x+J(A_\varepsilon^0)\right)=\int_D|D\widetilde{y}^0|_{\mathbb{R}^n}^p\,\mathrm{d}x+J(A^0).$$

Taking into account the lower semicontinuity property of the functional J, we have

$$\lim_{\varepsilon\to 0}J(A_\varepsilon^0)=J(A^0),$$
$$\lim_{\varepsilon\to 0}\int_D|D\widetilde{y}_\varepsilon^0|_{\mathbb{R}^n}^p\,\mathrm{d}x=\int_D|D\widetilde{y}^0|_{\mathbb{R}^n}^p\,\mathrm{d}x.$$

Then the last equality together with the weak convergence in $W_0^{1,\,p}(D)$ implies (16.111).

It remains only to prove the convergence of energies (16.113). To do this, we use (16.48) and (16.75) replacing therein y by y^0 and y_ε by y_ε^0, respectively. Then for the corresponding integral identities, we take as the test functions $\widetilde{y}^0$ and $\widetilde{y}_\varepsilon^0$, respectively. As a result, passing to the limit in (16.75), we get

$$\lim_{\varepsilon\to 0}\Big(\int_D \big(A^0_\varepsilon[D(\widetilde{y}^0_\varepsilon)^{p-2}]D\widetilde{y}^0_\varepsilon, D\widetilde{y}^0_\varepsilon\big)_{\mathbb{R}^n}\,\mathrm{d}x + \int_D a_0|\widetilde{y}^0_\varepsilon|^p\,\mathrm{d}x\Big)$$
$$= \lim_{\varepsilon\to 0}\int_D f\widetilde{y}^0_\varepsilon\,\mathrm{d}x = \int_D f\widetilde{y}^0\,\mathrm{d}x$$
$$= \int_D \big(A^0[D(\widetilde{y}^0)^{p-2}]D\widetilde{y}^0, D\widetilde{y}^0\big)_{\mathbb{R}^n}\,\mathrm{d}x$$
$$+ \int_D a_0|\widetilde{y}^0|^p\,\mathrm{d}x.$$

To this end, it remains only to use the equality

$$\lim_{\varepsilon\to 0}\int_D a_0|\widetilde{y}^0_\varepsilon|^p\,\mathrm{d}x = \int_D a_0|\widetilde{y}^0|^p\,\mathrm{d}x$$

(see (16.115)). The proof is completed. □

In conclusion, we note that the characteristic feature of the above problem is the fact that the "real" lattice structure (junction structure, thing structure) Ω is never perfectly smooth but contains microscopic asperities of the size significantly smaller than characteristic length scale of the domain. So a direct numerical computation of the solutions of OCPs in such domains is extremely difficult. Usually, a very fine discretization mesh is needed, which means significant computation time, and such a computation is often irrelevant. In view of the variational properties of Mosco-stable problems (see Theorem 16.34 and Proposition 16.37), the above approach allows us to replace the "rough" domain Ω by a family of more "regular" domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ forming some admissible perturbation and to approximate the original problem by the corresponding perturbed problems.

References

1. F. Abergel and R. Temam. On some control problems in fluids machanics. *Theoret. Comput. Fluid Dynam.*, 1:303–325, 1990.
2. R. Adams. *Sobolev Spaces*. Academic Press, New York, 1975.
3. L. D. Akulenko. *Asymptotic Methods of Optimal Control*. Nauka, Moskow, 1987. (in Russian)
4. V. Alekseev, V. Tikchomirov, and S. Fomin. *Optimal Control*. Counsultant Bureau, New York, 1987.
5. G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I,II. *Arch. Rat. Mech. Anal.*, 113:209–259 and 261–298, 1991.
6. G. Allaire. Homogenization and two–scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.
7. G. Allaire. *Shape Optimization by the Homogenization Method*. Springer, New York, 2002.
8. I. I. Argatov and S. A. Nazarov. Junction problem of shashlik (skewer) type. *C. R. Acad. Sci. Paris Sér. I*, 316(6):627–632, 1993.
9. I. I. Argatov and S. A. Nazarov. Asymptotic analysis of problems in junctions of domains of different limit dimensions. An elastic body, pierced by thin rods. *J. Math. Sci.*, 102(5):4349–4387, 2000.
10. H. Attouch. *Variational Convergence for Functionals and Operators*. Pitman, London, 1984.
11. V. I. Averbuch and O. G. Smoljanov. Diferentiation theory in linear topological spaces. *Usp. Matem. Nauk*, 22(6):201–260, 1967. (in Russian)
12. I. Babuška. The finite element method with penalty. *Math Commun.*, 27:221–228, 1973.
13. I. Babuška. Homogenization and its application, mathematical and computational problems. In B. Hubbard, editor, *Numerical Solution of Partial Differential Equations-III (Proceedings of Third SYNSPADE, 1975)*, pages 89–116. Academic Press, New York, 1976.
14. N. S. Bachvalov and G. P. Panasenko. *Homogenization: Averaging Processes in Periodic Media*. Kluwer, Dordrecht, 1989.
15. T. Bagby. Quasi topologies and rational approximation. *J. Funct. Anal.*, 10:259–268, 1972.

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4,

© Springer Science+Business Media, LLC 2011

16. M. V. Balaschov and E. S. Polovinkin. *Elements of Convex and Strongly Convex Analysis*. Fizmatlit, Moskow, 2004.
17. J. M. Ball and V. J. Mizel. One-dimentional variational problems whose minimizers do not satisfy the Euler–Lagrange equation. *Arch. Rational Mech. Anal.*, 90:325–388, 1985.
18. N. V. Banichuk. *Problems and Methods of Optimal Structural Desing*. Plenum Press, New York, 1983.
19. V. Barbu. *Analysis and Control of Infinite Dimensional Systems*. Academic Press, New York, 1993.
20. V. Barbu and Th. Precupanu. *Convexity and Optimization in Banach Spaces*. Reidel, Dordrecht, 1986.
21. E. N. Barron. Averaging in Lagrange and minimax problems of optimal control. *SIAM J. Control Optim.*, 31(6):1630–1652, 1993.
22. A. Bensoussan, J.-L. Lions, and G. Papanicolau. *Asymptotic Analysis for Periodic Structures*. North–Holland, Amsterdam, 1978.
23. L. Boccardo, T. Gallouët, and P. Marcellini. Anisotropic equations in L^1. *Diff. Integr. Eq.*, 9:209–212, 1996.
24. J. F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems*. Springer, New York, 2000.
25. G. Bouchitté, G. Buttazzo, and P. Seppecher. Energies with respect to a measure and applications to low-dimensional structures. *Calc. Var. Partial Diff. Eqs.*, 5(1):37–54, 1997.
26. G. Bouchitté and I. Fragala. Homogenization of thin structures by two-scale method with respect to measures. *SIAM J. Math. Anal.*, 32(6):1198–1226, 2001.
27. G. Bouchitté and I. Fragala. Homogenization of elastic thin structures: A measure-fattening approach. *Journal of Convex Analysis*, 9(2):339–362, 2002.
28. N. Bourbaki. *Les Structures Fondamentales de L'Analyse. Livre IV. Intégration*. Hermann and Co., Editeurs, Paris, 1964.
29. A. Braides. *Γ-convergence for Beginners*. Oxford University Press, Oxford, 2002.
30. A. Braides and A. Defranceschi. *Homogenization of Multiple Integrals*. Clarendon Press, Oxford, 1998.
31. H. Brezis and E. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 88:486–490, 1983.
32. M. Briane, A. Damlamian, and P. Donato. H-Convergence for perforated domains. In D. Cioranescu and J. L. Lions, editors, *Nonlinear Partial Differential Equations and their Applications*, pages 61–100. Longman, London, 1996.
33. A. Briani. *Hamilton-Jacobi-Bellman Equations and Γ-Convergence for Optimal Control Problems*. PhD thesis, Università di Pisa, 1999.
34. H. C. Brinkman. A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Appl. Sci. Res.*, A1:27–34, 1947.
35. R. Brizzi and J. P. Chalot. Homogenization and Neumann boundary value problem. *Ric. Mater.*, 46:347–387, 1997.
36. D. Bucur. Characterization for the Kuratowski limits of a sequence of Sobolev spaces. *J. Diff. Eqs.*, 151:1–19, 1999.
37. D. Bucur. Characterization of the shape stability for nonlinear elliptic problems. *J. Diff. Eqs.*, 226(1):99–117, 2006.
38. D. Bucur and G. Buttazzo. *Variational Methodth in Shape Optimization Problems*. Birkhäuser, Boston, 2005.

39. D. Bucur and P. Trebeschi. Shape optimization problem governed by nonlinear state equations. *Proc. Roy. Soc. Edinburgh*, 128(Ser. A):943–963, 1998.
40. D. Bucur and J. P. Zolésio. N-Dimensional shape optimization under capacitary constraints. *J. Diff. Eqs.*, 123(2):504–522, 1995.
41. G. Buttazzo. γ-convergence and its applications to some problem in the calculus of variations. In *School on Homogenization, ICTP, Trieste, September 6–17, 1993*, pages 38–61, 1994.
42. G. Buttazzo and E. Cavazzuti. Limit problems in optimal control theory. *Ann. Inst. H.Poincaré Anal. Non Linéaire*, 6:151–160, 1989.
43. G. Buttazzo and G. Dal Maso. Γ-convergence and optimal control problems. *J. Optim. Theory Appl.*, 32:385–407, 1982.
44. G. Buttazzo and G. Dal Maso. Γ-convergence et problémes de perturbation singuliere. *C.R. Acad. Sci. Paris, Ser. I*, 296:649–651, 1983.
45. G. Buttazzo and G. Dal Maso. Shape optimization for Dirichlet problems: Relaxed formulation and optimality conditions. *Appl. Math. Optimiz.*, 23:17–49, 1991.
46. G. Buttazzo, G. Dal Maso, A. Garroni, and A. Malusa. On the relaxed formulation of some shape optimization problems. *Adv. Math. Sci. Appl.*, 1(7):1–24, 1997.
47. G. Buttazzo and L. Freddi. Optimal control problems with weakly converging input operators. *Discrete Contin. Dynam. Syst.*, 1:401–420, 1995.
48. G. Buttazzo and L. Freddi. Relaxed optimal control problems and applications to shape optimization. In F. H. Clarke, R. J. Stern and G. Sabidussa, editors, *Nonlinear Analysis, Differential Equations and Control*, pages 159–206, Dordrecht, London, 1999.
49. G. Buttazzo, M. Giaquinta, and S. Hildebrandt. *One-dimentional Variational Problems. An Introduction*. Clarendon Press, Oxford, 1998.
50. C. Calvo-Jurado and J. Casado-Diaz. Results on existence of solution for an optimal design problem. *Extr. Math.*, 18(3):263–273, 2003.
51. C. Calvo-Jurado and J. Casado-Diaz. An existence result for control problems on the coefficients and the domain for nonlinear parabolic Dirichlet problems. *Commun. Nonlinear Sci. Numer. Simul.*, 9:157–166, 2004.
52. C. Calvo-Jurado and J. Casado-Diaz. Optimization by the homogenization method for nonlinear elliptic Dirichlet problems. *Mediterranian J. Math.*, 4:53–63, 2007.
53. L. Carbone and R. De Arcangelis. *Unbounded Functionals in the Calculus of Variations. Representation, Relaxation, and Homogenization*. Chapman & Hall/CRC, New York, 2002.
54. G. Cardone, C. D'Apice, and U. De Maio. Homogenization in perforated domains with mixed conditions. *Nonlinear Diff. Eqs. Appl.*, 9:325–246, 2002.
55. J. Casado-Diaz. Existence of a sequence satisfying Cioranescu-Murat conditions in homogenization of Dirichlet problems in perforated domains. *Rend. Mat., Serie VII*, 16:387–413, 1996.
56. J. Casado-Diaz, M. Luna-Laynez, and J. D. Marin. An adaption of the multiscale methods for the analysis of very thin reticulated structures. *C.R. Acad. Sci. Paris, Serie I*, 332:223–228, 2001.
57. C. Castaign and M. Valadier. *Convex Analysis and Measurable Multifunctions*. Birkhäuser, Boston, 1977.
58. G. A. Chechkin, V. V. Jikov, D. Lukkassen, and A. L. Piatnitski. On homogenization on networks and junctions. *J. Asymp. Anal.*, 30:61–80, 2000.

59. G. A. Chechkin, A. L. Piatnitski, and A. S. Shamaev. *Homogenization. Methods and Applications*. AMS, Providence, RI, 2007.
60. D. Chenais. On the existence of a solution in a domain identification problem. *J. Math. Anal. Appl.*, 52:189–219, 1975.
61. D. Chenais and E. Zuazua. Controllability of an elliptic equation and its finite difference approximation by the shape of the domain. *Numer. Math.*, 85:63–99, 2003.
62. P. G. Ciarlet. *Mathematical Elasticity. II. Theory of Plates*. Elsevier, Amsterdam, 1997.
63. D. Cioranescu, P. Donato, F. Murat, and E. Zuazua. Homogenization and corrector for the wave equation in domains with small holes. *Ann. Sc. Norm. Sup., Pisa*, 18:251–293, 1991.
64. D. Cioranescu, P. Donato, and E. Zuazua. Exact boundary controllability for the wave equation in domains with small holes. *J. Math. Pures Appl.*, 69:1–31, 1990.
65. D. Cioranescu and F. Murat. Un terme étrage venu d'ailleurs. In F. H. Clarke, R. J. Stern and G. Sabidussa, editors, *Partial Differential Equations and their Applications. Collége de France Seminar*, volume II,III, pages 58–138, 157–178, Pitman, London, 1981.
66. D. Cioranescu and D. Saint Jean Paulin. *Homogenization of Reticulated Structures*. Springer, New York, 1999.
67. D. Ciorenescu and Saint Jean Paulin. Homogenization in open sets with holes. *J. Math. Anal. Appl.*, 71:590–607, 1979.
68. F. H. Clarke. *Optimization and Nonsmooth Analysis*. Wiley, New York, 1983.
69. R. Clausius. Die mech. wärme theorie. *Vieweg*, 2:1–62, 1879.
70. C. Conca and P. Donato. Nonhomogeneous Neumann problems in domains with small holes. *Modél. Math. Anal. Numér.*, 22(4):561–608, 1988.
71. C. Conca, A. Osses, and J. Saint Jean Paulin. A semilinear control problem involving in homogenization. *Electr. J. Diff. Eqs.*, 6:109–122, 2001.
72. C. Conca, A. Osses, and J. Saint Jean Paulin. Approximate cantrollability and homogenization of a semilinear elliptic problem. *J. Math. Anal. Appl.*, 285(1):17–36, 2003.
73. A. Corbo Esposito, C. D'Apice, and A. Gaudiello. A homogenization problem in a perforated domain with both Dirichlet and Neumann conditions on the boundary of the holes. *Asymp. Anal.*, 31:297–316, 2002.
74. A. Corbo Esposito, P. Donato, A. Gaudiello, and C. Piccard. Homogenization of the p-Laplacian in a domain with oscillating boundary. *Commun. Appl. Nonlinear Anal.*, 4(4):1–23, 1997.
75. J.-M. Coron and E. Trélat. Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM J. Control Optim.*, 43(2):549–569, 2004.
76. G. Dal Maso. Some necessary and sufficient conditions for the convergence of unilatered convex sets. *J. Funct. Anal.*, 62:119–159, 1985.
77. G. Dal Maso. Γ-convergence and μ-capacities. *Ann. Scuola Norm. Sup. Pisa*, 14:423–464, 1987.
78. G. Dal Maso. *An Introduction to Γ-Convergence*. Birkhäuser, Boston, 1993.
79. G. Dal Maso and A. Defranceschi. Limits of nonlinear Dirichlet problems in varing domains. *Manuscr. Math.*, 61:251–278, 1988.

80. G. Dal Maso, F. Ebobisse, and M. Ponsiglione. A stability result for nonlinear Neumann problems under boundary variations. *J. Math. Pures Appl.*, 82:503–532, 2003.
81. G. Dal Maso and U. Mosco. Wiener criteria and energy decay for relaxed Dirichlet problems. *Arch. Rational Mech. Anal.*, 95:345–387, 1986.
82. G. Dal Maso and F. Murat. Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous montone operators. *Ann. Sc. Norm. Sup. Pisa, Cl. Sci.*, 24(2):239–290, 1997.
83. E. N. Dancer. The effect of domains shape on the number of positive solutions of certain nonlinear equations. *J. Diff. Eqs.*, 87:316–339, 1990.
84. D. Daners. Domain perturbation for linear and nonlinear parabolic equations. *J. Diff. Eqs.*, 129(2):358–402, 1996.
85. C. D'Apice, U. De Maio, and P. I. Kogut. Gap phenomenon in homogenization of parabolic optimal control problems. *IMA J. Math. Control and Inform.*, 25:461–489, 2008.
86. C. D'Apice, U. De Maio, and P. I. Kogut. Subopimal boundary control for elliptic equation in critically perforated domain. *Ann. Ist. Henri Poincare (C). Nonlinear Anal.*, 25(6):1073–1101, 2008.
87. C. D'Apice, U. De Maio, and P. I. Kogut. Boundary velocity suboptimal control of incompressible flow in cylindrically perforated domain. *Discrete and Contin. Dynam. Syst., B*, 11(2):283–314, 2009.
88. U. De Maio, A. Gaudiello, and C. Lefter. Optimal control for a parabolic problem in a domain with highly oscillating boundary. *Applic. Anal.*, 83(12):1245–1264, 2004.
89. U. De Maio and T. A. Mel'nyk. Homogenization of the robin problem in a thick multi-structure of the type 3:2:2. *Asymp. Anal.*, 41(2):161–177, 2005.
90. U. De Maio, T. A. Mel'nyk, and C. Perugia. Homogenization of the robin problem in a thick multilevel junction. *Nonlinear Oscill.*, 7(3):336–355, 2004.
91. M. C. Delfour and J.-P. Zolésio. *Shapes and Geometries: Analysis, Differential Calculus, and Optimization*. SIAM, Philadelphia, 2001.
92. Z. Denkowski and S. Migorski. Control problems for parabolic and hyperbolic equations via the theory of G- and Γ-convergence. *Ann. Mat. Pure Applic.*, 149:23–39, 1987.
93. Z. Denkowski and S. Mortola. Asymptotic behaviour of optimal solutions to control problems for system described by differential inclusions corresponding to partial differential equations. *J. Optimiz. Theory Appl.*, 78(2):365–391, 1993.
94. Z. Denkowski and V. Staicu. Asymptotic behaviour of minima to a class of optimization problems for differential inclusions. *J. Optimiz. Theory Appl.*, 81(1):35–54, 1994.
95. R. DiPerna and A. Majda. Reduced Hausdorff dimension and concentration cancellation for two-dimensional incompressible flow. *J. Amer. Math. Soc.*, 1:59–95, 1988.
96. P. Donato and A. Nabil. Homogenization and correctors for the heat equation in perforated domains. *Ric. Math.*, L(1):115–144, 2001.
97. Ju. A. Dubinskiy. Strong convergence in nonlinear elliptic and parabolic equations. *Math. Sbornik*, 67(4):299–365, 1965. (in Russian)
98. A. D. Dubovitski and A. A. Miljutin. *Necessary Conditions of the Weak Minimum for General Optimal Control Problem*. Nauka, Moskow, 1971. (in Russian)

99. N. Dunford and J. T. Schwartz. *Linear Operators*. Wiley, New York, 1957.
100. T. Durante, L. Faella, and C. Perugia. Optimal control for the wave equation in domain with oscillating boundary. *Nonlinear Diff. Eqs. Applic.*, 14(5-6):455–489, 2007.
101. T. Durante and L. A. Mel'nyk. Asymptotic analysis of an optimal control problem involving a thick two-level junction with alternate type of controls. *J. Optim. Theory Appl.*, 144(2):205–225, 2010.
102. G. Duvaut and J.-L. Lions. *Les Inéquations en Méchanique et en Physique*. Dunon, Paris, 1971.
103. A. Egorov. *Optimal Control of Heat and Diffusion Processes*. Nauka, Moskow, 1978.
104. I. Ekeland and R. Temam. *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
105. L. C. Evans. *Weak Convergence Methods for Nonlinear Partial Differential Equations*. AMS, Providence, RI, 1990.
106. L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC Press, New York, 1992.
107. K. J. Falconer. *The Geometry of Fractal Sets*. Cambridge University Press, Cambridge, 1985.
108. H. O. Fattorini. *Infinite Dimensional Optimization and Control Theory*. Cambridge University Press, New York, 1999.
109. H. Federer. *Geometric Measure Theory*. Springer, Berlin, 1969.
110. I. Fragalà and C. Mantegazza. On some notion of tangent space to a measure. *Proc. Roy. Soc. Edinburgh*, 129A:331–342, 1999.
111. A. V. Fursikov. *Optimal Control of Distributed Systems. Theory and Applications*. AMS, Providence, RI, 2000.
112. A. V. Fursikov, M. D. Gunzburger, and L. S. Hou. Boundary value problems and optimal boundary control for the Navier-Stokes system. *SIAM J. Control Optim.*, 36(3):852–894, 1998.
113. H. Gaevskii, K. Greger, and K. Zaharias. *Nonlinear Operator Equations and Operator-Differential Equations*. Mir, Moskow, 1978. (Russian translation)
114. L. Gibson and M. Ashby. *Cellular Solids*. Pergamon Press, New York, 1997.
115. E. Giusti. *Minimal Surfaces and Function of Bounded Variation*. Birhäuser, Boston, 1984.
116. M. D. Gunzburger, L. S. Hou, and T. Svobodny. Boundary velocity control of incompressible flow with an application to viscous drag reduction. *SIAM J. Control Optim.*, 30:167–181, 1992.
117. A. Haraux and F. Murat. Perturbations singulieres et problèmes de contrŏle optimal: deux cas bien poses. *C.R. Acad. Sc. Paris, Serie I*, 297:21–24, 1983.
118. A. Haraux and F. Murat. Perturbations singulieres et problèmes de contrŏle optimal: Un cas mal pose. *C.R. Acad. Sc. Paris, Serie I*, 297:93–96, 1983.
119. J. Haslinger and P. Neittaanmäki. *Finite Element Approximation of Optimal Shape. Material and Topology Design*. Wiley, Chichester, UK, 1996.
120. J. Heinonen, T. Kilpelainen, and O. Martio. *Nonlinear Potential Theory of Degenerate Elliptic Equations*. Clarendon Press, Oxford, 1993.
121. A. Henrot and M. Pierre. *Variation et Optimisation de Formes. Une Analyse Géométrique*. Springer Verlag, New York, 2005.
122. M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *Optimization with PDE Constraints*. Springer, Dordrecht, 2009.

123. L. S. Hou and S. S. Ravindran. A penalized Neumann control approach for solving an optimal Dirichlet control problem for the Navier-Stokes equations. *SIAM J. Control Optim.*, 36(5):1795–1814, 1998.
124. L. S. Hou and T. Svobodny. Optimization problems for the Navier-Stokes equations with regular boundary controls. *J. Math. Anal. Appl.*, 177:342–367, 1993.
125. A. D. Ioffe and V. M. Tikhomirov. *Theory of Extremal Problems.* Nauka, Moskow, 1974. (in Russian)
126. V. I. Ivanenko and V. S. Mel'nik. *Variational Methods in Optimal Control Problem for Distributed Systems.* Naukova Dumka, Kyiv, 1988. (in Russian)
127. V. K. Kalantarov and O. A. Ladizhenskaja. Collapse phenomenon for quasilinear parabolic and hyperbolic equations. *Zapiski LOMI*, 69:77–102, 1977. (in Russian)
128. L. V. Kantorovich and G. P. Akilov. *Functional Analysis.* Nauka, Moskow, 1976. (in Russian)
129. V. Y. Kapustyan and L. A. Panic. Asymptotics of optimal distributed controls for singular perturbed elliptic problems with local constraints. *Nonlinear Bound. Value Probl.*, 13:89–99, 2003. (in Russian)
130. V. Y. Kapustyan and E. V. Shamaev. Optimal control of the order of asymptotics for elliptic equations with fast oscillating coefficients: I. Formal constructions. *Cybernet. Syst. Anal.*, 40(5):699–709, 2004. (in Russian)
131. V. Y. Kapustyan, P. O. Kasyanov, and O. P. Kogut. On solvability of one class of parametrized operator inclusions. *Ukrain. Math. J.*, 60(12):1619–1630, 2008. (in Ukrainian)
132. P. O. Kasyanov, V. S. Mel'nik, and V. V. Yasinsky. *Evolution Inclusions and Inequalities in Banach Spaces with w_λ-Pseudomonotone Maps.* Naukova Dumka, Kyiv, 2007.
133. M. V. Keldysh. On the solvability and stability of the Dirichlet problem. *Amer. Math. Soc. Transl.*, 51-52:1–73, 1963.
134. S. Kesavan and J. Saint Jean Paulin. Homogenization of an optimal control problem. *SIAM J. Control Optim.*, 35(5):1557–1573, 1997.
135. S. Kesavan and J. Saint Jean Paulin. Optimal control on perforated domains. *J. Math. Anal. Appl.*, 229:563–586, 1999.
136. S. Kesavan and J. Saint Jean Paulin. Optimal control problem in homogenization. In *Équations aux Dérivées Partialles et Applications. Articles Dédiés à J.-L. Lions*, pages 597–609. Gavthier–Villars, Paris, 1999.
137. S. Kesavan and M. Vanninathan. L'homogénéisation d'un problème de contrŏle optimal. *C.R. Acad. Sci., Serie A*, 285:441–444, 1977.
138. E. Ya. Khruslov. Asymptotic behavior of solutions of the second boundary value problem with fragmentation of the boundary of domain. *Math. USSR Sbornik*, 35(2):604–621, 1978.
139. O. P. Kogut. *Qualitative Analysis of One Class of Optimization Problems for Nonlinear Elliptic Problems.* PhD thesis, V.M. Glushkov Institute of Cybernetics of NAS of Ukraine, Kyiv, 2010. (in Ukrainian).
140. P. I. Kogut. *S*-convergence in homogenization theory of optimal control problems. *Ukrain. Mat. Zh.*, 49(11):1488–1498, 1997. (in Russian)
141. P. I. Kogut. *S*-convergence of constrained minimization problems and its variational properties. *Probl. Upravlenia Inform.*, 4:64–79, 1997. (in Russian)
142. P. I. Kogut. Γ-representation of variational *S*-litis of constrained minimization problems in normal Hausdorff spaces. *Cybern. Syst. Anal.*, 34(1):86–96, 1998. (in Russian)

143. P. I. Kogut. *Homogenization of Optimal Control Problems for Distributed Systems.* PhD thesis, Cybernetics Institute of Ukrainian National Academic Science, Kyiv, 1998. (in Russian)
144. P. I. Kogut. On the structural representation of S-homogenized optimal control problems. *Nonlinear Bound. Valued Probl.*, 9:84–91, 1999.
145. P. I. Kogut and G. Leugering. Homogenization of optimal control problems in variable domains. Principle of the fictious homogenization. *Asymp. Anal.*, 26:37–72, 2001.
146. P. I. Kogut and G. Leugering. On S-homogenization of an optimal control problem with control and state constraints. *J. Anal. Applic.*, 20(2):395–429, 2001.
147. P. I. Kogut and G. Leugering. **S**-homogenization of optimal control problems in Banach spaces. *Math. Nachr.*, 233–234:141–169, 2002.
148. P. I. Kogut and G. Leugering. Asymptotic analysis of state constrained semilinear optimal control problems. *J. Optim. Theory Applic.*, 135(2):301–321, 2007.
149. P. I. Kogut and G. Leugering. On the homogenization of optimal control problems on periodic graphs. In R. Glowinski and J.-P. Zolesio, editors, *Free and Moving Boundaries: Analysis, Simulation and Control*, pages 55–74. Chapman & Hall/CRC, New York, 2007.
150. P. I. Kogut and G. Leugering. Homogenization of Dirichlet optimal control problems with exact partial controllability constraints. *Asymp. Anal.*, 57(3–4):229–249, 2008.
151. P. I. Kogut and V. S. Mel'nik. On one class of extremal problems for nonlinear operator systems. *Cybernet. Syst. Anal.*, 6:121–132, 1998. (in Russian)
152. P. I. Kogut and V. S. Mel'nik. On the weak compactness of bounded sets in Banach and loccally convex spaces. *Ukrain. Math. J.*, 52(6):837–846, 2001. Translated from Ukrain. Mat. Zh. 52(6):731–739, 2000.
153. P. I. Kogut and T. A. Mel'nyk. Asymptotic analysis of optimal control problems in thick multi-structures. In *Generalized Solutions in Control Problems*, pages 265–275. Proceeding of the IFAC Workshop GSCP-2004, Pereslavl–Zalesky, Russia, September 21–27, 2004. Fizmatlit, Moskow, 2004.
154. P. V. Kokotovic. Applications of singular perturbation techniques to control problems. *SIAM Rev.*, 26(4):501–550, 1984.
155. A. A. Kovalevskii. G-convergence and homogenization of nonlinear elliptic operators in divergence form with variable domain. *Russ. Acad. Sci. Izv. Math.*, 44(3):431–460, 1995.
156. A. A. Kovalevskii. On the Γ-convergence of integral functionals defined on Sobolev weakly connected spaces. *Ukrain. Math. J.*, 48(5):683–698, 1996.
157. A. A. Kovalevskii. An effect of double homogenization for Dirichlet problems in variable domains of general structure. *C. R. Acad. Sci. Paris, Serie I.*, 328(12):1151–1156, 1999.
158. A. A. Kovalevskii. Averaging of integral functionals associated with domains of periodic structure of framework type with thin channels. *Ukrain. Math. J.*, 53(5):703–714, 2000.
159. A. A. Kovalevskii and Yu. S. Gorban. Degenerate anisotropic variational inequalities with L^1-data. *C.R. Acad. Sci. Paris*, 345:441–444, 2007.
160. P. Kuchmer. Graph models for waves in thing structures. *Waves Random Media*, 12:1–24, 2002.

161. K. Kuratowski. *Topology I, II.* PWN, Warszawa, 1966.
162. O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and Quasilinear Elliptic Equations.* Academic Press, New York, 1968.
163. J. E. Lagnese and G. Leugering. *Domain Decomposition Methods in Optimal Control of Partial Differential Equations.* Birkhäuser, Basel, 2004.
164. J. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt. *Modelling, Analysis and Control of Multi-Link Flexible Structures.* Birkhäuser, Basel, 1994.
165. M. Lenczner and G. Senouci-Bereski. Homogenization of electrical networks including voltage to voltage amplifiers. *Math. Methods Appl. Sci.*, 9(6):899–932, 1999.
166. G. Leugering and E. J. P. G. Schmidt. On the modelling and stabilization of flows in networks of open canals. *SIAM J. Control Optim.*, 41:164–180, 2002.
167. J.-L. Lions. *Équations Différentielles Opérationnelles.* Springer, Berlin, 1961.
168. J.-L. Lions. *Some Methods of Solving Non-Linear Boundary Value Problems.* Dunod-Gauthier-Villars, Paris, 1969.
169. J.-L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations.* Springer, Berlin, 1971.
170. J.-L. Lions. Asymptotic methods in optimal control of distributed systems. *Automatica*, 14(3):199–211, 1978.
171. J.-L. Lions. *Some Methods in the Mathematical Analysis of System and their Control.* Gordon and Breach, New York, 1981.
172. J.-L. Lions. *Control of Distributed Singular Systems.* Gauthier-Villars, Paris, 1985.
173. J.-L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications.* Springer, Berlin, 1972.
174. P. L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. Part I. *Rev. Mat. Ibero*, 1(1):145–201, 1985.
175. P. L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. Part II. *Rev. Mat. Ibero*, 1(2):45–121, 1985.
176. V. G. Litvinov. *Optimization in Elliptic Boundary Value Problems with Applications to Mechanics.* Mir, Moskow, 1987.
177. W. B. Liu. *Optimal Shape Desing for Variational Inequalities.* PhD thesis, University of Leeds, 1991.
178. W. B. Liu, P. Neittaanmäki, and D. Tiba. Existence for shape optimization problems in arbitrary dimension. *SIAM J. Control Optim.*, 41(5):1440–1454, 2003.
179. D. Lukkassen and P. Wall. Two-scale convergence with respect to measures and homogenization of monotone operators. *J. Function Spaces Applic.*, 3(2):125–161, 2005.
180. G. Lumer. Connecting of local operators and evolution equations on networks. In *Potential Theory*, pages 219–234. Springer, New York, 1980.
181. K. A. Lurie. *Applied Optimal Control Theory of Distributed Systems.* Plenum Press, New York, 1993.
182. V. A. Marchenko and E. Ya. Khruslov. *Boundary Value Problems in Domain with Fine-Grainy Boundary.* Naukova Dumka, Kyiv, 1974. (in Russian)
183. C. Maxwell. *Treatise on Electricity and Magnetismus.* Oxford University Press, Oxford, 1873.
184. A. Mazja and A. Slutskij. Averaging of a differential operator on thick periodic grid. *Math. Nachr.*, 133:107–133, 1987.

185. V. G. Maz'ya. *Sobolev Spaces*. Leningrad University Press, Leningrad, 1986. (English transl.: Springer-Verlag, Berlin, 1985)
186. V. S. Mel'nik. Method of monotone operators in the theory of constrained optimal system. *Rep. Ukrain. Acad. Sci.*, A(7):64–67, 1984.
187. T. A. Mel'nyk and S. A. Nazarov. Asymptotic structure of the spectrum of the Neumann problem in a thin comb-like domain. *C.R. Acad Sci. Paris*, 319:1343–1348, 1994.
188. V. P. Michajlov. *Partial Differential Equations*. Nauka, Moscow, 1973. (in Russian)
189. S. Mizohata. *Theory of Partial Differential Equations*. Mir, Moskow, 1977. (in Russian)
190. Boris S. Mordukhovich. *Variational Analysis and Generalized Differentiation. I: Basic Theory. II: Applications*. Springer, Berlin, 2006.
191. U. Mosco. Converges of convex sets and of solutions of variational inequalities. *Adv. Math.*, 3:510–585, 1969.
192. O. F. Mosotti. Discussione analitica sul in uenze che l'azione di in mezzo dielettrico ha sulla distribuzione dell' electtricita alla superfcie di pin corpi electtici disseminati in esso. *Mem. Math. Fisica Modena*, 24(2):49, 1850.
193. F. Murat. Un contre-example pour le problème de contrôle dans les coefficients. *C.R. Acad. Sci. Paris, Sér. A*, 273:1003–1007, 1971.
194. F. Murat. Téorèmes de non-existence pour des problèmes de contrôle dans le coefficients. *C.R. Acad. Sci. Paris, Sér. A*, 274:395–398, 1972.
195. F. Murat and L. Tartar. On the control of coefficients in partial differential equations. In A. Cherkaev and R. Kohn, editors, *Topics in the Mathematical Modelling of Composite Materials*, pages 139–173. Birkhäuser, Boston.
196. F. Murat and L. Tartar. H-convergence. In A. Cherkaev and R. Kohn, editors, *Topics in the Mathematical Modelling of Composite Materials*, pages 21–43. Birkhäuser, Boston, 1997.
197. T. Muthukumar. *Asymptotic Behaviour of Some Optimal Control Problems*. PhD thesis, University of Madras, 2006.
198. I. P. Natanson. *Theory of Functions of a Real Variable*, volume 1. Frederick Unger, New York, 1964.
199. S. A. Nazarov. Junctions of singularly degenerating domains with different limit dimensions. *Trudy Sem. Petrovsk.*, 18:3–78, 1995.
200. S. A. Nazarov. *Asymptotic Analysis of Thin Plates and Beams. 1. Reduction of the Dimension and Integral Estimates*, volume 1. Nauchnaya Kniga, Novosibirsk, 2002. (in Russian)
201. S. A. Nazarov and A. S. Slutskii. Arbitrary plane systems of anisotropic beams. *Tr. Mat. Inst. Steklov.*, 236:234–261, 2002. (in Russian)
202. S. A. Nazarov and A. S. Slutskii. Korn inequality for an arbitrary system of thin curvilinear rods. *Sibir. Mat. Zh.*, 43(6):1319–1331, 2002. (in Russian)
203. S. A. Nazarov and A. S. Slutskii. Asymptotic analysis of an arbitrary spatial system of thin rods. *Tr. St. Peterb. Mat. Obshch.*, 10:63–117, 2004. (in Russian)
204. P. Neittaanmäki and D. Tiba. A variational inequality approach in constrained control problem. *Appl. Math. Optimiz.*, 17:185–201, 1988.
205. J. Nečas. *Les Méthodes Directes en Théorie des Équations Elliptiques*. Masson, Paris, 1967.
206. G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20:608–623, 1989.

207. G. P. Panasenko. Asymptotic solutions of the elasticity theory system of equations for lattice and skeletal structures. *Russ. Acad. Sci. Sb. Math.*, 75(1):85–110, 1993.
208. G. P. Panasenko. Method of asymptotic partial decomposition of rod structures. *J. Comput. Civil Struct. Eng.*, 1(2):57–70, 2000.
209. G. P. Panasenko. *Multi-Scale Modelling for Structures and Composites*. Kluver Academic, Dordrecht, 2004.
210. A. A. Pankov. *G-convergence and Homogenization on Nonlinear Partial Differential Equations*. Kluver Academic, Dordrecht, 1997.
211. S. E. Pastuchova. Homogenization of problems of elasticity theory on periodic box and rod frames of critical thickness. *J. Math. Sci.*, 130(5):4954–5004, 2005.
212. S. E. Pastuchova. On the convergence of hyperbolic semigroups in variable Hilbert spaces. *J. Math. Sci.*, 127(5):2263–2283, 2005.
213. S. E. Pastuchova. Homogenization in elasticity problems on combined structures. *J. Math. Sci.*, 132(3):313–330, 2006.
214. O. Pironneau. *Optimal Shape Design for Elliptic Systems*. Springer, Berlin, 1984.
215. S. D. Poisson. Second mêm. sur la théorie de magnetisme. *Mem. Acad. France*, 5, 1822.
216. Yu. V. Pokornyi, O. M. Penkin, V. I. Pryadiev, A. V. Borovskikh, K. P. Lazarev, and S. A. Shabrov. *Differential Equations on Geometrical Graphs*. Fizmatlit, Moskow, 2004. (in Russian)
217. J. Rauch and M. Taylor. Potential and scattering theory on wildly perturbed domains. *J. Funct. Anal.*, 18:27–59, 1975.
218. J. W. Rayleigh. On the in uence of obstacles in rectangular order upon the properties of a medium. *Philos. Mag.*, 34(5):481, 1892.
219. U. E. Raytum. *Optimal Control Problems for Elliptic Equations*. Zinatne, Riga, 1989. (in Russian)
220. T. Roubiček. *Relaxation in Optimization Theory and Variational Calculus*. Walter de Gruyter, Berlin, New York, 1997.
221. W. Rudin. *Functional Analysis*. Mc Graw-Hill, New York, 1991.
222. J. Saint Jean Paulin and H. Zoubairi. Optimal control and "strange term" for the Stokes problem in perforated domains. *Portug. Math.*, 59(2):161–178, 2002.
223. G. Savaré and G. Schimperna. *Domain Perturbations and Estimates for the Solutions of Second Order Elliptic Equations*. Istituto Analisi Numerica, Pavia, 2001.
224. L. Schwartz. *Théorie des Distributions*. Hermann, Paris, 1951.
225. I. V. Scrypnik. Averaging nonlinear Dirichlet problems in domains with channels. *Sov. Math. Dokl.*, 42:853–857, 1991.
226. S. Ya. Serovaĭskiĭ. Variational inequalities in nonlinear optimal contril problems. *Methods Facilities Math. Model.*, 1:156–169, 1977. (in Russian)
227. S. Ya. Serovaĭskiĭ. Optimal control of nonlinear system with fixed terminal state. *Diff. Uravnenija*, 33(8):299–365, 1997. (in Russian)
228. S. Ya. Serovaĭskiĭ. Approximate solution of optimization problems for infinite-dimensional singular problems. *Siber. Math. J.*, 44(3):519–528, 2003.
229. S. Ya. Serovaĭskiĭ. Optimal control for singular evolutional equation with non-smooth operator and fixed finite state. *J. Diff. Eq.*, 43(2):251–258, 2007. (in Russian)

230. V. V. Shumilova. On approximate properties for thing structures. *J. Math. Sci.*, 126(6):1669–1671, 2005.
231. V. I. Smirnov. *A Course in Higher Mathematics*. Gostekhizdat, Moskow, 1947. (in Russian)
232. S. L. Sobolev. *Some Applications of Functional Analysis in Mathematical Physics*. Nauka, Moskow, 1988. (English transl.: AMS Translation of Mathematical Monographs, Vol. 10, 1991)
233. J. Sokolowski and J. P. Zolesio. *Introduction to Shape Optimization*. Springer, Berlin, 1991.
234. A. P. Suetov. A shape optimization algorithm for an elliptic system. *Control Cybernet*, 23(3):565–573, 1994.
235. M. A. Sychev and V. J. Mizel. A condition on the value function both necessary and sufficient for full regularity of minimizers of one-dimensional variational problems. *Trans. Amer. Math. Soc.*, 350(1):119–133, 1998.
236. L. Tartar. Estimations fines de coefficients homogénéisés. In *Ennio De Giorgi colloquium (Paris, 1983)*, pages 168–187. Pitman, Boston, MA, 1985.
237. R. Temam. *Navier-Stokes Equations, Theory and Numerical Methods*. Elsevier Science, New York, 1984.
238. D. Tiba. Une approache par inéquations variationnelles pour les problemes de contrôle avec contraintes. *C.R. Acad. Sci. Paris*, 302:29–31, 1986.
239. D. Tiba. *Optimal Control of Nonsmooth Distributed Parameter Systems*. Springer Verlag, Berlin, 1990.
240. D. Tiba. *Lectures on the Control of Elliptic Systems*. Department of Mathematics, University of Jyväskylä, Finland, 1995.
241. F. Treves. *Topological Vector Spaces, Distributions and Kernels*. Academic Press, New York, 1967.
242. Fredi Tröltzsch. *Optimal Control of Partial Differential Equations. Theory, Methods and Applications*. AMS, Providence, RI, 2010.
243. G. V. Uspenskii. On trace of functions from Sobolev space $W_p^{l_1,\dots,l_n}$ on smooth manyfolds. *Siberian Math. J.*, 13(2):429–451, 1972. (in Russian)
244. G. V. Uspenskii, G. V. Demidenko, and V. G. Perepelkin. *Embedding Theorems and Applications to Differential Equations*. Nauka, Moskow, 1984. (in Russian)
245. M. Valadier. Oscillations et compacité forte dans l^1. *Sém. Anal. Convexe*, 21(7):1–10, 1991.
246. M. Vogelius. A homogenization result for planar, polygonal networks. *Math. Model. Numer. Anal.*, 25:483–514, 1991.
247. A. I. Vol'pert. Differential equations on graphs. *Math. Sbornik*, 88(4):578–588, 1972.
248. J. von Below. *Parabolic Network Equations*. PhD thesis, Eberhard–Karis–Universität, Tübingen, 1993.
249. V. Šverák. On optimal shape design. *J. Math. Pures Appl.*, 72:537–551, 1993.
250. J. Warga. *Optimal Control of Differential and Functional Equations*. Academic Press, New York, 1972.
251. K. Yosida. *Functional Analysis*. Springer, Berlin, 1965.
252. L. C. Young. *Lectures on the Calculus of Variations and Optimal Control Theory*. W.B. Saunders Company, Philadelphia, 1969.
253. M. Z. Zgurovsky and V. S. Mel'nik. *Nonlinear Analysis and Control of Physical Processes and Fields*. Springer, Berlin, 2004.
254. V. V. Zhikov. Connection and averaging. Examples of fractal conductivity. *Sbornik Math.*, 187(8):3–40, 1996.

255. V. V. Zhikov. Weighted Sobolev spaces. *Sbornik Math.*, 189(8):27–58, 1998.
256. V. V. Zhikov. On an extension of the method of two-scale convergence and its applications. *Sbornik Math.*, 191(7):973–1014, 2000.
257. V. V. Zhikov. Homogenization of elasticity problems for singular structures. *Izv. Russ. Akad. Nauk., Ser. Mat.*, 66(2):81–148, 2002.
258. V. V. Zhikov. On two-scale convergence. *Trudy Semin. Petrovskii*, 23:149–187, 2003.
259. V. V. Zhikov. On two-scale convergence. In *Trudy Seminara Imeni I.G. Petrovskogo*, volume 23, pages 149–187. Moscow State University, 2003.
260. V. V. Zhikov. A note on Sobolev spaces. *J. Math. Sci.*, 129(1):3593–3595, 2005.
261. V. V. Zhikov, S. M. Kozkol, and O. A. Oleinik. *Homogenization of Differential Operators and Integral Functionals*. Springer, Berlin, 1994.
262. V. V. Zhikov, S. M. Kozlov, O. A. Oleinik, and K. Ngoan. Averaging and G-convergence of differential operators. *Russ. Math. Surv.*, 34:69–147, 1979.
263. V. V. Zhikov and D. Lukkassen. On two types of effective conductivities. *J. Math. Anal. Appl.*, 256:339–343, 2001.
264. V. V. Zhikov and S. E. Pastuchova. Korn inequalities on thin periodic structures. *Dokl. Russ. Akad. Nauk*, 388(54):588–592, 2003.
265. V. V. Zhikov and S. E. Pastuchova. Derivation of the limit equations of elasticity theory on thin nets. *J. Math. Sci.*, 135(1):2637–2665, 2006.
266. V. V. Zhikov and S. E. Pastuchova. Homogenization of degenerate elliptic equations. *Siberian Math. J.*, 49(1):80–101, 2006.
267. W. P. Ziemer. *Weakly Differentiable Functions*. Springer, Berlin, 1989.
268. H. Zoubairi. Optimal control and homogenization in a mixture of fluids separated by a rapidly oscillating interface. *Electr. J. Diff. Eqs.*, 2:1–32, 2002.

Index

G-convergence, 152
H-convergence, 592
H^c-admissible perturbation of a domain, 602
K-lower limit of sets, 234
K-upper limit of sets, 234
ε-approximate solution, 114
ε-suboptimal solution, 116
γ-convergence of Borel measures, 260
γ_p-convergence of sets, 258
$\mathcal{D}$-weak compactness, 59
$\mathcal{D}$-weak convergence, 59
μ-Laplacian, 55
$\sigma(\mathbb{X}^*, \mathbb{X})$-locally convex topology, 58
ε-periodic graphlike structures, 410
p-admissible perturbation of a domain, 610
p-connected measure, 182
p-stable domain, 602
t-admissible perturbation of a domain, 602

approximation of singular measure, 189

Banach–Alaoglu theorem, 58
Borel measure, 15
Borel measure space, 15
Borel subsets, 15
Brinkman-type law, 550
by-component homogenization, 134

capacity of sets, 44
Clarkson inequalities, 178
coerciveness condition, 65
compactness condition, 65
constrained Γ-limit in a Banach space, 271
constrained Γ-limit in a metric space, 269
convergence in the sense Kuratowski, 234
cylindrically perforated domain, 547

distributed singular system, 110
divergence, 25
divergence with respect to the measure, 54

Ekeland metric in $L^\infty(D)$, 218
epigraph of a cost functional, 157

finely Borel subset, 46
Friedrichs inequality, 23, 33

Gâteaux derivative, 102
Green formula, 23

Hölder inequality, 71
Hausdorff distance, 236
Hausdorff outer measure, 18

ill-posed control object, 83

Lagrange function, 101
Lagrange multiplier principle, 101
Lebesgue points, 17
Lemma on Compensated Compactness, 595

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4,

© Springer Science+Business Media, LLC 2011

mean value of periodic function, 33
mean value property, 33
Mosco-convergence of Sobolev spaces, 238
Mosco-stable optimal control problem, 613

nondegenerate measure, 184

parametrical convergence of open sets, 243
Poincaré inequality, 24
Pompeiu–Hausdorff distance, 237
potential vector field, 25
problem of concentration, 38
problem of wild oscillation, 38

quasi-continuous functions, 44
quasi-open subsets, 44

Radon positive measure, 16
Radon–Nikodym decomposition, 17
Radon–Nikodym Theorem, 17
reduced defect measure, 38
regular set of the measure, 46
regularity condition, 65
rigid optimal control, 86

scheme of homogenization of OCP, 158
set convergence in the Hausdorff complementary topology, 238
set convergence in the Hausdorff metric, 236
singular set of the measure, 46
smoothing operator, 190
solenoidal periodic field, 34
solenoidal vector field, 25
starting control, 91
state functional, 159
steady-state Navier–Stokes equation, 549
strong χ-convergence, 219
strong two-scale convergence, 174
structure of critical thickness, 6
sufficiently thick structure, 5
sufficiently thin structure, 5

tangential gradient, 53
tangential space, 53
the weak approximation property, 491
thick multistructure, 478
thin structure, 4
Tikhonov minimizing sequence, 115
topological set convergence, 240
total variation of a measure, 16
trace operator, 22

uniform set convergence, 236

variational limit of constrained minimization problems, 276
variational limit of constrained minimization problems in variable spaces, 284

weak ε-approximate solution, 115
weak approximation property with respect to the scale of spaces, 284
weak minimizing sequence, 115
weak two-scale convergence, 174
weakened ε-approximate solution, 116
weakened minimizing sequence, 116
weakly convergent sequence, 35
weakly-$*$ convergent sequence, 36